Tiling of regular polygon with similar right triangles I

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A tiling is a decomposition of a polygon into finitely many non-overlapping triangles. (Note that a tiling is not a triangulation, i.e. a vertex of a triangle can lie on a side of a triangle.)

Clearly, for each n there is the 'trivial' tiling of a regular n-gon obtained by joining the center of the n-gon to each vertex of the n-gon. We obtain n congruent isosceles triangles with angles $\frac{2\pi}{n}$ at their vertices opposite to their bases. The 'trivial' tiling with isosceles triangles gives a 'trivial' tiling with congruent right triangles having angles $\frac{\pi}{n}$.

Theorem 1 (M.Laczkovich)**.** *If a regular n-gon, $n \geq 25, n \neq 30, 42$ is tiled with similar right triangles, then the smaller angle of these triangles equals to $\frac{\pi}{n}$.*

Theorem 1 follows from [L20, Theorem 1], since for any tiling of the regular n-gon with similar right triangles, the angles of the triangles are rational multiples of π. (Let us present a simple deduction of this fact. By p, q, r we denote the number of smaller acute angles α, bigger acute angles $\frac{\pi}{2} - \alpha$ and angles $\frac{\pi}{2}$ at the vertex of the regular n-gon, respectively. Take $a = 2\alpha/\pi$, then by Lemma 7 it follows that α is a rational multiple of π, which satisfies the hypothesis of [L20, Theorem 1]).

Theorem 2. *If a regular n-gon, $n \geq 5$, $n \neq 28$, can be tiled with similar right triangles, then one of the angles of these triangles is in $\left\{\frac{\pi}{n}, \frac{2\pi}{n}, \frac{\pi}{6} + \frac{2\pi}{3n}\right\}$.*

Because of Theorem 1, Theorem 2 is a new result only for $5 \leq n \leq 24$ or $n \in \{30, 42\}$ (see Remark 9 (e)). Theorem 2 was announced in [V19].

Theorem 2 and [V. Theorem 1] imply the following corollary.

Corollary 3. *If a regular n-gon, $n \geq 9$, $n \neq 12, 14, 20, 32, 44$, can be tiled by similar right triangles then one of the angles of these triangles is in $\frac{\pi}{n}$ or $\frac{2\pi}{n}$.*

We do not know if the remaining value $\frac{2\pi}{n}$ from Corollary 3 is 'realizable' by some tilings for $7 \leq n \leq 24$ or $n = 30, 42$.

Our proof of Theorem 2 is based on showing that for angles of the triangles other than mentioned in statement *the number of smaller acute angles* is greater

*This work was prepared in frame of math circle "Math and Olympiads". High School 67, Kutuzovsky prospect, 10, Moscow, Russia.

than the *number of larger acute angles*. This idea reduces the amount of cases to search through.

We illustrate the idea by a proof of the following corollary.

Corollary 4. *If a regular 8-gon can be tiled with similar right triangles, then one of the angles of these triangles is in* $\left\{\frac{\pi}{8}, \frac{\pi}{4}\right\}$.

If a reader is only interested in the proof of Theorem 2, he could skip Lemma 5 and the proof of Corollary 4.

By the side of a triangle we mean side of a triangle without its vertex.

Lemma 5. *If* $\frac{3}{2} = pa + q(1-a) + r$, *where* $0 < a < \frac{1}{2}$, $a \neq \frac{1}{4}$ *and* p, q, r *are non-negative integers, then* $p > q$ *and* $a = \frac{3}{2s}$ *for some positive integer* s.

Proof. Assume to the contrary that $q - p \geq 0$. We have

$$\frac{3}{2} = pa + q(1-a) + r = (q-p)(1-a) + p + r.$$

If $q - p = 0$ then $\frac{3}{2} = p + r$, but $p + r$ is an integer. Then $q - p \geq 1$.

Since $\frac{1}{2} < 1 - a = \frac{\frac{3}{2} - p - r}{q-p} \leq \frac{3}{2} - p - r$, we have $r = p = 0$.

Thus $\frac{3}{2} = (1-a)q$. $q \neq 0$ since $\frac{3}{2} \neq 0$. Since $0 < a < \frac{1}{2}$, it follows that $\frac{q}{2} < \frac{3}{2} < q$. Then $q = 2$. Hence $a = \frac{1}{4}$. A contradiction.

Thus $p > q$. We have

$$\frac{3}{2} = pa + q(1-a) + r = (p-q)a + q + r\ .$$

Since $0 < a$ and q, r are non-negative integers, it follows that $0 \leq q + r < 2$. Then $q + r \in \{0, 1\}$. Hence $a = \frac{3 - 2(q+r)}{2(p-q)} \in \{\frac{3}{2(p-q)}, \frac{1}{2(p-q)}\}$.

Thus $a = \frac{3}{2s}$, where s is a positive integer. □

Lemma 6. *If* $4 = pa + q(1-a) + r$, *where* $0 < a < \frac{1}{2}$, $a \notin \{\frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{7}, \frac{1}{3}\}$, p, q, r *are non-negative integers, then* $p \geq q$.

Proof. Assume to the contrary that $q - p > 0$. Then $q - p \geq 1$. We have

$$4 = pa + q(1-a) + r = (q-p)(1-a) + r + p.$$

If $r + p > 2$, then $1 - a = \frac{4-p-r}{q-p}$ is either greater than or equal to 1 or is less than or equal to $\frac{1}{2}$.

If $r + p = 2$, then $1 - a = \frac{2}{q-p}$. Since $0 < a < \frac{1}{2}$ it follows that $a = \frac{1}{3}$.

If $r + p = 1$, then $1 - a = \frac{3}{q-p}$. Hence $a \in \{\frac{1}{4}, \frac{2}{5}\}$.

If $r + p = 0$, then $1 - a = \frac{4}{q-p}$. Hence $a \in \{\frac{1}{5}, \frac{1}{3}, \frac{3}{7}\}$.

A contradiction. Thus, $p \geq q$. □

Proof of Corollary 4 independent of Theorem 2. Suppose there is a tilling of a regular 8-gon with right triangles of angles $\alpha < \frac{\pi}{4}$, $\frac{\pi}{2} - \alpha$ and $\frac{\pi}{2}$, and $\alpha \neq \frac{\pi}{8}$.

Take a vertex of the 8-gon. By p, q, r we denote the number of smaller acute angles α, bigger acute angles $\frac{\pi}{2} - \alpha$ and angles $\frac{\pi}{2}$ at this vertex, respectively. Then $\frac{3\pi}{4} = p\alpha + q\left(\frac{\pi}{2} - \alpha\right) + r\frac{\pi}{2}$. Divide this equality by $\frac{\pi}{2}$ and get $\frac{3}{2} = pa + q(1-a) + r$, where $0 < a < \frac{1}{2}$.

Then by Lemma 5

• at each vertex of an 8-gon the number of smaller acute angles α is greater than the number of bigger acute angles $\frac{\pi}{2} - \alpha$.

• $\alpha = \frac{3\pi}{4s}$ for some positive integer s.

Hence $\alpha \notin \{\frac{\pi}{10}, \frac{\pi}{5}, \frac{3\pi}{14}, \frac{\pi}{6}\}$.

For the triangles that have same vertices inside of the 8-gon and not on the side of a triangle we use Lemma 6. Then at any point inside of the 8-gon and not on the side of a triangle the number of smaller acute angles is greater than or equal to the number of bigger acute angles.

For the triangles that have same vertices on the side of 8-gon or on the side of a triangle, we use Lemma 6, substituting $r - 2$ for r. Then at any point on the side of 8-gon or on the side of a triangle the number of smaller acute angles is greater or equal to the number of bigger acute angles.

Hence the number of smaller acute angles is greater than the number of bigger acute angles. A contradiction. $\square$

Lemma 7. *For any integer $n \geq 5$ if $2 - \frac{4}{n} = pa + q(1-a) + r$, where $0 < a \leq \frac{1}{2}$, $a \notin \{\frac{2}{n}, \frac{4}{n}, \frac{1}{3} + \frac{4}{3n}\}$, p, q, r are non-negative integers, then $p > q$ and $a \in \{\frac{2-\frac{4}{n}}{s}, \frac{1-\frac{4}{n}}{s}\}$ for some positive integer s.*

Lemma 8. *For any integer $n \geq 5, n \neq 28$ if $a \in \{\frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{7}, \frac{1}{3}\}$ and $a \in \{\frac{2-\frac{4}{n}}{s}, \frac{1-\frac{4}{n}}{s}\}$ for some positive integer s, then either a or $1-a$ is in $\{\frac{2}{n}, \frac{4}{n}, \frac{1}{3} + \frac{4}{3n}\}$.*

Lemma 7 and Lemma 8 will be proved after the proof of Theorem 2.

Proof of Theorem 2. Suppose to the contrary there is a tilling of a regular n-gon with right triangles of angles $\alpha \leq \frac{\pi}{4}$, $\frac{\pi}{2} - \alpha$ and $\frac{\pi}{2}$, and $\alpha \notin \left\{\frac{\pi}{n}, \frac{2\pi}{n}, \frac{\pi}{6} + \frac{2\pi}{3n}\right\}$. Denote $a = \frac{2\alpha}{\pi}$.

If $\alpha = \frac{\pi}{4}$ then the angle of regular n-gon should be a rational multiple of $\frac{\pi}{4}$, hence only regular $4, 8$ - gons can be tiled with right isosceles triangles. For $n = 8$, $\alpha = \frac{2\pi}{8} = \frac{\pi}{4}$.

Take a vertex of the n-gon. By p, q, r we denote the number of smaller acute angles α, bigger acute angles $\frac{\pi}{2} - \alpha$ and angles $\frac{\pi}{2}$, respectively at this vertex. Then $2 - \frac{4}{n} = pa + q(1-a) + r$, where $0 < a \leq \frac{1}{2}$. Then by Lemma 7, $p > q$ and $\alpha \in \{\frac{\pi - \frac{2\pi}{n}}{s}, \frac{\frac{\pi}{2} - \frac{2\pi}{n}}{s}\}$ for some positive integer s.

Since $\alpha \notin \{\frac{\pi}{n}, \frac{2\pi}{n}, \frac{\pi}{6} + \frac{2\pi}{3n}\}$, it follows that by Lemma 8, $\alpha \notin \{\frac{\pi}{10}, \frac{\pi}{5}, \frac{3\pi}{14}, \frac{\pi}{6}, \frac{\pi}{8}\}$.

For the triangles that have same vertices inside of regular n-gon not on the side of a triangle, denote $a = \frac{2\alpha}{\pi}$, by p we denote the number of the smaller acute angles α, by q the number of greater acute angles $\frac{\pi}{2} - \alpha$, by r the number of right angles. Then by Lemma 6 at any point inside of n-gon not on the side of a triangle $p \geq q$.

For the triangles that have same vertices on the side of n-gon or on the side of a triangle,we use Lemma 6, substituting $r-2$ for r. Then at any point on the side of n-gon or on the side of a triangle $p \geq q$.

Hence the number of smaller acute angles is greater than the number of bigger acute angles. A contradiction.

□

Proof of Lemma 7. Assume to the contrary that $q-p \geq 0$.

We have

$$2-\frac{4}{n}=pa+q\left(1-a\right)+r=\left(q-p\right)\left(1-a\right)+p+r.$$

If $q-p=0$ then $2-\frac{4}{n}=p+r$, but $2>2-\frac{4}{n}>1$ and $p+r$ is integer. Then $q-p \geq 1$.

If $r+p>1$, then $1-a=\frac{2-\frac{4}{n}-p-r}{q-p}<0$. A contradiction. Thus either $r+p=1$ or $r+p=0$.

If $r+p=1$, then $1-a=\frac{1-\frac{4}{n}}{q-p}$. Hence $a=\frac{4}{n}$.

If $r+p=0$, then $1-a=\frac{2-\frac{4}{n}}{q-p}$. Hence $q-p \in \{2,3\}$. Then $a \in \{\frac{2}{n}, \frac{1}{3}+\frac{4}{3n}\}$. A contradiction. Thus $p>q$.

We have

$$2-\frac{4}{n}=pa+q\left(1-a\right)+r=\left(p-q\right)a+q+r\ .$$

Hence $0 \leq r+q<2$.

Let $u=q+r$ and $s=p-q$. Then $a=\frac{2-u-\frac{4}{n}}{s}$, where s is a non-negative integer.

Thus $a \in \{\frac{2-\frac{4}{n}}{s}, \frac{1-\frac{4}{n}}{s}\}$ for some positive integer s.

□

Proof of Lemma 8. We provide proof for general $a=\frac{z-\frac{4}{n}}{s}$ where z is either 1 or 2.

Suppose that $a=\frac{1}{4}=\frac{z-\frac{4}{n}}{s}$. Hence $s=4z-\frac{16}{n}$. Since s and n are positive integers, it follows that $n \in \{1,2,4,8,16\}$. But $n \geq 5$ and $a=\frac{1}{4}=\frac{2}{8}=\frac{4}{16}$. Thus for $a=\frac{1}{4}$, $a \in \{\frac{2}{n}, \frac{4}{n}, \frac{1}{3}+\frac{4}{3n}\}$ for some positive integer n.

Suppose that $a=\frac{1}{5}=\frac{z-\frac{4}{n}}{s}$. Hence $s=5z-\frac{20}{n}$. Since s and n are positive integers, it follows that $n \in \{1,2,4,5,10,20\}$. But $n \geq 5$ and $a=\frac{1}{5}=1-\frac{4}{5}=\frac{2}{10}=\frac{4}{20}$. Thus for $a=\frac{1}{5}$, $a \in \{\frac{2}{n}, \frac{4}{n}, \frac{1}{3}+\frac{4}{3n}\}$ for some positive integer n.

If $a=\frac{2}{5}=\frac{z-\frac{4}{n}}{s}$. Hence $2s=5z-\frac{20}{n}$. Since s and n are positive integers, it follows that $n \in \{1,2,4,5,10,20\}$. But $n \geq 5$ and $a=\frac{2}{5}=\frac{2}{5}=\frac{4}{10}=\frac{1}{3}+\frac{4}{3 \cdot 20}$. Thus for $a=\frac{2}{5}$, $a \in \{\frac{2}{n}, \frac{4}{n}, \frac{1}{3}+\frac{4}{3n}\}$ for some positive integer n.

If $a=\frac{3}{7}=\frac{z-\frac{4}{n}}{s}$. Hence $3s=7z-\frac{28}{n}$. Since s and n are positive integers, it follows that $n \in \{1,2,4,7,14,28\}$. But $n \geq 5$, $n \neq 28$ and $a=\frac{3}{7}=\frac{1}{3}+\frac{4}{3 \cdot 14}=1-\frac{4}{7}$. Thus for, $a=\frac{3}{7}$ $a \in \{\frac{2}{n}, \frac{4}{n}, \frac{1}{3}+\frac{4}{3n}\}$ for some positive integer n.

If $a = \frac{1}{3} = \frac{z-\frac{4}{n}}{s}$. Hence $s = 3z - \frac{12}{n}$. Since s and n are positive integers, it follows that $n \in \{1, 2, 3, 4, 6, 12\}$. But $n \geq 5$ and $a = \frac{1}{3} = \frac{2}{6} = \frac{4}{12}$. Thus for $a = \frac{1}{3}$, $a \in \{\frac{2}{n}, \frac{4}{n}, \frac{1}{3} + \frac{4}{3n}\}$ for some positive integer n.

$\square$

Remark 9. *(Relation to earlier results)*

(a) A tiling of a regular n-gon with right non-isosceles triangles with angle $\alpha \neq \frac{\pi}{4}$ *is called regular if either*

(1) *at each point of the (convex hull of the) n-gon the number of angles* α *is equal to the number of angles* $\frac{\pi}{2} - \alpha$*, or*

(2) *at each point of the n-gon the number of angles* α *is equal to the number of angles* $\frac{\pi}{2}$*, or*

(3) *at each point of the n-gon the number of angles* $\frac{\pi}{2} - \alpha$ *is equal to the number of angles* $\frac{\pi}{2}$*.*

Otherwise the tiling is called irregular.

(b) From [L12, Theorem 2.1] it follows that a regular n-gon, $n \geq 5, n \neq 6$ *cannot be regularly tiled with congruent right triangles .*

(c) Let us present a simple deduction of (b) from [L12, Theorem 2.1]. If a regular n-gon, $n \geq 5, n \neq 6$*, is regularly tiled with triangles with angles* $\alpha, \frac{\pi}{2} - \alpha, \frac{\pi}{2}$ *then one of the properties (1), (2), (3) of (a) is true. Since* $n \geq 5, n \neq 6$ *cases (i, iv, vi, ix) listed in [L12, Theorem 2.1] cannot be true. Since the n-gon is regular and triangles are right, cases (vii, viii) cannot be true.*

In case (ii) we have either $\alpha = \pi - \frac{2\pi}{n}$ *or* $\frac{\pi}{2} - \alpha = \pi - \frac{2\pi}{n}$ *or* $\frac{\pi}{2} = \pi - \frac{2\pi}{n}$*. Since* $n \geq 5$ *it follows that* $\pi - \frac{2\pi}{n} > \frac{\pi}{2}$*. A contradiction.*

In case (iii) we have either $\alpha = \frac{2\pi}{n}$ *or* $\frac{\pi}{2} - \alpha = \frac{2\pi}{n}$ *and* $\sin\alpha, \sin(\frac{\pi}{2} - \alpha), \sin(\frac{\pi}{2})$ *should be pairwise commensurable. Since* $\sin(\frac{\pi}{2}) = 1$ *it follows that* $\sin\alpha = \cos(\frac{\pi}{2} - \alpha)$ *and* $\sin(\frac{\pi}{2} - \alpha) = \cos\alpha$ *should be rational. Since* $n \notin \{1, 2, 3, 4, 6\}$*, it follows that* $\cos(\frac{2\pi}{n})$ *is not rational, according to [M, Theorem 4.1]. Hence case (iii) cannot be true.*

In case (v) we have either $\alpha = \frac{2\pi}{n}$ *or* $\frac{\pi}{2} - \alpha = \frac{2\pi}{n}$ *and* $\frac{\sin\alpha}{\sin(\frac{\pi}{2} - \alpha)} = 1$*. Thus* $\alpha = \frac{\pi}{2} - \alpha = \frac{\pi}{4}$*, but* $n > 4$*. Hence case (v) cannot be true.*

(d) In Theorem 1 we have the assumption that, $n \geq 25, n \neq 30, 42$*. In [L12, Theorem 2.1] we have the assumption that the tiling is regular, we do not know if for any* $5 \leq n \leq 24, n \in 30, 42$ *there are irregular tilings with right triangles with angle* $\alpha \neq \frac{\pi}{n}$*.*

(e) For the cases $n < 5$ *and* $n = 6$ *see [L12, Theorem 2.1] and [S, Theorem].*

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