Towards higher-dimensional combinatorial geometry*

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Introduction

Many fields of science and technology, primarily mathematics, computer science and physics, often work with higher-dimensional space. Solving the following problems allows to acquire basic skills of such work. You will both develop spatial imagination and intuition, and learn how to check them by rigorous arguments. This is useful for further learning of computer graphics, as well as for necessary methods of linear algebra and geometry.

The main ideas are presented as 'olympic' examples in easiest specific cases, free of technical details and only with minimal amount of scientific terminology (see the problems). This makes the project accessible for beginners though still containing beautiful and complicated results. The project does not require preliminary knowledge of stereometry. Geometric intuition in space and ability to solve systems of linear equations will be useful (see Problem 1.2).

In this project we generalize the following result (see the following problems S, D, SD and §3).

Radon theorem in the plane. *For any 4 points in the plane either one of them belongs to the triangle with vertices at the others, or they can be decomposed into two pairs such that the segment joining the points of the first pair intersects the segment joining the points of the second pair.*

S = same size. From any 5 points in the plane one can choose two disjoint pairs such that segments joining them intersect.

In this text a triangle Δ is the part of plane bounded by the outline $\partial\Delta$. This part can be a line segment.

D = dimension. For any 5 points in the space either one of them belongs to the tetrahedron with vertices at the others, or the segment joining some two of them intersects the triangle formed by the remaining three of them.

In this text interesting non-trivial problems are called theorems.

Theorem SD: Radon theorem for sets of almost the same size.

(3) From any 6 points in the space one can choose a disjoint pair and a triple such that the segment joining points of the pair intersects the triangle formed by the triple.

(4) (linear van Kampen-Flores theorem, 1932) From any 7 points in 4-dimensional space one can choose two disjoint triples such that the corresponding triangles intersect.

Definition of 4-dimensional space and of notions required for proofs are presented in §1 and in §2. For a proof of Theorem SD.4 you would also need Theorem S'D below.

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Take two non-degenerate triangles in the space no 4 of whose vertices lie in a plane. Such triangles are **linked** if the outline of the first triangle intersects second triangle at exactly one point.

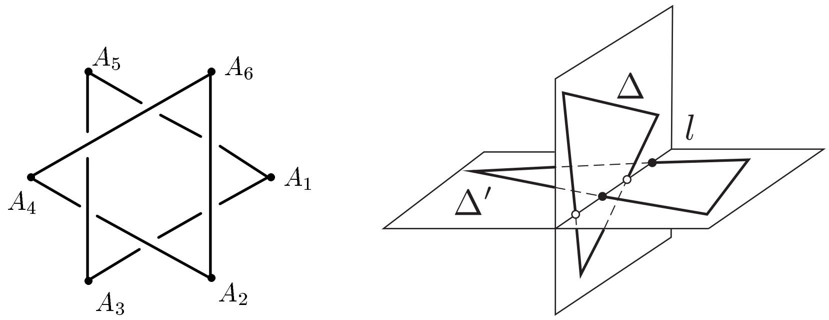

Figure 1: Linked triangles and linked pairs of points

Theorem S'D: spatial Radon theorem on linking of same size sets; linear Conway-Gordon-Sachs theorem, 1981-1983.

If no 4 of 6 points in the space lie in a plane then there are two linked triangles with vertices in these points.

For a proof you would need assertion QS of §3.

Other 'olympic' problems are Theorem D(d) in §1, 2.3.c, 2.5.b, 2.7.b, 3.2, 3.6.(4'-3). Open problems — 3.6.(4-3),(4-2),(4'-2).

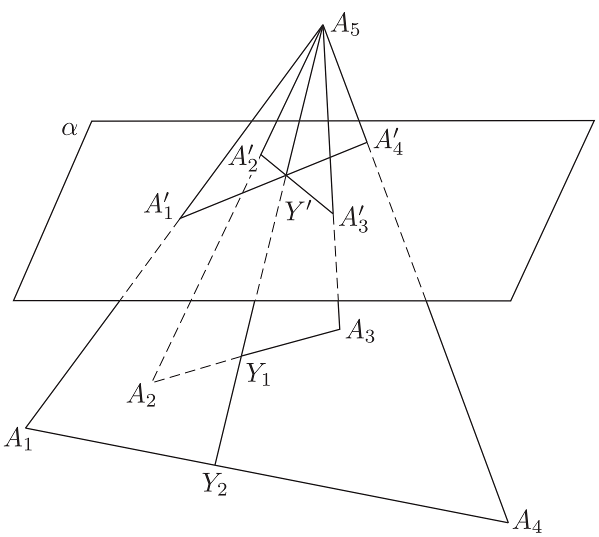

Figure 2: Hint to problem D

Recommendations for participants

If a mathematical statement is formulated as a problem, then the objective is to prove this statement. If a problem is named 'theorem' ('lemma', 'corollary', etc.), then this statement is considered to be more important. Usually we *formulate* a beautiful or important statement *before* giving a sequence of results (lemmas, assertions, etc) which constitute its proof. In this case, in order to prove this statement, one may need to solve some of the subsequent problems. We give hints on that after the statements but we do not want to deprive you of the pleasure of finding the right moment when you finally are ready to prove the statement. In general, if you are stuck on a certain problem, try looking at the next ones. They may turn out to be helpful. *Remarks* and problems marked by star are not used in the sequel. Important definitions are highlighted in **bold** for easy navigation.

For every **solution written for a user** (see recommendations below) marked with either '+' or '+.' a student (or a group of students) gets a 'bean'. The jury may also award extra beans for beautiful solutions, solutions of hard problems, or solutions typeset in TEX. The jury has infinitely many beans. Every participant (or group of participants) initially has 1 bean. One may submit a solution **in oral form** or as **written for a developer**, and one loses a bean with each 5 attempts (successful or not).

Participants (or teams) can submit their solutions by a personal communication to Egor Riabov at `https://mattermost.turgor.ru`. Please also send him questions and requests for hints on problems which you are stuck with. Students who successfully work on the project are entitled to ask interesting *extra problems for investigation*.

Participants (or teams) from Serbia and Croatia may submit their solutions to Prof. Rade Živaljević at `rade@mi.sanu.ac.rs`.

How to write a proof for a user

We give some recommendations on how to write a proof that could be included in a mathematical book or research paper (which is a 'reliable reference', cf. `https://arxiv.org/pdf/2101.03745.pdf`, p. 2). These recommendations are by no means complete. You can learn to write proofs (solutions of problems) by trying to write them and discussing your text with a teacher.

See also `https://en.wikipedia.org/wiki/KISS_principle`
`http://people.apache.org/~fhanik/kiss.html`

A genius makes his own rules, but a 'how to' article is written by one ordinary mortal for the benefit of another... Most things that an article such as this one can say have at least one counterexample in the practice of some natural born genius. Authors of articles such as this one know that, but in the first approximation they must ignore it, or nothing would ever get done.

(P. Halmos, How to talk mathematics.)

(1) Only write sentences that make sense.[1]

(1a) In particular, long sentence usually does not make sense because it is unclear which exactly parts of long sentence words 'and', 'or', 'then' are tying together. So break long sentences into short ones.

(2) Introduce notation and each definition explicitly with 'define', 'denote', 'let', 'set', 'put'. For example, the phrase '$a = b+c$' without these words means 'the previously defined object a equals to the sum of the previously defined objects b and c'.

(5) Do not put any part of your solution in parentheses. Parentheses do not make clear the logical relation between the phrase in and outside the parentheses. (Parentheses are used for remarks which are not part of the solution.)

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[1] For example, none of the following two sentences makes sense:

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2},$$

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \text{ for a positive integer } n,$$

because it is not written for which n the statement is stated. The following statements do make sense:

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \text{ for } \textit{every} \text{ positive integer } n,$$

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \text{ for } \textit{some} \text{ positive integer } n,$$

$$1 + 2 + \ldots + n = 100 \text{ for } \textit{some} \text{ positive integer } n.$$

However, the second of them is not interesting and the third is not correct.

1 How to work with four-dimensional space?

For Problems 1.1, 1.4.abc, 1.5.abcd, 1.8.cd and 1.9.be below it suffices to give correct answers. (Solution of Problem 1.8.a is already presented in this text.)

1.1. How many intersection points can a line and a plane have in the space?

1.2. How many solutions can have a system of linear equations
(a) 2×2; (b) 2×3 (2 equations, 3 variables); (c) 3×2?

We define
• the *line* as the set of all real numbers;
• the *plane* $\mathbb{R}^2$ as the set of all ordered pairs (x, y) of real numbers x and y;
• *three-dimensional space* (*3-space*) $\mathbb{R}^3$ as the set of all ordered triples (x, y, z) of real numbers;
• *four-dimensional space* (*4-space*) $\mathbb{R}^4$ as the set of all ordered quadruples (x, y, z, t) of real numbers.

Definition of *d-dimensional space* (*d-space*) $\mathbb{R}^d$ for $d > 4$ is analogous.

For points $A = (x_1, y_1, z_1, t_1), B = (x_2, y_2, z_2, t_2) \in \mathbb{R}^4$ and number $\lambda \in \mathbb{R}$ denote

$$\lambda A := (\lambda x_1, \lambda y_1, \lambda z_1, \lambda t_1) \quad \text{and} \quad A + B := (x_1 + x_2, y_1 + y_2, z_1 + z_2, t_1 + t_2).$$

1.3. A 2-dimensional plane does not split the 4-dimensional space. I.e., for each two points outside the plane $x = y = 0$ in 4-space, there exists a broken line which joins these points and does not intersect this plane.

For points $A, B \in \mathbb{R}^4$ a *segment* AB is the set $\{tA + (1-t)B \ : \ t \in [0, 1]\}$. A *broken line* is the union of segments $A_i A_{i+1}$ over all $i = 1, 2, \ldots, n-1$.

Hint. For points $A = (x_0, y_0, z_0, t_0)$ and B which do not lie in the plane $x = y = 0$ define points

$$A_x = A + (1, 0, 0, 0) = (x_0 + 1, y_0, z_0, t_0) \quad \text{and} \quad A_y = A + (0, 1, 0, 0) = (x_0, y_0 + 1, z_0, t_0).$$

Prove that one of the broken lines AB, AA_xB and AA_yB does not intersect the plane $x = y = 0$.

1.4. What is the intersection of the *2-dimensional sphere*

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

with the following sets:
(a) the line $x = y = 0$, containing the center of the sphere;
(b) the plane $x = 0$, containing the center of the sphere;
(c) the intersection of the positive octant of $\mathbb{R}^3$ and the union of the 2-dimensional coordinate planes, i.e.

$$\{(x, y, z) \in \mathbb{R}^3 \ : \ x \geq 0, \ y \geq 0, \ z \geq 0 \text{ and } xyz = 0\}.$$

1.5. What is the intersection of the *3-dimensional sphere*

$$S^3 := \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}$$

with the following sets:
(a) the line $x = y = z = 0$, containing the center of the sphere;
(b) the plane $x = y = 0$, containing the center of the sphere;

(c) the (3-dimensional) hyperplane $x = 0$, containing the center of the sphere;

(d) the intersection of the positive 'sixteenth' of $\mathbb{R}^4$ and the union of the 2-dimensional coordinate planes, i.e.

$$\{(x, y, z, t) \in \mathbb{R}^4 \ :$$

$$: \ x \geq 0, \ y \geq 0, \ z \geq 0, \ t \geq 0 \text{ and at least two of the four numbers } x, y, z, t \text{ are zeros}\}.$$

A subset $L \subset \mathbb{R}^4$ is called a **line** if L is not a point and there are points $A, B \in \mathbb{R}^4$ such that $L = \{A + Bt \ : \ t \in \mathbb{R}\}$.

A subset $L \subset \mathbb{R}^4$ is called a (2-dimensional) **plane** if L is neither a line nor a point, and there are points $A, B, C \in \mathbb{R}^4$ such that $L = \{A + Bt + Cu \ : \ t, u \in \mathbb{R}\}$.

We already introduced definition of a line before. However, below 'line' means a subset of $\mathbb{R}^d$ whose definition is similar to the one given above. Analogous remark applies to plane.

1.6. Write analogous definition of a (3-dimensional) **hyperplane** in $\mathbb{R}^4$.

In your solutions of the following problems on 4-space you can use without proof

- all rigorously formulated and correct facts on solutions of systems of linear equations;
- the results of Problem 1.7.

1.7. * (a) A subset $L \subset \mathbb{R}^4$ is a hyperplane if and only if $L \neq \emptyset$, $L \neq \mathbb{R}^4$ and there exist $a, b, c, d, e \in \mathbb{R}$ such that

$$L = \{(x, y, z, t) \in \mathbb{R}^4 \ : \ ax + by + cz + dt = e\}.$$

(b) A subset $L \subset \mathbb{R}^4$ is a plane if and only if $L \neq \emptyset$, $L \neq \mathbb{R}^4$, L is not a hyperplane and there exist $a_1, b_1, c_1, d_1, e_1, a_2, b_2, c_2, d_2, e_2 \in \mathbb{R}$ such that

$$L = \{(x, y, z, t) \in \mathbb{R}^4 \ : \ a_1x + b_1y + c_1z + d_1t = e_1, \ a_2x + b_2y + c_2z + d_2t = e_2\}.$$

(c) State and prove analogous result for a line in $\mathbb{R}^4$.

1.8. What could be the intersection in $\mathbb{R}^4$ of:

(a) a line and a hyperplane? (b) a line and a plane?

(c) a plane and a hyperplane? (d) two hyperplanes? (e) two planes?

Hint to (a). Answer. The empty set, a point, a line.

Examples. The empty set is the intersection of the line $x = y = z = 0$ and the hyperplane $x = 1$. A point is the intersection of the line $x = y = z = 0$ and the hyperplane $t = 0$. A line is the intersection of the line $x = y = z = 0$ and the hyperplane $x = 0$.

Proof that other intersections are impossible. It suffices to prove that if the intersection in $\mathbb{R}^4$ of a line l and a hyperplane contains at least two points, then the intersection coincides with the line l. This holds because *for any two points there exists a unique line containing both these points.* The latter fact is easily proved using the definition of a line. (In many other expositions this fact is accepted as an axiom.)

1.9. For different points $X, Y \in \mathbb{R}^4$ define the *line* XY as $\{X + (Y - X)t = (1-t)X + tY \ : \ t \in \mathbb{R}\}$. For points $X, Y, Z \in \mathbb{R}^4$ not belonging to any line define the *plane* XYZ as

$$\{X + (Y - X)t + (Z - X)u = (1 - t - u)X + tY + uZ \ : \ t, u \in \mathbb{R}\}.$$

No five of eight points 1,2,3,4,5,6,7,8 in $\mathbb{R}^4$ belong to a hyperplane. What could be the intersection of:

(b) the line 12 and the plane 567? (d) the hyperplanes 1234 and 5678?

(e) the planes 123 and 567?

The **convex hull** of a finite collection of points $A_1, \ldots, A_n \in \mathbb{R}^d$ is by definition the set

$$\langle A_1, \ldots, A_n \rangle := \{\lambda_1 A_1 + \ldots + \lambda_n A_n \ : \ \lambda_1, \ldots, \lambda_n \geq 0, \ \lambda_1 + \ldots + \lambda_n = 1\}.$$

1.10. The convex hull of a finite collection of points in the plane is the least (by inclusion, or by area) convex polygon containing all these points.

Theorem D(d): the Radon theorem, 1929.

Any $d+2$ points in d-space can be split into two subsets whose convex hulls intersect.

Edification. Usually only the simplest properties of planar and spatial geometric objects are deduced from the analytic definition (or just accepted as axioms). More complicated properties can be deduced in a 'synthetic' ('geometric') way from the simplest ones (i.e., as in school geometry, without using the analytic definition). Often it is convenient to reduce a planar problem to a linear one (i.e., to a problem in a line), and a spatial problem to a planar one. Similarly, the best approach to the following four-dimensional problems is an analogy to, or a reduction to, spatial ones.

2 Parity Lemmas

As a specific goal of this subsection one can consider propositions 2.3.c, 2.5.b, 2.7.b, which illuminate the non-triviality of the material.

Lemma 2.1 (parity). *If out of 6 vertices of two triangles in the plane, no 3 lie on a line, then the outlines of these triangles intersect each other at an even number of points.*

Proof. The outline of a triangle splits the plane.[2] The polygonal line formed by the sides of one triangle goes *inside* the other triangle as many times as it goes *outside*. □

Some points in the plane **are in general position**, if no 3 of them lie in the same line, and no three segments joining them have a common interior point.

2.2. (a) Are all the points of some circle in general position?

(b) If the vertices of two polygonal lines in the plane are in general position, then the polygonal lines intersect at a finite number of points.

Hint: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

2.3. (a) The plane is not a union of a finite number of lines.

(b) Are there 100 general position points in the plane?

(c) There are 14 general position points in the plane: 7 red and 7 yellow. Then the number of all the intersection points of the red segments (i.e., the segments joining the red points) with the yellow segments is even.

Lemma 2.4 (Parity). *If no 4 among the 7 vertices of a triangle and a tetrahedron in 3-space lie in the same plane, then the outline of the triangle and the surface of the tetrahedron intersect at an even number of points.*

Some points in 3-space **are in general position**, if no 4 of them lie in the same plane, and no segment, triangle and triangle spanned by them have a common interior point. E.g. in general position are the 6 points in Figure 3.

2.5. (a) Are there 100 general position points in the 3-space?

(b) In the 3-space there are 17 general position points: 7 red and 10 yellow. Then the number of all the intersection points of the red segments (i.e., the segments joining the red points) with yellow triangles is even.

Lemma 2.6 (Parity). *If no 5 among the 8 vertices of two tetrahedra in 4-space lie in the same hyperplane, then the surfaces of the tetrahedra intersect at an even number of points.*

[2] This fact, in contrast to the *piecewise linear Jordan theorem* [Sk20, §1.4], is proved without using the Parity Lemma.

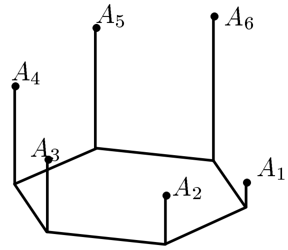

Figure 3: Six general position points in 3-space

Hint: Take the section by the hyperplane containing one of the tetrahedra.

Some points in 4-space **are in general position**, if no 5 of them lie in the same hyperplane, and no three triangles spanned by them have a common interior point.

2.7. (a) Are there 100 general position points in the 4-space?

(b) In the 4-space there are 16 general position points: 8 red and 8 yellow. We call *red/yellow* 2-dimensional triangles spanned by red/yellow points. Then the number of all the intersection points of the red triangles with the yellow triangles is even.

3 Quantitative versions

Q = quantitative. If no 3 of 4 points in the plane lie on a line, then there exists exactly one partition of these points into two subsets as in the above Radon theorem in the plane.

QS: the quantitative Radon theorem for same size sets in the plane; linear van Kampen-Flores theorem in the plane.

If no 3 of 5 points in the plane lie on a line, then the number of common interior points of the segments joining these 5 points is odd.

QD: the quantitative Radon theorem.

(3) If no 4 of 5 points in 3-space lie in a plane, then there exists exactly one partition of these points into two subsets whose convex hulls intersect.

(4) State and prove analogous result for 4-space and for d-dimensional space.

Theorem QSD: the quantitative Radon theorem for sets of almost the same size.

(3) If no 4 of 6 points in 3-space lie in a plane, then the number of intersection points of interiors of segments joining pairs of points, and (2-dimensional) triangles spanned by these points, is even.

(4) If no 5 of 7 points in 4-space lie in a hyperplane, then the number of intersection points of (2-dimensional) triangles spanned by these 7 points is even.

For a proof of Theorem QSD.4 you would need Theorem QS'D below.

The property of being linked is not symmetric a priori.

Lemma 3.1 (Symmetry)**.** *Triangles Δ and Δ' in 3-space are linked if and only if Δ' and Δ are linked.*

3.2. There are 13 points in 3-space: 3 red and 10 yellow. No 4 of them lie in a plane. Then the number of yellow triangles linked with red triangle is even. We call triangle *red (yellow)* if all of its vertices are red (yellow). Triangles which differ only by a permutation of vertices are considered to be the same.

3.3. In 3-space *a segment p is below a segment q* (looking from point O), if there exists a half-line OX with the endpoint O that intersects the segment p at a point $P := p \cap OX$, the segment q at a point $Q := q \cap OX$, $P \neq Q$, so that Q is in the segment OP.

Assume that no 4 of points $O, A_1, \dots, A_5$ in 3-space lie in a plane, and there is a plane splitting O from $A_1, \dots, A_5$. The triangles OA_1A_2 and $A_3A_4A_5$ are linked if and only if A_1A_2 is below an odd number of sides of the triangle $A_3A_4A_5$.

Theorem QS'D: the quantitative spatial Radon theorem on linking of same size sets.

If no 4 of 6 points in 3-space lie in a plane, then the number of non-ordered pairs of linked triangles with vertices in these points is odd.

Theorems SD and QSD show that under transition from dimension 2 to dimension 3 the property of the existence of intersection is preserved, while the parity of the number of intersections change. The odd-dimensional version of SD and QSD have a stronger form: Theorems S'D and QS'D.

The following unlinking properties are related to intersection properties QS, QSD.

3.4. (2) There are 5 points in the plane such that no 3 of them lie in a line, and every segment joining 2 of them intersects the outline of the triangle formed by the 3 remaining points at an even number of points.

(2') If no 3 of 5 points in the plane lie in a line, then the number of those segments joining 2 of them that intersect the outline of the triangle formed by the remaining 3 points exactly at one point, is even.

Assertion 3.4.2 means that every pair of points is 'unlinked' with the triangle formed by the remaining points. We do not spell out analogous interpretations of properties 3.5.3, 3.6.(4-2),(4-3) below.

In 3-space instead of unlinking properties 3.4 there is a linking property (Theorem QS'D from §3) and the following unlinking properties.

3.5. (3) There are 6 points in 3-space such that no 4 of them lie in a plane, and every segment joining 2 of them intersects the surface of the tetrahedron formed by the remaining 4 points at an even number of points.

(3') If no 4 of 6 points in 3-space lie in a plane, then the number of intersection points of segments joining them and surfaces of tetrahedra formed by the remaining 4 points, is even.

One can make a remark analogous to the one after Theorem QSD.

It would be interesting to prove the following statement 3.6.(4'-3), conjectures 3.6.(4-3),(4'-2),(4-2) and their higher-dimensional analogues. (We are grateful to M. Tancer for sending me proof of the PL version of 3.6.(4-3).)

3.6. (4'-3) If no 5 of 7 points in 4-space lie in a hyperplane, then the number of those triangles spanned by 3 of them that intersect exactly at one point the surface of the tetrahedron formed by the 4 remaining points, is even.

(4-3) There are 7 points in 4-space such that no 5 of them lie in a hyperplane, and every triangle formed by 3 of them intersects the surface of the tetrahedron formed by the 4 remaining points at an even number of points.

(4'-2) If no 5 of 7 points in 4-space lie in a hyperplane, then the number of intersection points of segments joining them and 3-dimensional surfaces of 4-dimensional simplices formed by the 5 remaining points, is even.

(4-2) There are 7 points in 4-space such that no 5 of them lie in a hyperplane, and every segment joining 2 of them intersects the 3-dimensional surface of the 4-simplex formed by the 5 remaining points at an even number of points.

4 Multiple versions (M)

M = multiplicity. Any 17 points in the plane can be split into three sets whose convex hulls have a common point.

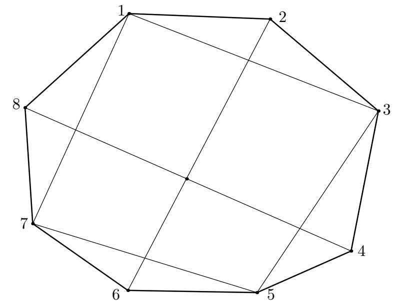

Figure 4: The common point of three convex hulls

Proof. The vertices of any convex octagon in the plane can be split into three sets whose convex hulls have a common point (fig. 4).

Suppose that the convex hull of the given set of 17 (or even 11) points has at least 8 vertices. Then split these 8 vertices into three sets as above. If the convex hull has less than 8 vertices, then denote by S_1 the set of these vertices. There remain at least 4 points. Thus they can be split into two sets whose convex hulls intersect. This intersection lies in the convex hull of S_1 as well.

Theorem 4.1 (M: r-fold Radon theorem in the plane; the Tverberg theorem in the plane, 1965)**.** *Any 7 points in the plane can be split into three sets whose convex hulls have a common point.*

Any $3r-2$ *points in the plane can be split into* r *sets whose convex hulls have a common point.*

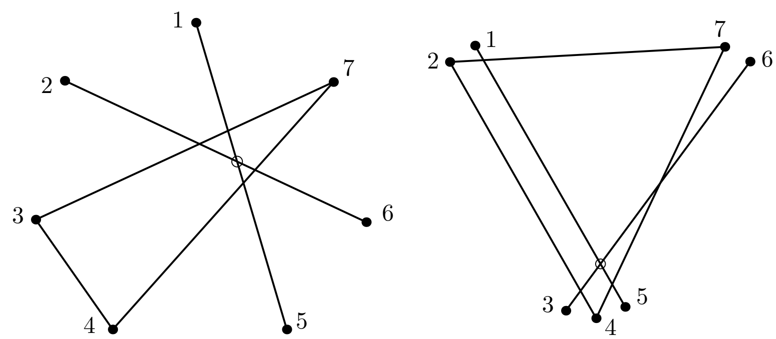

Figure 5: The common point of three convex hulls

Before proving Theorems M and DM, it is advisable to solve problems 4.2, 4.3, 4.4, 4.6.

4.2 (cf. Theorem M)**.** (a) There are 6 points in the plane such that any partition of the points into three sets the convex hulls of these sets do not have a common point.

Hint. Take a pair of points near each vertex of a triangle.

(b) There are 7 points in the plane no one of which lies in any triangle formed by any triple of the remaining points.

Hint. Take the vertices of a convex heptagon.

(c) There are 7 points in the plane with the following property. Take any two segments joining two disjoint pairs of the given 7 points. Then either the segments do not intersect or the point of their intersection does not lie in the triangle formed by the three remaining given points.

Hint. Take the vertices of an equilateral triangle and its center. Also take the middle-points of the segments joining the vertices and the center.

(d) State and prove r-fold versions of these examples.

The quantitative 3-fold Radon theorem in the plane (QM) **is unknown!**

4.3. Are there 6 points in the plane such that for any partition of these points into three sets the convex hulls of some two of these sets are disjoint?

4.4. (a) For the vertices of regular heptagon the number of partitions from Theorem M is 7.

Hint. Every such partition looks like a rotated partition of fig. 5, left.

(b) For the points in fig. 5, right, the number of partitions from Theorem M is 4.

Hint. This follows because for every such partition one of the convex hulls is a triangle with one vertex 4, another vertex 1 or 2, and the third vertex 6 or 7.

Remark. Hence the following sum has different parity for the two above 7-element sets M_a, M_b

$$v(M_i) := \sum_{\{R_1,R_2,R_3\}\ :\ M_i=R_1\sqcup R_2\sqcup R_3} |\,\langle R_1\rangle \cap \langle R_2\rangle \cap \langle R_3\rangle\,|.$$

See further [Sk18, §2].

Theorem 4.5 (DM: spatial r-fold Radon theorem; the spatial Tverberg theorem, 1965)**.** *Any $4r-3$ points in 3-space can be split into r sets whose convex hulls have a common point.*

4.6 (cf. Theorem DM)**.** (a) There are 8 points in 3-space such that for any partition of these points into three sets the convex hulls of these sets do not have a common point.

(b) There are $4r-4$ points in 3-space such that for any partition of these points into r sets the convex hulls of these sets do not have a common point.

(c) There are $(r-1)(d+1)$ points in d-space such that for any partition of these points into r sets the convex hulls of these sets do not have a common point.

4.7. * State and solve the spatial version of Problem 4.3.

In the proof of Theorems M and DM you can use without proof the Coloured Caratheodory Theorem (whose proof is not a part of this project).

Theorem 4.8 (Barany; the Coloured Caratheodory Theorem)**.** *Suppose that the point $0 \in \mathbb{R}^n$ lies in the convex hull of every set among finite sets $M_0, M_1, \dots, M_n \subset \mathbb{R}^n$. Then there are points $m_i \in M_i$ such that $0 \in \langle m_0, m_1, \dots, m_n\rangle$.*

Theorem 4.9 (SM: low-dimensional 3-fold Radon theorem for same size sets; the linear Sarkaria theorem, 1991)**.** ** From any 11 points in 3-space one can choose three pairwise disjoint triples such that the three triangles formed by these triples have a common point.*

4.10 (cf. Theorem SM)**.** There are 10 points in 3-space such that there are no three pairwise disjoint triples of these points for which the three triangles formed by these triples have a common point.

Theorem 4.11 (SDM: r-fold Radon theorem for arbitrary dimension for same-sized sets; the linear Sarkaria-Volovikov theorem, 1991-1996)**.** ** Let r be a prime power. From any $(kr+2)(r-1)+1$ points in $\mathbb{R}^{kr}$ one can choose r pairwise disjoint sets such that each set contains $k(r-1)+1$ points and the convex hulls of these sets have a common point.*

It **is unknown** whether the version of this theorem for $r=6$ is true!

Quantitative versions (QSM), (QDM), (QSDM) **are unknown**, even for r a prime power! *The version of (S'M) on 3-fold linking* **is unknown**, see [Skr].

For topological versions see [Sk18, §2], [Sk16].

Answers, hints, solutions

1.1. 0, if they are parallel; 1, if the line intersects the plane; ∞, if the line is contained in the plane.

1.4. (a) The pair of the points $(0,0,1)$ and $(0,0,-1)$.

(b) The circumference $\begin{cases} x=0, \\ y^2+z^2=1 \end{cases}$.

(c) The union of three quarters of circumferences:

$$\begin{cases} x=0, y\geqslant 0, z\geqslant 0 \\ y^2+z^2=1 \end{cases}, \quad \begin{cases} y=0, x\geqslant 0, z\geqslant 0 \\ x^2+z^2=1 \end{cases} \quad \text{and} \quad \begin{cases} z=0, x\geqslant 0, y\geqslant 0 \\ x^2+y^2=1 \end{cases}.$$

1.5. (a) The pair of the points $(0,0,0,1)$ and $(0,0,0,-1)$.

(b) The circumference $\begin{cases} x=y=0, \\ z^2+t^2=1 \end{cases}$.

(c) The sphere $\begin{cases} x=0, \\ y^2+z^2+t^2=1 \end{cases}$.

(d) The graph K_4 formed by the union of six quarters of circumferences, one of the quarters being $\begin{cases} x=y=0,\ z\geqslant 0,\ t\geqslant 0 \\ z^2+t^2=1 \end{cases}$.

1.8. (b) The empty set, a point (if the line intersects the plane), a line (if the line is contained in the plane).

(c) The empty set, a line (if the plane intersects the hyperplane), a plane (if the plane is contained in the hyperplane).

(d) The empty set, a plane (if they intersect), a hyperplane (if they coincide).

(e) The empty set, a point or a line (if they intersect), a plane (if they coincide).

1.9. (b) The empty set; (e) a point.

2.2. (a) No.

2.3. (c) The union of the red segments is the sum modulo 2 of the outlines of the red triangles. Also $(A\oplus B)\cap C=(A\cap C)\oplus(B\cap C)$.

3.1. Consider $\Delta\cap\Delta'$.

3.3. Since no 4 of the given points $O, A_1, \dots, A_5$ lie in the same plane, the number of those sides of the triangle $A_3A_4A_5$ that are higher than A_1A_2 equals to the number of intersection points of the outline of the triangle $A_3A_4A_5$ with the triangle OA_1A_2. Also, a segment cannot intersect a triangle by more than 2 points. All this implies the required assertion.

3.5. (3) Take points on a helix, see fig. 3.

(3') The property (3') is equivalent to Theorem QSD.3.

QSD. (3) We may assume that there is a unique 'highest' point O among the given ones. Consider a 'horizontal' plane slightly below the point O. Take the intersection of this plane with the segment OA_j, for every given point A_j (see figure 6). Then by QS there are 4 given points A, B, C, D such that the triangles OAB and OCD have a common point other than O. Now QSD.3 follows.

S'D, QS'D. We may assume that there is a unique 'highest' point O among the given ones. Consider a 'horizontal' plane α slightly below the point O. Denote by $A_1', \dots, A_5'$ the intersection points of α and segments joining O to other given points. In the plane α we obtain a picture analogous to fig. 1, left.

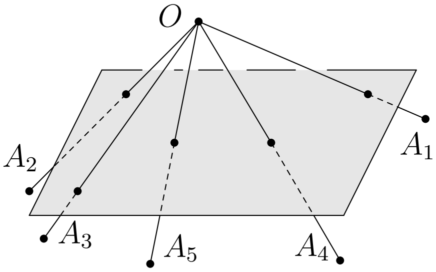

Figure 6: A plane in 3-space intersects the segments OA_j by 5 points.

Then the following numbers have the same parity:

• the number of linked unordered pairs of triangles formed by given points;

• the number of segments A_iA_j that are below an odd number of sides of their 'complementary' triangles $A_kA_lA_m$, $\{i,j,k,l,m\}=\{1,2,3,4,5\}$;

• the number of ordered pairs (A_iA_j, A_kA_l) of segments of which the first is below the second;

• the number of intersection points of segments whose vertices are $A'_1,\ldots,A'_5$.

By QS the latter number is odd.

SD. (4) We may assume that there is a unique 'highest' point O among the given ones. Consider a 'horizontal' 3-dimensional hyperplane α such that O and the other given points $A_1,\ldots,A_6$ lie on different sides of α. For $i=1,\ldots,6$ denote by A'_j the intersection point of α and the segment OA_j (see figure 7, left). Clearly, no 4 of the 6 points $A'_1,\ldots,A'_6$ lie in the same plane. Hence by Theorem S'D there are two linked triangles with vertices at these 6 points.

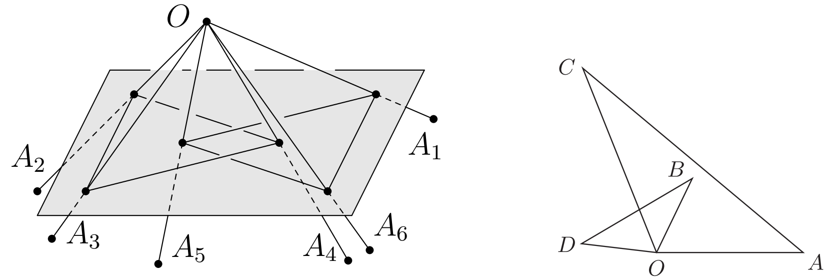

Figure 7: *Left:* a hyperplane in 4-space (shown as a plane in 3-space) intersects the segments $OA_1,\ldots,OA_6$ at 6 points $A'_1,\ldots,A'_6$ which are vertices of two linked triangles. *Right:* section by the plane γ: $\Delta_1^\gamma = OAC$, $\Delta_2^\gamma = OBD$.

Denote by Δ_1 and Δ_2 the triangles formed by $A_1,\ldots,A_6$ so that the linked triangles are the intersections $\alpha\cap O\Delta_1$ and $\alpha\cap O\Delta_2$ of the hyperplane α with tetrahedra $O\Delta_1$ and $O\Delta_2$ (e.g. $\Delta_1 = A_2A_3A_4$ and $\Delta_2 = A_1A_5A_6$). Denote by γ the plane containing O and the intersection line of the planes of the linked triangles. Then $\gamma\cap\alpha$ is a line and $\Delta_j^\gamma := \gamma\cap O\Delta_j$ is a triangle for $j=1,2$ (see figure 7, right). The side of Δ_j^γ not containing O is $\gamma\cap\Delta_j$. The two sides of Δ_j^γ containing O form the intersection of γ and the lateral surface of the tetrahedron $O\Delta_j$ (whose base is Δ_j).

Since the triangles $\alpha\cap O\Delta_1$ and $\alpha\cap O\Delta_2$ are linked, the intersection points of the line $\gamma\cap\alpha$ and the outlines of Δ_1^γ and Δ_2^γ alternate along the line (see figure 1, right). Hence the outlines have a common point distinct from O.

This point is either the intersection of the sides $\gamma\cap\Delta_1$ and $\gamma\cap\Delta_2$ or, without loss of generality, of the side $\gamma\cap\Delta_1$ and the union of the two sides of Δ_2^γ containing O. In the first case Δ_1 intersects Δ_2. In the second case Δ_1 intersects the lateral surface of the tetrahedron $O\Delta_2$.

QSD. (4) The result is reduced to Theorem QS'D analogously to the proof of SD.4, by a simple additional counting analogous to the proof of Theorem QS'D below. We need the following assertion.

For triangles Δ_1 and Δ_2 formed by $A_1, \ldots, A_6$ and having disjoint vertices, the number of intersection points of the surfaces of the tetrahedra $O\Delta_1$ and $O\Delta_2$ is even if and only if the triangles $\alpha \cap O\Delta_1$ and $\alpha \cap O\Delta_2$ are linked in α.

4.6. (a) Take a pair of points at each vertex of a tetrahedron.

(c) Take the union of sets of $r-1$ points at each vertex of a d-dimensional simplex.

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