OLYMPIC AND RESEARCH PROBLEMS

www.mccme.ru/circles/oim/oimpeng.pdf

N. N. Konstantinov and A. B. Skopenkov

Introduction. High-school teachers working with talented pupils often face the problem: *how to prepare the pupils to math olympiads or to 'serious' mathematics?*

Some people think that for the former one has to solve the last years' olympiad problems, while for the latter one has to read university textbooks, and that it is unreal to achieve both aims because they are so diverse. The authors share the widespread belief that these approaches are insufficiently effective and lead to harmful 'side effects': the pupils either become too keen on emulation or study the *language* of mathematics instead if its substance.

We believe that the basis of mathematical education should constitute *solution and discussion of motivated problems, during which a student learns important mathematical ideas and theories.* This would prepare the pupil to study and research in mathematics, as well as to math olympiads. This would also be good to his/her general development. This would be more effective to achieve success in olympiads alone or in research alone (if we are to disregard plenty of other factors besides organization of studies).

How to find problems for teaching? The teachers asked themselves this question over the last ten thousands years:

- Plato; Phaedo.
- S. Suzuki; Zen Mind, Beginner's Mind.
- D. O. Shklyarskiy, N. N. Chentsov, I. M. Yaglom; Selected Problems and Theorems of Elementary Mathematics.
- H. Polya, G. Szegö; Aufgaben und Lehrsatze aus der Analysis.
- D. Fomin, S. Genkin, I. Itenberg; Mathematical Circles (Russian Experience).

Math circles. We would present a brief account of our own teaching of 'olympic' problems, continuing the experience of the cited authorities.

The first author leads a math circle for beginners at the school under supervision of Moscow Institute of Open Education. The second author teaches (or, rather, catechizes) in math circles at Moscow Center for Continuous Mathematical Education and at the Kolmogorov College. These circles are for 14-17 years old students.

Any high-school pupil has the right to participate in the circles; the participation is free of charge. However, the level of the material studied is pretty high; many participants are winners of Moscow Math Olympiade. At the lessons the pupils solve interesting problems and discuss them with teachers. The problems are chosen in such a way that in the course of the solution and discusion the pupils learn important mathematical ideas and theories. We spend much time in *indiviual teaching*, discussing with each pupil his/her solutions or giving hints.

Our lessons aim at building a bridge (by showing that there is no gap) between ordinary high school excercises and more sophisticated, intricate and abstract concepts in undergraduate mathematics. The focus is on engaging a wide audience of students to think creatively in applying techniques and strategies to problems in the real world. The students are encouraged to express their ideas, conjectures and conclusions in writing. The goal is to help students develop a host of new mathematical tools and strategies that will be useful beyond the classroom and in a number of disciplines.

Our assistants are Moscow State University students who are ex-winners of International and All-Russian Math Olympiads. Most of them excel in studies, some of them already have written research papers.

Research problems for high-school students. Many talented high-school students are interested in solving research problems. Such problems are usually suggested as complicated problems split into several steps. The final result can even be unknown at the beginning and would naturally appear in the course of thinking over the problem. Thinking over such problems is good in itself and is similar to scientific research. Therefore it is useful if a teacher can support and develop this interest.

We shall describe organization of research activity for high-school students. The experience of mathematical circles and high-school students' conferences will be presented. We shall describe as main examples

• *Summer Conference of International Tournament of Towns*, which is the most important among events in Russia joining high-school pupils interested in research problems;

• *mathematical circles* (see above) in which high-school pupils study research problems; and

• *the INTEL ISEF affiliated conference of high-school pupils* (where high-school students supervised by the second author regularly win prizes).

We shall give several examples of research problems in which students achieved as excellent and serious results as to be published in good research journals [Ku00]. Most of these examples will be from the realm of topological graph theory. Due to the beauty and visual clarity of this theory it is possible to expose serious results and problems from higher mathematics in a language accesible to high-school pupils [CRS98].

An example of a research problem. In the Kolmogorov-Arnold papers solving Hilbert's 13th problem on superpositions implicitly appeared the notion of a *basic embeding* (explicitly formulated by Y. Sternfeld). We define this notion for the particular case of the plane. An embedding $K \subset R^2$ is *basic* if for each continuous function $f : K \to R$ there exist continuous functions $g, h : R \to R$ such that $f(x, y) = g(x) + h(y)$ for any point $(x, y) \in K$.

The following result [Sk95] solves the problem of Y. Sternfeld from 1988. *A finite graph is basically embeddable into the plane if and only if it does not contain subgraphs homeomorphic to circle S^1, five-point star T_5, and a cross with branched endpoints.*

More generally, an embedding $K \subset X \times Y$ is called *basic* if for any continuous function $f : K \to R$ there exist continuous functions $g : X \to R$ and $h : Y \to R$ such that $f(x, y) = g(x)+h(y)$ for any point $(x, y) \in K$. Let T_n be an n-od, i.e., the union of n arcs having an only common vertex. The problem of characterization of graphs basically embeddable into products $T_n \times T_m$ was posed in [CRS98] and solved for $m = 2$ [Ku00]. The *defect* of a graph K is defined as the sum $\delta(K) = (degA_1 - 2) + \ldots + (degA_k - 2)$, where $A_1, \ldots, A_k$ are all vertices either of degree greater than 4 or those of degree 4 that are not endpoints of a hanging edge. *A finite graph K is basically embeddable into $R \times T_n$ if and only if K is a tree and either $\delta(K) < n$ or $\delta(K) = n$ and K has a vertex of degree greater than 4 with a hanging edge.*

These results were presented as a sequence of problems at the 1997 International Tounament of Towns Summer Conference. At this Conference some further development occured.

References

[CRS98] A. Cavicchioli, D. Repovš and A. B. Skopenkov. Open problems on graphs, arising from geometric topology. Topol. Appl. 1998. 84. P. 207–226.

[Ku00] V. A. Kurlin, Basic embeddings into products of graphs, Topol. Appl. 102 (2000) 113–137.

[Sk95] A. B. Skopenkov, A description of continua basically embeddable in R^2, Topol. Appl. 65 (1995), 29–48.

[Sk08] A. Skopenkov, Some reflections on research problems for high-school students (in Russian), Mat. Prosveschenie, 13 (2008).