PIERRE DELIGNE SCHOLARSHIP REPORT. DMITRY CHELKAK, 2011.

1. Results

The most important results obtained by myself in the year 2011 are related to the development of a "discrete complex analysis toolbox" allowing one to work with discrete counterparts of various classical conformal invariants "staying on the microscopic level". In particular, this is crucial for several problems related to the fine structure of the critical Ising model interfaces, corresponding random cluster model, etc (see the list of preprints given in Sect. 2 for more details and corollaries related to the critical Ising model).

In order to describe these results, we need several preliminary definitions:

Definition 1. *Let Ω be a discrete domain (e.g., connected subset of $\mathbb{Z}^2$) and $A, B \subset \partial\Omega$. We denote by $\mathrm{Z}_\Omega(A;B)$ the* **total partition function of the simple random walk** *running from A to B inside Ω. Namely,*

$$\mathrm{Z}_\Omega(A;B) = \mathrm{Z}_\Omega(B;A) \quad := \sum_{\gamma\in S_\Omega(a;b)} 4^{-\mathrm{L}ength(\gamma)},$$

where $S_\Omega(A;B) = \{\gamma = (u_0 \sim u_1 \sim \cdots \sim u_n) : u_0 \in A,\ u_1, \ldots, u_{n-1} \in \mathrm{Int}\Omega,\ u_n \in B\}$ is the set of all nearest-neighbor paths connecting A and B inside Ω, and $\mathrm{L}ength(\gamma) = n$.

Definition 2. *Let Ω be a simply-connected discrete domain and $a, b, c, d \in \partial\Omega$ be four boundary points listed counterclockwise. We define their* **discrete cross-ratio** *by*

$$\mathrm{Y}_\Omega(a,b;c,d) := \left[\frac{\mathrm{Z}_\Omega(a;d)\cdot\mathrm{Z}_\Omega(b;c)}{\mathrm{Z}_\Omega(a;b)\cdot\mathrm{Z}_\Omega(c;d)}\right]^{\frac{1}{2}}.$$

Note that the continuous analogue of the partition function $\mathrm{Z}_\Omega(a;b)$ for the upper half-plane $\Omega = \mathbb{H}$ (up to a multiplicative constant) is given by $(b-a)^{-2}$, so the quantity introduced above is a natural discrete analogue of the usual cross-ratio $y_\mathbb{H}(a,b;c,d) := [(b-a)(d-c)]/[(d-a)(c-b)]$, and for other continuous domains Ω the corresponding cross-ratio could be easily defined via the proper conformal map onto $\mathbb{H}$. It is important that the discretizations Y_Ω are defined for any discrete domain Ω without appealing to the conformal mappings (which have no proper discrete analogue for the subsets of the fixed grid).

At this point, for any discrete simply-connected domain Ω with four boundary points a, b, c, d listed counterclockwise, we have two quantities

$$\mathrm{Y} := \mathrm{Y}_\Omega(a,b;c,d) \quad \text{and} \quad \mathrm{Z} := \mathrm{Z}_\Omega([ab];[cd]),$$

where $[ab], [cd] \subset \partial\Omega$ denote the corresponding boundary arcs. In the continuous setup, both Y and Z have natural counterparts which are conformally invariant, and so can be expressed via each other *explicitly* (since the configuration consisting of a simply-connected domain with four marked boundary points has only one conformal parameter (modulus) to be completely characterized). For sure, this exact relation cannot survive on the discrete level but one can wonder whether there exist some *universal* (i.e., completely independent of the discrete domain Ω under consideration) *double-sided estimates* relating Y and Z, or not?

Before we give a (positive) answer to this question, it is worthwhile to introduce the third quantity related to the configuration $(\Omega; a, b, c, d)$ which is a discrete analogue of the extremal length notion. Let Ω be a discrete domain and $E(\Omega)$ be the set of all edges of Ω. For a given nonnegative function ("discrete metric") $w : E(\Omega) \to \mathbb{R}_+$ we define the "w-area" of Ω by

$$A_w(\Omega) := \textstyle\sum_{e\in E(\Omega)} (w(e))^2.$$

Similarly, for a lattice path $\Gamma \subset E(\Omega)$ we define its "w-length" by

$$L_w(\Gamma) := \textstyle\sum_{e\in E(\Omega)} w(e).$$

Further, for $\mathcal{E}$ being some family of lattice paths in Ω, we set

$$L_w(\mathcal{E}) := \inf_{\Gamma\in\mathcal{E}} L_w(\Gamma).$$

Definition 3. *The* **discrete extremal length** *of the family $\mathcal{E}$ is given by*

$$\mathrm{L}[\mathcal{E}] := \sup_{w:E(\Omega)\to\mathbb{R}_+} \frac{(L_w(\mathcal{E}))^2}{A_w(\Omega)},$$

where the supremum is taken over all w's such that $0 < A_w(\Omega) < +\infty$. In particular, if Ω is simply connected and $a, b, c, d \in \partial\Omega$ are listed in the counterclockwise order, then we define $\mathbf{L}_\Omega([ab]; [cd])$ *as the discrete extremal length of the family $(\Omega; [ab] \leftrightarrow [cd])$ of lattice paths joining the boundary arcs $[ab]$ and $[cd]$ inside Ω.*

Note that Definition 3 is quite useful if one needs to estimate the extremal length, since for this purpose it is sufficient to take any "discrete metric" w_0 in Ω (possibly, having some natural geometric meaning) and estimate $A_{w_0}(\Omega)$, $L_{w_0}(\Omega; [ab] \leftrightarrow [cd])$ for this particular w_0.

Theorem 4. *Let Ω be a simply connected discrete domain and $a, b, c, d \in \partial\Omega$ be listed in the counterclockwise order. Denote $\mathrm{Y} := \mathrm{Y}_\Omega(a, b; c, d)$, $\mathrm{Y}^* := \mathrm{Y}_\Omega(b, c; d, a)$, $\mathrm{Z} := \mathrm{Z}_\Omega([ab]; [cd])$, $\mathrm{Z}^* := \mathrm{Z}_\Omega([bc]; [da])$, and $\mathrm{L} := \mathrm{L}_\Omega([ab]; [cd])$, $\mathrm{L}^* := \mathrm{L}_\Omega([bc]; [da])$.*

(i) Then, $\mathrm{Y} \cdot \mathrm{Y}^* = 1$ and $\mathrm{L} \cdot \mathrm{L}^* \asymp 1$ (here and below we denote by $\asymp$ the double-sided estimates with absolute constants independent of the particular configuration $(\Omega; a, b, c, d)$).

(ii) If at least one of the estimates

$$\begin{aligned} \mathrm{Y} \leqslant \mathrm{const}, \quad \mathrm{Z} \leqslant \mathrm{const}, \quad \mathrm{L} \geqslant \mathrm{const}, \\ \mathrm{Y}^* \geqslant \mathrm{const}, \quad \mathrm{Z}^* \geqslant \mathrm{const}, \quad \mathrm{L}^* \leqslant \mathrm{const} \end{aligned} \tag{1.1}$$

holds true (for some absolute constant), then all these estimates hold true (with some absolute constants independent of Ω, a, b, c, d but depending on the initial bound).

(iii) Moreover, if at least one of the quantities Y, Y^*, Z, Z^*, L, L^* is of order 1 (i.e., admits the double-sided estimate $\asymp 1$), then all these quantities are of order 1.

(iv) If (1.1) holds true, then the following double-sided estimates are fulfilled:

$$\mathrm{Z} \asymp \mathrm{Y} \asymp \mathrm{X} \qquad \textit{and} \qquad \log(1+\mathrm{Y}^{-1}) \asymp \mathrm{L}.$$

In particular, for any $\lambda > 0$ there exist some absolute constants $C = C(\lambda)$, $\beta = \beta(\lambda)$ such that the uniform estimate

$$\mathrm{Z}_\Omega([ab]; [cd]) \leqslant C \cdot \exp[-\beta \cdot \mathrm{L}_\Omega([ab]; [cd])] \tag{1.2}$$

holds true for any simply connected discrete domain $(\Omega; a, b, c, d)$ with $\mathrm{L}_\Omega([ab]; [cd]) \geqslant \lambda$.

Note that Theorem formulated above is valid not only for the square grid but also for the wide class of so-called isoradial planar graphs (in particular, for triangular an hexagonal grids). Our proof are independent of the lattice and based only on a few facts concerning corresponding random walks (the most involved of them is the free Green's function asymptotics which is known for isoradial graphs due to R.Kenyon). We refer the reader interested in the applications of these double-sided estimates in the critical Ising model theory to the preprints listed in Sect. 2, and give here only one corollary which is of independent interest.

Corollary 5. *Let Ω be a simply connected discrete domain, $u \in \Omega$ be an inner vertex and $a, b \in \partial\Omega$. Let Γ be a straight (say, vertical) segment passing through u and cutting Ω into two halves such that the boundary arc $[ab] \subset \partial\Omega$ belongs to one of the halves. Further, let $A_1, A_2 \ldots, A_N$ denote the vertical columns of width 1 separating $[ab]$ from u in Ω, and let k_j denote the number of horizontal edges passing through A_j. Then,*

$$\mathbf{P}[\textit{random walk started at } u \textit{ hits } \partial\Omega \textit{ on } [ab]] \leqslant C \cdot \exp\left[-\beta \cdot \sum_{j=1}^{N} \frac{1}{k_j}\right]. \tag{1.3}$$

with some **absolute** *constants $C, \beta > 0$ independent of Ω, u, a, b and Γ.*

The classical analogue of (1.3) is well-known and goes back to Carleman. Nevertheless, to the best of our knowledge, the corresponding uniform "discrete" bound was unknown until recently (even for the square grid).

2. Publications

Published:

- Dmitry Chelkak, Stanislav Smirnov, Discrete complex analysis on isoradial graphs. *Advances in Mathematics*, 228 (2011), no. 3, 1590–1630
- Dmitry Chelkak, Stanislav Smirnov, Universality in the 2D Ising model and conformal invariance of fermionic observables. Final version (52pp., last revised May 16, 2011, `http://arxiv.org/abs/0910.2045`) has been accepted for *Inventiones Mathematicae.*
- Dmitry Chelkak, Konsantin Izyurov, Holomorphic Spinor Observables in the Critical Ising Model, `http://arxiv.org/abs/1105.5709`, submitted.

Preprints in preparation:

- D.Chelkak, Discrete complex analysis: a toolbox. Preprint 2011, to appear at `arxiv.org`.
- D.Chelkak, H.Duminil-Copin and C.Hongler, Crossing probabilities in topological rectangles for the critical planar FK-Ising model. Preprint 2011, preliminary version available at `http://www.unige.ch/~duminil/publicationlist.html`
- D.Chelkak, H.Duminil-Copin, C.Hongler, A.Kemppainen and S. Smirnov, Convergence of Ising interfaces to Schramm's SLEs. Preprint 2011, preliminary version available at `http://www.unige.ch/~duminil/publicationlist.html`

3. Conferences, seminar talks etc.

- Conferences, workshops etc (invited talks):
  "Third Northern Triangular Seminar", April 11–13, 2011, St.Petersburg;
  "The 3rd St.Petersburg Conference in Spectral Theory (dedicated to the memory of M.Sh.Birman)", July 1–6, 2011, St.Petersburg;
  "Geometry Days in Novosibirsk, 2011", September 1–4, 2011, Novosibirsk;
  "Random Processes, Conformal Field Theory and Integrable Systems", September 19–23, 2011, Poncelet Laboratory, Moscow.
- Seminar talks, colloquia etc.:
  Barcelona, Moscow, Novosibirsk, St.Petersburg: various research seminars.
- Workshops, schools etc (participation):
  "Krein - de Branges spaces of entire functions and old and new spectral problems", May 2–6, 2011, CRM, Barcelona;
  "Hilbert spaces of entire functions and spectral theory of self-adjoint differential operators", May 30 – June 4, 2011, CRM, Barcelona.

4. Participation in international research projects

- Research visit to CRM (Barcelona): May 1 – June 4, 2012 (research program "Complex Analysis and Spectral Problems").
- I'm a member of an international team working on a rigorous approach to the conformal invariance in critical two-dimensional lattice models (particularly, spin and random cluster representations of the critical Ising model) via discrete complex analysis tools. I have several ongoing projects with H.Duminil-Copin (Geneva), C.Hongler (New York), K.Izyurov (Geneva-St.Petersburg), A.Kemppainen (Helsinki), K.Kytölä (Helsinki) and S.Smirnov (Geneva-St.Petersburg).

5. Pedagogical activity

Having no obligatory teaching in the year 2011, I taught the lecture course "Dimer model on planar graphs" (10 lectures, video available at `http://chebyshev.spb.ru/course/?id=22792`) at the Chebyshev Laboratory (St.Petersburg State University).

I was an advisor of diploma project "Characterization of Loewner chains generated by continuous driving forces" by Pavel Lepekhin (Math. Analysis Dept., St.Peersburg State University). I am a co-advisor of 3rd year graduate student Sergey Matveenko (PhD project "Sharp spectral theory of Schrodinger-type operators with matrix potentials") and several undergraduate students.

6. Other activity

Organization of the *Chebyshev Laboratory* (`http://chebyshev.spb.ru`) at the St.Petersburg State University under the RF Government "megagrant" project 11.G34.31.0026 (principal investigator: Stanislav Smirnov): Dec 2010 - Apr 2011: Vice Head; Sep - Dec 2011: Acting Head.

7. Overall summary comparing to the original proposal

Two completely different research directions were formulated in the original proposal: (a) conformal invariance in 2D lattice models and (b) spectral theory of 1D Schrödinger operators with matrix potentials.

Research direction (a). The core problems related to the critical Ising model were solved successfully. Now we have much better understanding and sharper results than three years ago. Moreover, the careful analysis of spinor holomorphic observables introduced by K.Izyurov and myself eventually allows one to prove the convergence of spin-spin correlations to conformally covariant limits (predicted by CFT) in arbitrary planar domain, which seemed out of reach at the beginning of the project. Also, the "discrete analysis toolbox" (see Sect. 1) derived originally for the Ising model purposes, seems to be quite useful for the analysis of other lattice models, providing a solid background for possible related research directions listed in the original proposal.

Research direction (b). Being unrelated to the successfully going project (a), basically remained untouched. Nevertheless, jointly with my PhD student S.Matveenko, we obtained several results leading to the complete characterization theorems for Sturm-Liouville operators on the unit interval with matrix L^p-potentials and general separated boundary conditions (being a combination of Dirichlet and Neumann-type b.c. on each end of the interval). At the moment, these results are unpublished (in particular, due to activity mentioned in Sect. 6), but we hope to finish the construction of the unified "characterization" theory including all p's as well as general separated b.c. soon.