SUMMARY OF THE RESEARCH PROPOSAL
by Maxim Vsemirnov

Research topic: Hurwitz and (2,3)-generated matrix groups and related problems.

This proposal concerns the problem of determining $(2,3)$-generated and Hurwitz groups. A group is called $(2,3)$-*generated* if it can be generated by an involution and an element of order 3. These groups were the subject of intensive study since XIX century. The reason to study these groups arises from the fact that the class of $(2,3)$-generated groups together with three cyclic groups $\{1\}$, C_2 and C_3 is exactly the class of quotients of the classical modular group $\mathrm{PSL}_2(\mathbb{Z})$.

Hurwitz groups form a subclass of the $(2,3)$-generated groups. They can be characterized as finite groups that are generated by an involution and an element of order 3 such that their product has order 7. The study of Hurwitz groups goes back to papers of Hurwitz and Klein. These groups arose first in the topological context as automorphism groups of compact Riemann surfaces attaining the famous Hurwitz upper bound.

In the proposed research programme it is expected to concentrate on the algebraic properties of Hurwitz and $(2,3)$-generated groups. Despite of its long history the problem is still far from completion. However, recent developments give an evidence that the programme is realistic and we can expect further advances. The problem is chosen not only for its historical importance but also because its solution (or even significant steps towards it) will have a strong interdisciplinary theoretical impact including applications to the theory of finite simple groups, the theory of Riemann surfaces, representation theory and combinatorial group theory.

The main aim of this project is to study which simple and quasi-simple finite groups are Hurwitz. The answer is known for sporadic groups, some exceptional groups of Lie type and for the most series of classical matrix groups of large rank. The low-rank case meets principal difficulties and it is planned to concentrate on that case. A special attention is paid to the exact determination of the corresponding Hurwitz generators.

The applicant already has some important results in this area. He completed the solution of the problem on $(2,3)$-generation of the groups $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Z})$. His method of building new matrix Hurwitz group from known ones has already led to the discovery of Hurwitz special linear groups in 60 new dimensions. Another result concerns exact Hurwitz generators for exceptional Lie groups of type G_2. It was the first time when explicit Hurwitz generators for exceptional Lie groups were found. In the opposite direction, the applicant proved that orthogonal groups $\Omega_7(q)$ are never Hurwitz. Finally, he proved the conjecture of Holt and Plesken about Hurwitz generators of $\mathrm{PSL}_2(q)$ that satisfy an extra condition on their commutator.

It is planned to continue this research and study related problems. The following results are highly expected:

• determination of new Hurwitz series among the classical groups over finite fields; for special linear groups the interesting dimensions are less than 250, for other classical series they are less than 400;

• a full description of Hurwitz subgroups in $\mathrm{PGL}_6(\mathbb{F})$; this will complete the project on finding all Hurwitz groups in dimensions ≤ 7.

• parameterization (up to conjugation) of all irreducible matrix Hurwitz generators in dimensions 8, 9, 10, and 11 and identification of the corresponding groups;

• determination of exact Hurwitz generators for other series of exceptional groups of Lie type;

• the complete solution of an open question on $(2,3)$-generation of $\mathrm{Sp}_{2n}(\mathbb{Z})$.

• a generalization of the Holt–Plesken conjecture to $(2,3,k)$-generated groups and to the case, when the commutator is replaced by an arbitrary word in the two generators.

It is also planned to cover other related aspects of the above problems, including interactions with representation theory, number theory and combinatorial group theory.