REPORT ON THE DYNASTY FELLOWSHIP 2016

ARKADIY SKOPENKOV

Papers

(Abstracts of conference talks are not listed)

[1] A. Skopenkov, How do autodiffeomorphisms act on embeddings, Proceedings A of The Royal Society of Edinburgh, to appear. http://arxiv.org/abs/1402.1853

We work in the smooth category. For an n-manifold N denote by $E^m(N)$ the set of isotopy classes of embeddings $N \to \mathbb{R}^m$. The following problem was suggested by E. Rees in 2002: describe the action of self-diffeomorphisms of $S^p \times S^{n-p}$ on $E^m(S^p \times S^{n-p})$.

Let $g : S^p \times S^{n-p} \to \mathbb{R}^m$ be an embedding such that $g|_{a \times S^{n-p}} : a \times S^{n-p} \to \mathbb{R}^m - g(b \times S^{n-p})$ is null-homotopic for some different points $a, b \in S^p$ and $m \geq n + 2 + \frac{1}{2}\max\{p, n-p\}$.

Theorem. *For a map $\varphi : S^p \to SO_{n-p}$ define an autodiffeomorphism φ' of $S^p \times D^{n-p}$ by $\overline{\varphi}(a, b) := (a, \varphi(a)b)$. Let φ'' be the S^{n-p-1}-symmetric extension of φ to an autodiffeomorphism of $S^p \times S^{n-p}$. Then for each map $\varphi : S^p \to SO_{n-p}$ embedding $g \circ \varphi''$ is isotopic to embedded connected sum $g\#u$ for some embedding $u : S^n \to S^m$.*

Let N be an oriented n-manifold and $f : N \to \mathbb{R}^m$ an embedding. Denote by $E^m(N)/\#$ the quotient set of $E^m(N)$ by embedded connected sum with embeddings $S^n \to \mathbb{R}^m$. As a corollary we obtain that under certain conditions *for orientation-preserving embeddings $s : S^p \times D^{n-p} \to N$ the class of S^p-parametric embedded connected sum $f\#_s g$ in $E^m(N)/\#$ depends only on f, g and the isotopy (the homotopy or the homology) class of $s|_{S^p \times 0}$.*

[2] S. Avvakumov, I. Mabillard, A. Skopenkov, U. Wagner, Eliminating Higher-Multiplicity Intersections, III. Codimension 2. http://arxiv.org/abs/1511.03501 Submitted to Geometry and Topology.

We study conditions under which a finite simplicial complex K can be mapped to $\mathbb{R}^d$ without higher-multiplicity intersections. An *almost r-embedding* is a map $f\colon K \to \mathbb{R}^d$ such that the images of any r pairwise disjoint simplices of K do not have a common point. We show that if r is not a prime power and $d \geq 2r + 1$, then there is a counterexample to the topological Tverberg conjecture, i.e., *there is an almost r-embedding of the $(d+1)(r-1)$-simplex in $\mathbb{R}^d$.* This improves on previous constructions of counterexamples (for $d \geq 3r$) based on a series of papers by M. Özaydin, M. Gromov, P. Blagojević, F. Frick, G. Ziegler, and the second and fourth present author.

The counterexamples are obtained by proving the following algebraic criterion in codimension 2: *If $r \geq 3$ and if K is a finite $2(r-1)$-complex then there exists an almost r-embedding $K \to \mathbb{R}^{2r}$ if and only if there exists a general position PL map $f\colon K \to \mathbb{R}^{2r}$ such that the algebraic intersection number of the f-images* *of any r pairwise disjoint simplices of K is zero.* This result can be restated in terms of cohomological obstructions or equivariant maps, and extends an analogous codimension 3 criterion by the second and fourth author.

It follows from work of M. Freedman, V. Krushkal, and P. Teichner that the analogous criterion for $r = 2$ is false. We prove a beautiful lemma on singular higher-dimensional Borromean rings, yielding an elementary proof of the counterexample. As another application of our methods, we classify *ornaments* $f\colon S^3 \sqcup S^3 \sqcup S^3 \to \mathbb{R}^5$ up to *ornament concordance*.

[3] A. Skopenkov, Stability of intersections of graphs in the plane and the van Kampen obstruction. http://arxiv.org/abs/1609.03727 Submitted to Topology and its Applications

A map $\varphi : K \to \mathbb{R}^2$ of a graph K is *approximable by embeddings*, if for each $\varepsilon > 0$ there is an ε-close to φ embedding $f : K \to \mathbb{R}^2$. Analogous notions were studied in computer science under the names of *cluster planarity* and *weak simplicity*. This short survey is intended not only for specialists in the area, but also for mathematicians from other areas.

We present criteria for approximability by embeddings (P. Minc, 1997, M. Skopenkov, 2003) and their algorithmic corollaries. We introduce *the van Kampen (or Hanani-Tutte) obstruction* for approximability by embeddings and discuss its completeness. We discuss analogous problems of moving graphs in the plane apart (cf. S. Spież and H. Toruńczyk, 1991) and finding closest embeddings (H. Edelsbrunner). We present higher dimensional generalizations, including completeness of the van Kampen obstruction and its algorithmic corollary (D. Repovš and A. Skopenkov, 1998).

[4] A. Skopenkov, High codimension embeddings: classification, submitted to Bull. Man. Atl.
http://www.map.mpim-bonn.mpg.de/High_codimension_embeddings

This page is intended not only for specialists in embeddings, but also for mathematicians from other areas who want to apply or to learn the theory of embeddings.

This article gives a short guide to the Knotting Problem of compact manifolds N in Euclidean spaces and in spheres. After making general remarks we record some of the dimension ranges where no knotting is possible, i.e. where any two embeddings of N are isotopic. We then establish notation and conventions and give references to other pages on the Knotting Problem, to which this page serves as an introduction. We conclude by introducing connected sum and make some comments on codimension 1 and 2 embeddings.

[5] A. Skopenkov, Embeddings just below the stable range: classification, submitted to Bull. Man. Atl. http://www.map.mpim-bonn.mpg.de/Embeddings_just_below_the_stable_range:_classification

This page is intended not only for specialists in embeddings, but also for mathematician from other areas who want to apply or to learn the theory of embeddings.

Recall the Whitney-Wu Unknotting Theorem: if N is a connected manifold of dimension $n > 1$, and $m \geq 2n+1$, then every two embeddings $N \to \mathbb{R}^m$ are isotopic. In this page we summarize the situation for $m = 2n \geq 6$ and some more general situations.

[6] A. Skopenkov, 3-manifolds in 6-space, submitted to Bull. Man. Atl.
http://www.map.mpim-bonn.mpg.de/3-manifolds_in_6-space

This page is intended not only for specialists in embeddings, but also for mathematicians from other areas who want to apply or to learn the theory of embeddings.

The classification of 3-manifolds in 6-space is of course a particular case of the classification of n-manifolds in $2n$-space. In this page we recall the general results as they apply when $n = 3$ and we discuss examples and invariants peculiar to the case $n = 3$.

[7] A. Skopenkov, 4-manifolds in 7-space, submitted to Bull. Man. Atl. `http://www.map.mpim-bonn.mpg.de/4-manifolds_in_7-space`
This page is intended not only for specialists in embeddings, but also for mathematician from other areas who want to apply or to learn the theory of embeddings.
Basic results on 4-manifolds in 7-space are particular cases of results on n-manifolds in $(2n-1)$-space for $n = 4$. In this page we concentrate on more advanced results peculiar for $n = 4$.

[8] A. Skopenkov, High codimension links, submitted to Bull. Man. Atl. `http://www.map.mpim-bonn.mpg.de/High_codimension_links`
This page is intended not only for specialists in embeddings, but also for mathematician from other areas who want to apply or to learn the theory of embeddings.
We describe classification of embeddings $S^{n_1} \sqcup \ldots \sqcup S^{n_s} \to S^m$ for $m - 3 \geq n_i$.

[9] A. Skopenkov, Classification of knotted tori, `http://arxiv.org/abs/1502.04470` (the paper is rewritten in 2016, a new version uploaded to arxiv)
We describe the group of (smooth isotopy classes of smooth) embeddings $S^p \times S^q \to R^m$ for $p \leq q$ and $m \geq 2p + q + 3$. Earlier such a description was known only for $2m \geq 3p + 3q + 4$. We use a recent exact sequence of M. Skopenkov.

[10] D. Crowley and A. Skopenkov, Embeddings of non-simply-connected 4-manifolds in 7-space, I. Classification modulo knots. `http://arxiv.org/abs/1611.04738`
We work in the smooth category. Let N be a closed connected orientable 4-manifold with torsion free H_1, where $H_q := H_q(N; \mathbb{Z})$. The main result is *a complete readily calculable classification of embeddings $N \to \mathbb{R}^7$*, up to equivalence which is isotopy and embedded connected sum with embeddings $S^4 \to \mathbb{R}^7$. Such a classification was earlier known only for $H_1 = 0$ by Boéchat-Haefliger-Hudson 1970. Our classification involves Boéchat-Haefliger invariant $\kappa(f) \in H_2$, Seifert bilinear form $\lambda(f) : H_3 \times H_3 \to \mathbb{Z}$ and β-invariant assuming values in the quotient of H_1 defined by values of $\kappa(f)$ and $\lambda(f)$.
In particular, for $N = S^1 \times S^3$ we define geometrically a 1–1 correspondence between the set of equivalence classes of embeddings and an explicitly defined quotient of $\mathbb{Z} \oplus \mathbb{Z}$.

[11] D. Crowley and A. Skopenkov, Embeddings of non-simply-connected 4-manifolds in 7-space, II. On the smooth classification. `http://arxiv.org/abs/1612.04776`
We work in the smooth category. Let N be a closed connected orientable 4-manifold with torsion free H_1, where $H_q := H_q(N; \mathbb{Z})$. Our main result is *a readily calculable classification of embeddings $N \to \mathbb{R}^7$ up to isotopy*, with an indeterminancy. Such a classification was only known before for $H_1 = 0$ by our earlier work from 2008. Our classification is complete when $H_2 = 0$ or when the signature of N is divisible neither by 64 nor by 9.
The group of knots $S^4 \to S^7$ acts on the set of embeddings $N \to \mathbb{R}^7$ up to isotopy by embedded connected sum. In Part I we classified the quotient of this

action. The main novelty of this paper is the description of this action for $H_1 \neq 0$, with an indeterminancy.

Besides the invariants of Part I, the classification involves a refinement of the Kreck invariant from our work of 2008 which detects the action of knots.

For $N = S^1 \times S^3$ we give a geometrically defined 1–1 correspondence between the set of isotopy classes of embeddings and a quotient of the set $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{12}$.

[12] A. Skopenkov, A user's guide to topological Tverberg conjecture. `http://arxiv.org/abs/1605.05141`

The well-known *topological Tverberg conjecture* was considered a central unsolved problem of topological combinatorics. The conjecture asserts that *for each integers $r, d > 1$ and each continuous map $f\colon \Delta \to \mathbb{R}^d$ of the $(d+1)(r-1)$-dimensional simplex Δ there are pairwise disjoint subsimplices $\sigma_1, \dots, \sigma_r \subset \Delta$ such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

A proof for a prime power r was given by I. Bárány, S. Shlosman, A. Szűcs, M. Özaydin and A. Volovikov in 1981-1996. A counterexample for other r was found in a series of papers by M. Özaydin, M. Gromov, P. Blagojević, F. Frick, G. Ziegler, I. Mabillard and U. Wagner, most of them recent. The arguments form a beautiful and fruitful interplay between combinatorics, algebra and topology. In this expository note we present a simplified explanation of easier parts of the arguments, accessible to non-specialists in the area.

Expository publications for university students.

[13] A. Chernov, A. Daynyak, A. Glibichuk, M. Ilyinskiy, A. Kupavskiy, A. Raigorodskiy and A. Skopenkov, Elements of Discrete mathematics as a sequence of problems. 2016, Moscow, MCCME
`http://www.mccme.ru/circles/oim/discrbook.pdf`

In this book we present sequences of problems on combinatorics and graph theory (including random graphs).

[14] Mathematics via problems: from olympiads and math circles to a profession, editors: A. Zaslavsky, A. Skopenkov, and M. Skopenkov. 2016, Moscow, MCCME, to appear. `http://www.mccme.ru/circles/oim/sturm.pdf`

In this book we present an approach to 'university' mathematics as sequences of 'high-school' problems.

[15] A. Skopenkov, Embeddings into the plane of graphs with vertices of degree 4, Mat. Prosveschenie, 21 (2017), to appear, `http://arxiv.org/abs/1008.4940` (the paper is rewritten in 2016, a new version uploaded to arxiv)

In this expository note we present a proof of the V.A. Vassiliev conjecture on the planarity of graphs with vertices of degree 4 and certain additional structure. Both statement and proof are accessible to high-school students familiar with basic notions of graph theory. The conjecture was first proved by V.O. Manturov (such a proof was one of the main results of his habilitation thesis). In this note the exposition is made clearer and some comments for beginners are added.

[16] A. Volostnov, A. Skopenkov and Yu. Yarovikov, A study on recursive relations, Mat. Prosveschenie, submitted.

In this expository note we present and discuss a short proof of an estimation required for a proof of the Symmetric Local Lovasz Lemma.

[17] A. Skopenkov, How Fermat found extrema,
`http://arxiv.org/abs/1610.05968`

In this expository note we present a short elementary proof of the well-known criterion for a cubic polynomial to have three real roots. The proof is based on Fermat's approach to calculus for polynomials, and illustrates the idea of a derivative rigorously but without technical ε-δ language. The note is accessible to high-school students.

[18] A. Belov, I. Mitrofanov, A. Skopenkov, A. Chilikov, S. Shaposhnikov, 13th Hilbert Problem on superpositions of functions,
`http://www.turgor.ru/lktg/2016/5/index.htm`

In this expository note we present and discuss a structured proof of the Kolmogorov Superposition Theorem.

[19] A. Skopenkov, Algebraic Topology From Algorithmic Viewpoint, draft of a book, `http://www.mccme.ru/circles/oim/algor.pdf` (some sections are added or rewritten in 2016)

In this book we present an 'algorithmic' approach to algebraic topology.

Scientific conferences and seminar talks

[1] Mathematics of Jiri Matiusek, Prague, July, poster 'Eliminating Higher-Multiplicity Intersections, Codimension 2'

[2] Conference of Moscow Institute of Physics and Technology, Dolgoprudnyi, November,
Talk "Stability of intersections of graphs in the plane and the van Kampen obstruction"

[3] Discrete Geometry Seminar, Institute of Science and Technology, Austria,
Talk "Stability of intersections of graphs in the plane and the van Kampen obstruction"

[4] Topology seminar of PDMI, St Petersburg,
Talk "A user's guide to topological Tverberg conjecture"

[5] Joint seminar of Faculty of Computer Science, Higher School of Economics, and Faculty of Innovations and High Technology, Moscow Institute of Physics and Technology,
Talk "Stability of intersections of graphs in the plane and the van Kampen obstruction"

[6] Seminar on Lie groups, Independent University of Moscow,
Talk "A user's guide to topological Tverberg conjecture"

[7] Postnikov memorial seminar, Moscow State University,
Talk "Classification of knotted tori"

Teaching

A list of university courses taught by A. Skopenkov in 2016

[1] Discrete structures and algorithms in topology, III year students, September-December 2016, 4 hours per week. Moscow Institute of Physics and Technology (DIHT)

Program. It is shown how in the course of solution of interesting geometric problems (close to discrete mathematics and computer science) naturally appear main notions of algebraic topology (homology groups, obstructions and invariants). Thus main ideas of algebraic topology are presented with minimal technicalities.

Detailed information in Russian:
http://www.mccme.ru/circles/oim/home/combtop13.htm#combtop14

[2] Topological theory of vector fields on manifolds, Independent University of Moscow, February-May 2014, 2 hours per week.

Program. It is shown how in the course of solution of interesting geometric problems (close to dynamical systems and physics) naturally appear main notions of algebraic topology (homology groups, obstructions and invariants). Thus main ideas of algebraic topology are presented with minimal technicalities.

Detailed information in Russian:
http://www.mccme.ru/circles/oim/home/combtop13.htm#vefi

[3] Topological Tverberg conjecture: combinatorics, algebra and topology, Independent University of Moscow, September-December 2016, 2 hours per week.

Program. The well-known topological Tverberg conjecture was considered a central unsolved problem of topological combinatorics. A proof for a prime power r was given by I. Bárány, S. Shlosman, A. Szűcs, M. Özaydin and A. Volovikov in 1981-1996. A counterexample for other r was found in a series of papers by M. Özaydin, M. Gromov, P. Blagojević, F. Frick, G. Ziegler, I. Mabillard and U. Wagner, most of them recent. The arguments form a beautiful and fruitful interplay between combinatorics, algebra and topology. We present a simplified explanation of easier parts of the arguments, accessible to non-specialists in the area.

Detailed information in Russian:
http://www.mccme.ru/circles/oim/home/combtop13.htm#tver

[4] Discrete analysis (exercises), II year students, February-December 2016, 2 hours per week. Moscow Institute of Physics and Technology (DIHT)

Program. We study certain topics in combinatorics and graph theory (including random graphs).

Detailed information in Russian:
http://www.mccme.ru/circles/oim/home/discran1314.htm

Other educational activities by A. Skopenkov in 2016

[5] International Summer Conference of Tournament of Towns, Jury member, June-August, Pereslavl Detailed information:
http://www.turgor.ru/en/lktg/index.php

[6] Moscow Mathematical Conference of High-school Students, Programme Committee member, September-December, Moscow Detailed information in Russian:
http://www.mccme.ru/mmks/index.htm

[7] Conference on Advanced Education, October, invited speaker, Kirov Detailed information in Russian: http://cdoosh.ru/conf/conf.html

[8] A course on 'special' mathematics for high-school students, high-school 'Intellectual', January-December. Detailed information in Russian:
http://www.mccme.ru/circles/oim/index.htm#il

[9] Math circle 'Olympiads and Mathematics' for high-school students, MCCME, January-December. Detailed information in Russian:
http://www.mccme.ru/circles/oim/index.htm#oim

[10] Minicources on mathematics for high-school students, Kirov region summer school, July, Kirov region.

[11] Minicources on mathematics for high-school students, Moscow ‘olympic’ schools, April, June and November, Moscow region.

[12] Chebyshev lab lecture for high-school students, December, invited lecturer, St Petersburg.