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Oriented Cohomology Theories of Algebraic Varieties

I. PANIN (after I. PANIN and A. SMIRNOV)
St. Petersburg Branch, Laboratory of Algebra, V.A. Stecklov Institute of Maths (POMI), Russian Academy of Sciences, 191011 St. Petersburg, Russia. e-mail: Panin@pdmi.ras.ru

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Abstract. This article contains proofs of the results announced in [21] in the part concerning general properties of oriented cohomology theories of algebraic varieties. It is constructed one-to-one correspondences between orientations, Chern structures and Thom structures on a given ring cohomology theory. The theory is illustrated by motivic cohomology, algebraic K-theory, algebraic cobordism theory and by other examples.

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1. Introduction

The concept of oriented cohomology theory is well known in topology [1, Part II, p. 37], [27, Chapter 1, 4.1.1]. An algebraic version of this concept was introduced in [21] and is considered here. So in this article we consider a field k and the category of pairs (X, U) with a smooth variety X over k and its open subset U. By a cohomology theory we mean a contravariant functor A from this category to the category of Abelian groups endowed with a functor transformation $\partial\colon A(U) \to A(X, U)$ and satisfying the localization, Nisnevich excision and homotopy invariance properties (Definition 2.1).

We consider three structures a ring cohomology theory A can be equipped with: *an orientation on* A, *a Thom structure on* A and *a Chern structure on* A. *An orientation on* A is a rule ω assigning to each variety X and to each vector bundle E/X a two-sided $A(X)$-module isomorphism $A(X) \to A(E, E - X)$ satisfying certain natural properties (Definition 3.1) and called *Thom isomorphisms*. *A Thom structure on* A is a rule assigning to each smooth variety X and each line bundle L over X a class $\mathrm{th}(L) \in A(L, L - X)$ satisfying certain natural properties (Definition 3.3) and called *the Thom class*. *A Chern structure on* A is a rule assigning to each smooth variety X and each line bundle L over X a class $c(L) \in A(X)$ satisfying certain natural properties (Definition 3.2) and called *the first Chern class* (or some times called *the Euler class* [20]).

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It is proved in this paper that for a given A these structures are in natural bijections with each other. More precisely we construct the following diagram

$$\begin{array}{ccc} & \textit{Orientations on } A & \\ \overset{\delta}{\nearrow} & & \overset{\rho}{\searrow} \\ \textit{Chern structures on } A & \xleftarrow{\ \gamma\ } & \textit{Thom structures on } A \end{array} \qquad (1)$$

in which each arrow is *a bijection* and each round trip coincides with *the identity* (Theorems 3.5, 3.35 and 3.36). The constructions of these arrows are described briefly below in this section. One of the consequence of the theorem is this: *the existence at least one of these structures on A* implies *the existence of an orientation on A*; an orientation on A is *never defined by* the ring cohomology theory itself, even on usual singular cohomology there are plenty different orientations (see an example below Section 1).

However in practice certain ring cohomology theories come equipped with either a specific Chern structure or with a specific Thom structure. Thus they are equipped with distinguished orientations. Say, usual singular cohomology with integral coefficients (on complex algebraic varieties) come equipped with the known Chern structure, algebraic K-theory is equipped with a Chern structure as well ($L \mapsto [\mathbf{1}] - [L^\vee]$). The motivic cohomology $H^*(-, \mathbb{Z}(*))$ is equipped with a Chern structure. The algebraic cobordism theory $\mathrm{MGL}^{*,*}$ is equipped with a natural Thom structure. Thus these two theories are equipped with the corresponding orientations. The algebraic cobordism theory $\mathrm{MGL}^{*,*}$ of Voevodsky [31] is one of *the main motivating example* for this article, but it is expected to be written in details later.

An oriented ring cohomology theory is a cohomology theory equipped with an orientation in the sense above. As it was already mentioned to orient a ring theory A is the same as to fix a Thom structure on A or to fix a Chern structure on A. An orientation is usually denoted ω. The Thom structure corresponding to ω via ρ is written often as $L \mapsto \mathrm{th}_\omega(L)$. The Chern structure corresponding to ω via $\gamma \circ \rho$ is written often as $L \mapsto c_\omega(L)$. *An orientable ring cohomology theory* is a ring cohomology theory which can be equipped with an orientation. An oriented cohomology theory is as well an oriented cohomology pretheory in the sense of [20] because the integration constructed in [21], [23] is perfect in the sense of [20].

An example of a *non-standard Chern structure* on the usual singular cohomology with rational coefficients is given by the assignment $L \mapsto 1 - \exp(-c_1(L))$. This Chern structure gives an integration on $H^*(-, \mathbb{Q})$ which, via the Chern character, respects the Chern structure on the algebraic K-theory given by $L \mapsto [1] - [L^\vee]$.

Following [19] and [24] each orientation ω on A gives rise to a commutative formal group law over the coefficient ring $A(pt)$ of the theory. This is a formal power series $F_\omega \in A(pt)[[u_1, u_2]]$ in two variables such that for each smooth variety X and each pair of line bundles L_1, L_2 over X one has the relation

$c_\omega(L_1 \otimes L_2) = F_\omega(c_\omega(L_1), c_\omega(L_2))$ in $A(X)$. The formal group law plays a key rule in constructing push-forwards on A [21, 23] and is described in Section 3.9.

In the topological setting there is the universal oriented theory. It is the complex cobordism equipped with a distinguished orientation [16, 25]. The corresponding formal group law is the universal one [24]. Other examples of oriented theories are the singular cohomology, the complex K-theory, the Brown–Peterson theory, Morava K-theories, elliptic cohomology. Stable cohomotopy is a typical example of an unorientable theory.

We now sketch the structure of the text. In Section 1.1 certain general notation are introduced. In Section 2 the notion of cohomology theory introduced in [21] is recalled and general properties of a cohomology theory are proved. The deformation to the normal cone construction is recalled in Section 2.2.7 as well. An analog of the purity theorem from [18] is proved (Theorem 2.2). The canonical isomorphism $A(X, X-Z) \cong A(N, N-Z)$ from Theorem 2.2 one should consider as a replacement of the excision isomorphism for a tubular neighborhood (well known in the topology). Although Theorem 2.2 is not used in the present article, it is very useful for the construction of an integration on a given oriented cohomology theory. In the end of the section we recall the notion of a ring cohomology theory [21].

In Section 3 we construct the triangle of correspondences ρ, γ, δ mentioned above (see Theorems 3.5, 3.35 and 3.36). Proofs of these three theorems use Theorem 3.9, the Splitting principle (Lemma 3.24) and higher Chern classes Theorem (3.27). In the very end of the section it is shown how an orientation ω on the theory A gives rise to the formal group law F_ω over the coefficient ring of A.

Since the text is rather long it is reasonable to sketch here our constructions of the assignments γ, ρ and δ.

Suppose we are given with a Thom structure $L/X \mapsto \mathrm{th}(L) \in A(L, L-X)$ on A. For the zero section $z\colon X \to L$ of a line bundle L over X set $c(L) = z^A(\mathrm{th}(L)) \in A(X)$ (the pull-back of the element $\mathrm{th}(L)$ under the inclusion $(X, \emptyset) \hookrightarrow (L, L-X)$). The assignment $L \mapsto c(L)$ is the Chern structure on A corresponding via γ to the Thom structure (see Theorem 3.5).

Suppose we are given with an orientation ω. For a line bundle L over a smooth X consider the image $\mathrm{th}(L) \in A(L, L-X)$ of the unit $1 \in A(X)$ under the Thom isomorphism $A(X) \to A(L, L-X)$ determined by the orientation ω. The assignment $L \mapsto \mathrm{th}(L) \in A(L, L-X)$ is the Thom structure corresponding via ρ to the orientation ω (see Theorem 3.36).

Suppose we are given with a Chern structure $L \mapsto c(L)$ on A. In this case the Projective bundle theorem holds (see Theorem 3.9) and there is a Chern class theory $E \mapsto c_n(E)$ with values in A. To produce an orientation ω on A we associate to each vector bundle E/X its Thom class $\mathrm{th}(E) \in A(E, E-X)$. Firstly for a rank n vector bundle E we consider the vector bundle $F = \mathbf{1} \oplus E$, the projective bundle $p\colon \mathbf{P}(F) \to X$, the tautological line bundle $\mathcal{O}_F(1)$ on it and set

$$\bar{\mathrm{th}}(E) := c_n(\mathcal{O}_F(1) \otimes p^*(E)) \in A(\mathbf{P}(F)).$$

It turns out that the element $\bar{\mathrm{th}}(E)$ belongs to the subgroup $A(\mathbf{P}(F), \mathbf{P}(F) - \mathbf{P}(\mathbf{1}))$ of the group $A(\mathbf{P}(F))$. The class $\mathrm{th}(E)$ is defined as the image of the element $\bar{\mathrm{th}}(E)$ under the excision isomorphism identifying $A(\mathbf{P}(F), \mathbf{P}(F) - \mathbf{P}(\mathbf{1}))$ with $A(E, E - X)$. The required orientation ω on A is given by the assignment which associate to a vector bundle $p: E \to X$ the map $(\cup\, \mathrm{th}(E)) \circ p^A: A(X) \to A(E, E - X)$. Details are given in the proof of Theorem 3.35.

There is another class $\mathrm{th}^{\mathrm{naive}}(E) \in A(E, E - X)$ which is quite often used in literature. It is constructed as the image under the mentioned excision isomorphism of the class

$$\bar{\mathrm{th}}^{\mathrm{naive}}(E) = \sum_{k=0}^{n} (-1)^{n-k} c_{n-k}(E) \xi^k \in A(\mathbf{P}(F)),$$

where $\xi = c(\mathcal{O}_F(-1))$ is the first Chern class of the tautological line bundle on $\mathbf{P}(F)$. The assignment $E \mapsto \mathrm{th}^{\mathrm{naive}}(E)$ gives an orientation too. However in this case the assignment $L \mapsto c'(L) = z^A(\mathrm{th}^{\mathrm{naive}}(L)) \in A(X)$ is a Chern structure on A which is *in general different* of the one we began with. If the Chern structure $L \mapsto c(L)$ satisfies the additivity property, that is $c(L_1 \otimes L_2) = c(L_1) + c(L_2)$, then $\mathrm{th}^{\mathrm{naive}}(E) = \mathrm{th}(E)$ and $c(L) = c'(L)$.

To simplify technicalities a reader may assume through the text that

- all cohomology theories in the sense of Definition 2.1 take values in the category of $\mathbb{Z}/2$-graded Abelian groups and grade-preserving homomorphisms, the boundary operator ∂ is either grade-preserving or of the degree $+1$ and moreover,
- all ring cohomology theories in the sense of Definition 2.13 are $\mathbb{Z}/2$-graded-commutative ring theories, i.e. for any $a \in A^p(P)$ and $b \in A^q(Q)$ one has the relation $a \times b = (-1)^{pq} b \times a$ in $A^{p+q}(P \times Q)$,
- all Thom isomorphisms in the sense of Definition 3.1 are grade-preserving and all Thom and Chern classes are of even degree,
- ‘a universally central elements’ is just ‘an even degree element’.

Following these simplifications the reader should replace everywhere through the text the concept of ‘universally central elements’ (see Definition 2.15) by the concept of ‘even degree elements’. For instance the reader should replace the ring $A^{uc}(X)$ of all universally central elements by the ring $A^{\mathrm{ev}}(X)$ of all even degree elements.

1.1. TERMINOLOGY AND NOTATION

Let k be a field. The term ‘variety’ is used in this text to mean a reduced quasi-projective scheme over k. If X is a variety and $U \subset X$ is a Zariski open then $Z := X - U$ is considered as a closed subscheme with a unique structure of the reduced scheme, so Z is considered as a closed subvariety of X. We fix the following notation:

- $\mathcal{A}b$ – the category of Abelian groups;
- $\mathcal{S}m$ – the category of smooth varieties;
  $\mathcal{S}m\mathcal{O}p$ – the category of pairs (X, U) with smooth X and open U in X. Morphisms are morphisms of pairs.

We identify the category $\mathcal{S}m$ with a full subcategory of $\mathcal{S}m\mathcal{O}p$ assigning to a variety X the pair $(X, \emptyset)$:

- $pt = \mathrm{Spec}(k)$;
  For a smooth X and an effective divisor $D \subset X$ we write $L(D)$ for a line bundle over X whose sheaf of sections is the sheaf $\mathcal{L}_X(D)$ (see [9, Chapter II, Section 6, 6.13]).
  $\mathbf{P}(V) = \mathrm{Proj}(S^*(V^\vee))$ – the space of lines in a finite-dimensional k-vector space V; $L_V = \mathcal{O}_V(-1)$ – the tautological line bundle over $\mathbf{P}(V)$;
  $\mathbf{1}_X$ – the trivial rank 1 bundle over X, often we will write $\mathbf{1}$ for $\mathbf{1}_X$;
- $\mathbf{P}(E)$ – the space of lines in a vector bundle E;
  $L_E = \mathcal{O}_E(-1)$ – the tautological line bundle on $\mathbf{P}(E)$;
  E^0 – the complement to the zero section of E;
  $E^\vee$ – the vector bundle dual to E;
  $z\colon X \to E$ – the zero section of a vector bundle E;
- For a contravariant functor A on $\mathcal{S}m$ set
  $$A(\mathbf{P}^\infty) = \varprojlim A(\mathbf{P}(V)), \tag{2}$$
  where the projective system is induced by all the finite-dimensional vector subspaces $V \hookrightarrow k^\infty$.
  Similarly set
  $$A(\mathbf{P}^\infty \times \mathbf{P}^\infty) = \varprojlim A(\mathbf{P}(V) \times \mathbf{P}(W)),$$
  where the projective system is induced by all the finite-dimensional subspaces $V, W \subset k^\infty$.

2. Cohomology Theories

DEFINITION 2.1. A cohomology theory is a contravariant functor $\mathcal{S}m\mathcal{O}p \xrightarrow{A} \mathcal{A}b$ together with a functor morphism $\partial\colon A(U) \to A(X, U)$ satisfying the following properties

1. Localization: the sequence $A(X) \xrightarrow{j^A} A(U) \xrightarrow{\partial_P} A(X, U) \xrightarrow{i^A} A(X) \xrightarrow{j^A} A(U)$ is exact for each pair $P = (X, U) \in \mathcal{S}m\mathcal{O}p$, where $j\colon U \hookrightarrow X$ and $i\colon (X, \emptyset) \hookrightarrow (X, U)$ are the natural inclusions.
2. Excision: the operator $A(X, U) \to A(X', U')$ induced by a morphism $e\colon (X', U') \to (X, U)$ is an isomorphism, if the morphism e is etale and for $Z = X - U$, $Z' = X' - U'$ one has $e^{-1}(Z) = Z'$ and $e\colon Z' \to Z$ is an isomorphism.

3. Homotopy invariance: the operator $A(X) \to A(X \times \mathbf{A}^1)$ induced by the projection $X \times \mathbf{A}^1 \to X$ is an isomorphism.

The operator ∂_P is called the boundary operator and is written usually as ∂.

A morphism of cohomology theories φ: $(A, \partial^A) \to (B, \partial^B)$ is a functor transformation φ: $A \to B$ commuting with the boundary morphisms in the sense that for every pair $P = (X, U) \in \mathcal{S}m\mathcal{O}p$ one has $\partial^B_P \circ \varphi_U = \varphi_P \circ \partial^A_P$.

We write also $A_Z(X)$ for $A(X, U)$, where $Z = X - U$, and call the group $A_Z(X)$ cohomology of X with the support on Z. The operator

$$A_Z(X) \xrightarrow{i^A} A(X) \tag{3}$$

is called the support extension operator for the pair (X, U).

We do not assume at all in this text that cohomology theories are graded and the boundary operator is of degree $+1$. We do not assume this in particular because it is never used below and it is even inconvenient to assume this for certain points.

One could replace in this definition the category of Abelian group by any Abelian category or even by additive one which is equipped with Kernels and Cokernels for projectors. We left such a replacement to a reader to avoid technicalities as much as it is possible.

2.1. EXAMPLES

Consider a number of examples.

2.1.1. *Classical Singular Cohomology*

Let $k = \mathbb{C}$ and let A be an Abelian group. Let $(X, U) \mapsto \oplus^{\infty}_{-\infty} H^p(X(\mathbb{C}), U(\mathbb{C}); A)$ be the usual singular cohomology (with coefficients in A) of the pair of the complex point sets with respect to the complex topology. Take as a boundary ∂ the usual boundary map ∂ (see for instance [29]).

2.1.2. *A Generalized Cohomology Theory*

Let $k = \mathbb{C}$ and let $(X, U) \mapsto \oplus^{\infty}_{p=-\infty} E^p(X(\mathbb{C}), U(\mathbb{C}))$ be a generalized cohomology theory say represented by a spectrum E with the usual boundary map ∂ (see for instance [1] or [29, 8.33]).

2.1.3. *Singular Cohomology of the Real Point Sets*

Let $k = \mathbb{R}$ and let A be an Abelian group $(X, U) \mapsto \oplus^{\infty}_{0} H^p((X(\mathbb{R}), U(\mathbb{R}); A)$ be the usual singular cohomology (with coefficients in A) of the pair of real points set considered with respect to the strong topology. Take as a boundary ∂ the usual boundary map for the pair $(X(\mathbb{R}), U(\mathbb{R}))$.

2.1.4. *Witt Theory of Balmer*
Let $(X, X - Z) \mapsto \oplus_{p=-\infty}^{\infty} W_Z^p(X)$ be the Witt functor defined in [2]. Clearly it is a cohomology theory in the sense of Definition 2.1.

2.1.5. *Bloch's Higher Chow Groups*
Let $(X, X - Z) \mapsto \oplus_{p=-\infty}^{\infty} \oplus_{q=-\infty}^{\infty} CH_Z^p(X, q)$ be the higher Chow groups defined in [3]. Clearly it is a cohomology theory in the sense of Definition 2.1.

2.1.6. *Motivic Cohomology of M. Levine*
Clearly it is a cohomology theory in the sense of Definition 2.1 [15].

2.1.7. *Etale Cohomology*
Let F be a locally constant torsion sheaf on the etale k-situs and assume that char(k) is prime to the torsion of F. In this example $A^n(X, U) = H_Z^n(X_{\text{et}}, F)$ [17, 3.1] and ∂ is defined in [17, 3.1.25]. The localization property for the pair (A, ∂) is proved in [17, 3.1.25], the excision property is proved in [17, 3.1.27] and the homotopy invariance is proved in [17, 6.4.20].

2.1.8. *K-theory*
Algebraic K-theory also can be fitted to Definition 2.1. To do this use, for instance, K-groups with support $K_n(XonZ)$ $(n \geqslant 0)$ of [30]. So set $A^n(X, U) = K_{-n}(XonZ)$, where $Z = X - U$. Further set $A(X, U) = \oplus_{n=0}^{\infty} A^n(X, U)$. The definition of ∂ and the exactness of the localization sequence are contained in [30, Theorem 5.1] (except the surjectivity of the restriction $A^0(X) \to A^0(U)$). If X is quasi-projective then $K(XonX)$ coincides with the Quillen's K-groups $K_n^Q(X)$ by [30, 3.9, 3.10]. This proves in particular the homotopy invariance $A^n(X)$ for smooth X. The excision property for A follows from [30, 3.19]. It remains now to check the surjectivity of the restriction $A^0(X) \to A^0(U)$. Clearly $A^0(X) = K_0^Q(X)$ coincides with the Grothendieck group of the vector bundles on X. Since X is smooth the desired surjectivity follows from [4, Section 8, Proposition 7]. Thus (A, ∂) satisfies Definition 2.1.

2.1.9. *Motivic Cohomology*
Here $A_Z^{\mathcal{C},p}(X) = H_Z^p(X, \mathcal{C}) := \mathrm{Hom}_{DM^-(k)}(M_Z(X), \mathcal{C}[p])$ is the motivic cohomology with coefficients in a motivic complex $\mathcal{C} \in DM^-(k)$ [28], where the motive $M_Z(X)$ with supports on Z is defined in [28, the text just below the proof of Theorem 4.8]. The motive $M_Z(X)$ is identified with the complex $C^*(\mathbb{Z}_{\text{tr}}(X)/\mathbb{Z}_{\text{tr}}(X - Z))$ in the proof of Lemma 4.11 in [28]. Set $A_Z^{\mathcal{C}}(X) = \oplus_{-\infty}^{\infty} A_Z^{\mathcal{C},p}(X)$. The boundary operator ∂ which we denote in this example $\partial^{\mathcal{C}}$ is defined in [28, ??].

The homotopy invariance property holds by [28, Proposition 4.2]. The excision property is proved in [28, the proof of Lemma 4.11]. The localization property follows from the exactness of the complex

$$0 \to \mathbb{Z}_{\text{tr}}(X - Z) \to \mathbb{Z}_{\text{tr}}(X) \to \mathbb{Z}_{\text{tr}}(X)/\mathbb{Z}_{\text{tr}}(X - Z) \to 0$$

because the functor C^* takes short exact sequences to exact triangles [28, Theorem 1.12].

2.1.10. *Semi-topological Complex and Real K-theories [6]*
If the ground field k is the field $\mathbb{R}$ of reals then the semi-topological K-theory of real algebraic varieties $K\mathbb{R}^{\mathrm{semi}}$ defined in [6] is a cohomology theory as it is proved in [6].

2.1.11. *Representable Theories*
Here $A^p(X, U) = \oplus_q E^{p,q}(X/U)$, where E is a T-spectrum [31]. Set $A(X, U) = \oplus_{-\infty}^{\infty} A^p(X, U)$. The boundary operator is described in [31] and is defined via the triangulated structure on the stable homotopy category [31]. In particular, in the case $E = \mathrm{MGL}$ [31, Section 6.3] we obtain the algebraic cobordism theory.

2.2. GENERAL PROPERTIES OF COHOMOLOGY THEORIES

We specify here certain properties of an arbitrary cohomology theory A which are useful below in the text.

2.2.1.
The localization property implies that $A_{\emptyset}(X) = A(X, X) = 0$. Therefore $A(\emptyset) = A_{\emptyset}(\emptyset) = 0$.

2.2.2.
If any two of morphisms $(X, U) \to (Y, V)$, $X \to Y$, $U \to V$, defined by a morphism $f\colon (X, U) \to (Y, V)$, induce isomorphisms on A-cohomology then the third of these morphisms induces an isomorphism on A-cohomology.

2.2.3. *Localization Sequence for a Triple*
Let $T \subset Y \subset X$ be closed subsets of a smooth variety X. Let $\partial\colon A(X - T) \to A_T(X)$ be the boundary map for the pair $(X, X - T)$. Consider the support extension map $e^A\colon A_{Y-T}(X - T) \to A(X - T)$ and set $\partial_{Y,T} = \partial \circ e^A\colon A_{Y-Z}(X - Z) \to A_T(X)$.

We claim that the sequence

$$\cdots \to A_T(X) \xrightarrow{\alpha} A_Y(X) \xrightarrow{\beta} A_{Y-T}(X - T) \xrightarrow{\partial_{Y,T}} A_T(X) \xrightarrow{\gamma} A_Y(X) \to \cdots$$

with the obvious mappings α, β and γ is a complex and moreover it is exact. We call this sequence *the localization sequence* for the triple $(X, X - T, X - Y)$. If $Y = X$, then this sequence coincides with the localization sequence for the pair $(X, X - T)$.

If $U \subset X - T$ is an open containing $Y - T$ then the pull-back $A_{Y-T}(X-T) \to A_{Y-T}(U)$ is an isomorphism by the excision property. So replacing $A_{Y-T}(X-T)$ by $A_{Y-T}(U)$ we get an exact sequence

$$\cdots \to A_T(X) \xrightarrow{\alpha} A_Y(X) \xrightarrow{\beta} A_{Y-T}(U) \xrightarrow{\partial_{Y,T}} A_T(X) \xrightarrow{\gamma} A_Y(X) \to \cdots.$$

We call it *the localization sequence* for the triple $(X, U, X - Y)$.

2.2.4. *Mayer–Vietoris Sequence*
If $X = U_1 \cup U_2$ is a union of two open subsets U_1 and U_2 and if Y is a closed subset in X, then set $T_i = Y - U_i$, $Y_i = Y \cap U_i = Y - T_i$, $U_{12} = U_1 \cap U_2$ and $Y_{12} = U_{12} \cap Y$. Consider the morphism of the localization sequences for the triples $(X, U_1, X-Y)$ and (U_2, U_{12}, U_2-Y) induced by the inclusion of the triples $(U_2, U_{12}, U_2 - Y_2) \subset (X, U_1, X - Y)$

$$\begin{array}{ccccccc} A_Y(X) & \xrightarrow{\alpha_1} & A_{Y_1}(U_1) & \longrightarrow & A_{T_1}(X) & \xrightarrow{e^A} & A_Y(X) \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \gamma} & & \downarrow \\ A_{Y_2}(U_2) & \xrightarrow{\beta_2} & A_{Y_{12}}(U_{12}) & \xrightarrow{\partial} & A_{T_1}(U_2) & \longrightarrow & A_{Y_2}(U_2). \end{array}$$

The map γ is an isomorphism by the excision property. Set $d = e^A \circ \gamma^{-1} \circ \partial\colon A_{Y_{12}}(U_{12}) \to A_Y(X)$. We claim that the sequence

$$\cdots \to A^n_Y(X) \xrightarrow{\alpha_1+\alpha_2} A^n_{Y_1}(U_1) \oplus A^n_{Y_2}(U_2)$$
$$\xrightarrow{(\beta_1,-\beta_2)} A^n_{Y_{12}}(U_{12}) \xrightarrow{d} A^{n+1}_Y(X) \to \cdots$$

is exact and call this sequence *the Mayer–Vietoris sequence* of the open covering $X = U_1 \cap U_2$. The proof of the exactness is straightforward and we skip it.

The Mayer–Vietoris sequence is natural in the following sense. If $f\colon X' \to X$ is a morphism and $X' = U'_1 \cup U'_2$ is a Zariski covering of X' such that $f(U'_i) \subset U_i$ and if Y' is a closed subset in X' containing $f^{-1}(Y)$, then the pull-back mappings $f^A\colon A_Y(X) \to A_{Y'}(X')$, $f^A\colon A_{Y_i}(U_i) \to A_{Y'_i}(U'_i)$, $f^A\colon A_{Y_{12}}(U_{12}) \to A_{Y'_{12}}(U'_{12})$ form a morphism of the corresponding Mayer–Vietoris sequences.

2.2.5.
Let $i_r\colon X_r \hookrightarrow X_1 \amalg X_2$ be the natural inclusion $(r = 1, 2)$. Let $Y_r \subset X_i$ be a closed subset for $(r = 1, 2)$. Then the induced map $A_{X_1 \amalg X_2}(X_1 \amalg X_2) \to A_{Y_1}(X_1) \oplus A_{Y_2}(X_2)$ is an isomorphism.

Proof. This follows from the Mayer–Vietoris property and the fact that $A_\emptyset(\emptyset) = 0$. □

2.2.6. *Strong Homotopy Invariance*
Let $p\colon T \to X$ be an affine bundle (i.e., a torsor under a vector bundle). Let $Z \subset X$ be a closed subset and let $S = p^{-1}(Z)$. Then the pull-back map $p^A\colon A_Z(X) \to$

$A_S(T)$ is an isomorphism. If $s\colon X \to T$ is a section then the induced operator $s^A\colon A_S(T) \to A_Z(X)$ is an isomorphism as well.

Proof. First consider the case $Z = X$. Then $S = T$ and we have to check that the pull-back map $p^A\colon A(X) \to A(T)$ is an isomorphism. Choose a finite Zariski open covering $X = \cup U_i$ such that $T_i = p^{-1}(U_i)$ is isomorphic to the trivial vector bundle over each U_i and then use the morphism of the Mayer–Vietoris sequences and the homotopy invariance property of the cohomology theory A.

To prove the general case consider the localization sequences for the pairs $(X, X - Z)$ and $(T, T - S)$. The pull-back mappings form a morphism of these two long exact sequences. The 5-Lemma completes the proof. □

2.2.7. *Deformation to the Normal Cone*

The deformation to the normal cone is a well-known construction (for example, see [7]). Since the construction and its property (6) play an important role in what follows we give here some details.

Let $i\colon Y \hookrightarrow X$ be a closed imbedding of smooth varieties with the normal bundle N. There exists a smooth variety X_t together with a smooth morphism $p_t\colon X_t \to \mathbf{A}^1$ and a closed imbedding $i_t\colon Y \times \mathbf{A}^1 \hookrightarrow X_t$ such that the map $p_t \circ i_t$ coincides with the projection $Y \times \mathbf{A}^1 \to \mathbf{A}^1$ and

- the fiber of p_t over $1 \in \mathbf{A}^1$ is canonically isomorphic to X and the base change of i_t by means of the imbedding $1 \hookrightarrow \mathbf{A}^1$ coincides with the imbedding $i\colon Y \hookrightarrow X$;
- the fiber of p_t over $0 \in \mathbf{A}^1$ is canonically isomorphic to N and the base change of i_t by means of the imbedding $0 \hookrightarrow \mathbf{A}^1$ coincides with the zero section $Y \hookrightarrow N$.

Thus we have the diagram

$$(N, N - Y) \xrightarrow{i_0} (X_t, X_t - Y \times \mathbf{A}^1) \xleftarrow{i_1} (X, X - Y). \tag{4}$$

Here and further we identify a variety with its image under the zero section of any vector bundle over this variety.

Let us recall a construction of X_t, p_t and i_t. For that take X_t' to be the blow-up of $X \times \mathbf{A}^1$ with the center $Y \times \{0\}$. Set $X_t = X_t' - \tilde{X}$, where $\tilde{X}$ is the proper preimage of $X \times \{0\}$ under the blow-up map. Let $\sigma\colon X_t \to X \times \mathbf{A}^1$ be the restriction of the blow-up map $\sigma'\colon X_t' \to X \times \mathbf{A}^1$ to X_t and set p_t to be the composition of σ and the projection $X \times \mathbf{A}^1 \to \mathbf{A}^1$.

The proper preimage of $Y \times \mathbf{A}^1$ under the blow-up map is mapped isomorphically to $Y \times \mathbf{A}^1$ under the blow-up map. Thus the inverse isomorphism gives the desired imbedding $i_t\colon Y \times \mathbf{A}^1 \hookrightarrow X_t$ (observe that $i_t(Y \times \mathbf{A}^1)$ does not cross $\tilde{X}$).

It is not difficult to check that the imbedding i_t satisfies the mentioned two properties (the preimage of $X \times 0$ under the map σ' consists of two irreducible components: the proper preimage of X and the exceptional divisor $\mathbf{P}(N \oplus 1)$. Their intersection is $\mathbf{P}(N)$ and $i_t(Y \times \mathbb{A}^1)$ crosses $\mathbf{P}(N \oplus 1)$ along $\mathbf{P}(\mathbf{1})$ = the zero section of the normal bundle N).

We claim that the diagram (4) consists of isomorphisms on the A-cohomology.

THEOREM 2.2. *The following diagram consists of isomorphisms*

$$A_Y(N) \xleftarrow{i_0^A} A_{Y\times\mathbf{A}^1}(X_t) \xrightarrow{i_1^A} A_Y(X). \tag{5}$$

Moreover for each closed subset $Z \subset Y$ the following diagram consists of isomorphisms as well

$$A_Z(N) \xleftarrow{i_0^A} A_{Z\times\mathbf{A}^1}(X_t) \xrightarrow{i_1^A} A_Z(X). \tag{6}$$

This theorem is analogous to the Homotopy Purity Theorem from [18, Theorem 3.2.3]. The proof is postponed until Section 2.3. Now we state and prove the following corollary.

COROLLARY 2.3. *Let j_0: $\mathbf{P}(\mathbf{1} \oplus N) \hookrightarrow X'_t$ be the imbedding of the exceptional divisor into X'_t and let $j_1 = e_t \circ i_1 : X \hookrightarrow X'_t$, where e_t: $X_t \hookrightarrow X'_t$ is the open inclusion. Then the diagram*

$$A_{\mathbf{P(1)}}(\mathbf{P}(\mathbf{1} \oplus N)) \xleftarrow{j_0^A} A_{Y\times\mathbf{A}^1}(X'_t) \xrightarrow{j_1^A} A_Y(X) \tag{7}$$

consists of isomorphisms.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} A_{\mathbf{P(1)}}(\mathbf{P}(\mathbf{1} \oplus N)) & \xleftarrow{j_0^A} & A_{Y\times\mathbf{A}^1}(X'_t) \\ {\scriptstyle e^A}\downarrow & & \downarrow{\scriptstyle e_t^A} \\ A_Y(N) & \xleftarrow{i_0^A} & A_{Y\times\mathbf{A}^1}(X_t), \end{array}$$

where the vertical arrows are the obvious pull-backs. These vertical arrows are isomorphisms by the excision property. The operator i_0^A is an isomorphism by the first item of Theorem 2.2. Thus the operator j_0^A is an isomorphism. □

2.2.8.
Let X be a smooth variety and let L be a line bundle over X. Let $E = \mathbf{1} \oplus L$ and let $\bar{i}_L$: $X = \mathbf{P}(L) \hookrightarrow \mathbf{P}(E)$ be the closed imbedding induced by the direct summand L of E. Let $A_{\mathbf{P(1)}}(\mathbf{P}(E)) \xrightarrow{i^A} A(\mathbf{P}(E))$ be the support extension operator and let $\bar{i}_L^A$: $A(\mathbf{P}(E)) \to A(\mathbf{P}(L))$ be the pull-back operator. We claim that the following sequence

$$0 \to A_{\mathbf{P(1)}}(\mathbf{P}(E)) \xrightarrow{i^A} A(\mathbf{P}(E)) \xrightarrow{\bar{i}_L^A} A(\mathbf{P}(L)) \to 0. \tag{8}$$

is exact.

To prove this consider $U = \mathbf{P}(E) - \mathbf{P}(\mathbf{1})$ with the open inclusion $j\colon U \hookrightarrow \mathbf{P}(E)$ and observe that U becomes a line bundle over X by means of the linear projection $q\colon U \to \mathbf{P}(L) = X$ (the line bundle is isomorphic to $L^\vee$) The obvious inclusion $i_L\colon \mathbf{P}(L) \hookrightarrow U$ is just the zero section of this line bundle, $\bar{i}_L = j \circ i_L$ and the pull-back operator $i_L^A\colon A(U) \to A(\mathbf{P}(L))$ is an isomorphism (the inverse to the one q^A).

Now consider the pair $(\mathbf{P}(E), U)$. By the localization property (Definition 2.1) the following sequence

$$\cdots \to A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E)) \xrightarrow{i^A} A(\mathbf{P}(E)) \xrightarrow{j^A} A(U) \to \cdots$$

is exact. If $\mathbf{P}(E) \xrightarrow{p} X$ is the natural projection then the operator $p^A \circ i_L^A\colon A(U) \to A(\mathbf{P}(E))$ splits j^A. This implies the surjectivity of j^A and the injectivity of i^A. To proof that the sequence (8) is short exact it remains to recall that the operator i_L^A is an isomorphism and $\bar{i}_L = j \circ i_L$.

2.2.9.

We use here the notation from Section 2.2.7. Let $e_t\colon X_t \hookrightarrow X_t'$ be the open inclusion and let $p\colon \mathbf{P}(\mathbf{1} \oplus N) \to Y$ be the projection and let $s\colon Y \to \mathbf{P}(\mathbf{1} \oplus N)$ be the section of the projection identifying Y with the subvariety $\mathbf{P}(\mathbf{1})$ in $\mathbf{P}(\mathbf{1} \oplus N)$. The following commutative diagram will be repeatedly used below in the text

$$\begin{array}{ccccc}
\mathbf{P}(\mathbf{1} \oplus N) & \xrightarrow{j_0} & X_t' & \xleftarrow{j_1} & X \\
{\scriptstyle s}\uparrow & & {\scriptstyle I_t}\uparrow & & \uparrow{\scriptstyle i} \\
Y & \xrightarrow{k_0} & Y \times \mathbf{A}^1 & \xleftarrow{k_1} & Y,
\end{array}$$

where $I_t = e_t \circ i_t$ and j_0 is the inclusion of the exceptional divisor and $j_1 = e_t \circ i_1$ and k_0, k_1 are the closed imbedding given by $y \mapsto (y, 0)$ and $y \mapsto (y, 1)$ respectively.

LEMMA 2.4. (**Useful lemma**). *Under the notation from Section* 2.2.7 *let* $j_t\colon V_t = X_t' - Y \times \mathbf{A}^1 \to X_t'$ *be the inclusion. If the support extension operator* $A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(\mathbf{1} \oplus N)) \to A(\mathbf{P}(\mathbf{1} \oplus N))$ *is injective then*

$$\mathrm{Ker}(j_0^A) \cap \mathrm{Ker}(j_t^A) = (0),$$

in the other words the operator

$$j_0^A \oplus j_t^A\colon A(X_t') \to A(\mathbf{P}(\mathbf{1} \oplus N)) \oplus A(V_t)$$

is monomorphism. In particular this holds if Y *is a divisor on* X.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} A(\mathbf{P}(\mathbf{1}\oplus N)) & \xleftarrow{j_0^A} & A(X'_t) \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\alpha_t} \\ A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(\mathbf{1}\oplus N)) & \xleftarrow{j_0^A} & A_{Y\times\mathbf{A}^1}(X'_t), \end{array}$$

where α and α_t are the support extension operators. The bottom operator j_0^A is an isomorphism by Corollary 2.3. The map α is injective by the very assumption (if Y is a divisor in X then α is injective by (8)). Since the composition $j_0^A \circ \alpha_t$ coincides with the one $\alpha \circ j_0^A$ it is injective as well.

The localization sequence for the pair (X'_t, V_t) shows that $\mathrm{Ker}(j_t^A) = \mathrm{Im}(\alpha_t)$. The lemma follows. □

2.2.10.
Let i: $\mathbf{P}(V) \hookrightarrow \mathbf{P}(W)$ and j: $\mathbf{P}(V) \hookrightarrow \mathbf{P}(W)$ be two linear imbeddings (imbeddings induced by linear imbeddings V into W). If the dimension of V is strictly less than the dimension of W, then $i^A = j^A$: $A(\mathbf{P}(W)) \to A(\mathbf{P}(V))$.

In fact, in this case there exists a linear automorphism ϕ of W which has the determinant 1 and such that $j = \phi \circ i$. Since ϕ is a composite of elementary matrices and each elementary matrix induces the identity automorphism $A(\mathbf{P}(W))$ (by the homotopy invariance of A) one gets the relation $\phi^A = \mathrm{id}$. Therefore $j^A = i^A \circ \phi^A = i^A$.

2.3. PROOF OF THEOREM 2.2

Proof. Basically the proof mimics the arguments used for the proof of the Homotopy Purity Theorem [18, Theorem 3.2.3]. Since the proof of the second assertion will be left to the reader the proof of the first one will be given in details. We start with certain observations concerning elementary properties of the deformation to the normal cone construction. Namely, if U and V are Zariski open subsets of X, then the following holds

(a) $U_t \cap V_t = (U \cap V)_t$;
(b) $U_t \cup V_t = (U \cup V_t)$;
(c) if an etale morphism e: $(\tilde{X}, \tilde{X} - \tilde{Y}) \to (X, X - Y)$ satisfies the hypotheses of the excision property (Definition 2.1), then the induced morphism e_t: $(\tilde{X}_t, \tilde{X} - \tilde{Y} \times \mathbf{A}^1) \to (X_t, X_t - Y \times \mathbf{A}^1)$ satisfies as well the hypotheses of the excision property.

To prove the theorem, we will need the lemma and four claims below.

LEMMA 2.5. *If* $X = Y \times \mathbf{A}^n$ *and* $Y = Y \times \{0\} \hookrightarrow Y \times \mathbf{A}^n$ *then* $i^A_{1,X}$: $A_{Y\times\mathbf{A}^1}(X_t) \to A_Y(X)$ *is an isomorphism.*

DEFINITION 2.6. An open subset U in X is called good if there exists a diagram

$$(U, U - Y_U) \xleftarrow{e} (T, T - S) \xrightarrow{f} (Y_U \times \mathbf{A}^n, Y_U \times \mathbf{A}^n - Y_U \times \{0\})$$

with $Y_U = Y \cap U$ and with morphisms e and f satisfying the hypotheses of the excision property (Definition 2.1).

NOTATION 2.7. We write below in this proof $i_{1,U}$ for the imbedding $U \hookrightarrow U_t$ from the deformation to the normal cone construction for the pair (U, Y_U). We will write below in the proof of theorem $i^A_{1,U}$ for the pull-back map $i^A_{1,U}\colon A_{Y_U \times \mathbf{A}^n}(U_t) \to A_{Y_U}(U)$.

CLAIM 2.8. *If $U \subset X$ is good then the pull-back map $i^A_{1,U}\colon A_{Y_U \times \mathbf{A}^n}(U_t) \to A_{Y_U}(U)$ is an isomorphism.*

CLAIM 2.9. *If an open subset U in X is good then each open subset V in U is good as well.*

CLAIM 2.10. *If an open subset U in X is good and if $V \subset X$ is an open subset such that the pull-back map $i^A_{1,V}$ is an isomorphism, then the pull-back map $i^A_{1,U\cup V}$ is an isomorphism as well.*

CLAIM 2.11. *For each point $x \in X$ there exists a good Zariski open neighborhood U of the point x.*

Assuming for a moment Lemma 2.5 and these four claims one can complete the proof of theorem as follows. By the fourth claim there exists a finite Zariski open covering $X = \cup_{i=1}^n U_i$ with good open subsets U_i. Claim 2.8 states that the pull-back map i^A_{1,U_1} is an isomorphism. Suppose for $V = \cup_{i=1}^k U_i$ the pull-back map $i^A_{1,V}$ is an isomorphism. Since the open subset U_{k+1} is good Claim 2.10 shows that the pull-back map $i^A_{1,W}$ is an isomorphism for $W = \cup_{i=1}^{k+1} U_i$. The induction by k shows that the pull-back map i^A_1 is an isomorphism.

It remains to prove Lemma 2.5 and four claims. □

Proof of Lemma 2.5. Let F/Y be a vector bundle and let F' be the blow-up of F at the zero section. The variety F' coincides with the total space of the line bundle $\mathcal{O}_F(-1)$ over $\mathbf{P}(F)$. Let $q_F\colon F' \to \mathbf{P}(F)$ be projection of the line bundle to its base $\mathbf{P}(F)$.

If $F = \mathbf{1} \oplus E$ for a vector bundle E over Y then one has the following commutative diagram

$$\begin{array}{ccccccc}
E' & \longrightarrow & F' & \longleftarrow & F' - E' & \longleftarrow & \mathbf{P}(\mathbf{1}) \times \mathbf{A}^1 \\
{\scriptstyle q_E}\downarrow & & {\scriptstyle q_F}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle \mathrm{pr}}\downarrow \\
\mathbf{P}(E) & \longrightarrow & \mathbf{P}(F) & \longleftarrow & \mathbf{P}(F) - \mathbf{P}(E) & \longleftarrow & \mathbf{P}(\mathbf{1}),
\end{array}$$

in which all the vertical arrows are the projections of the line bundles to their bases. Section 2.2.6 shows that the pull-back map $q^A\colon A_{\mathbf{P(1)}}(\mathbf{P}(F) - \mathbf{P}(E)) \to A_{\mathbf{P(1)}\times\mathbf{A}^1}(F' - E')$ is an isomorphism.

The projection q has two sections s_0 and s_1. The section s_0 is the zero section and the section s_1 is given by $x \mapsto (x, 1)$. Since $q \circ s_0 = \mathrm{id}$ the pull-back map $s_0^A\colon A_{\mathbf{P(1)}\times\mathbf{A}^1}(F' - E') \to A_{\mathbf{P(1)}}(\mathbf{P}(F) - \mathbf{P}(E))$ is an isomorphism. Since $q \circ s_1 = \mathrm{id}$ the pull-back map $s_1^A\colon A_{\mathbf{P(1)}\times\mathbf{A}^1}(F' - E') \to A_{\mathbf{P(1)}}(\mathbf{P}(F) - \mathbf{P}(E))$ is an isomorphism as well.

Now take $X = E$ and $Y =$ the zero section of E. Observe that the space X_t coincides with the variety $F' - E'$, the imbedding $i_1\colon X \hookrightarrow X_t$ coincides with the section $s_1\colon E \hookrightarrow F' - E'$. The normal bundle $N = N_{E/Y}$ to Y in E coincides with bundle E itself and the imbedding $i_0\colon N \hookrightarrow X_t$ coincides with the section $s_0\colon E \hookrightarrow X_t$. Finally the variety $Y \times \mathbf{A}^1$ coincides with $\mathbf{P(1)} \times \mathbf{A}^1$ and the imbedding $Y \times \mathbf{A}^1 \hookrightarrow X_t$ coincides with the imbedding $\mathbf{P(1)} \times \mathbf{A}^1 \hookrightarrow F' - E'$. Therefore both maps in the diagram

$$A_Y(E) = A_Y(N) \xleftarrow{i_0^A} A_{Y\times\mathbf{A}^1}(X_t) \xrightarrow{i_1^A} A_Y(E)$$

are isomorphisms. In particular these two maps are isomorphisms for the case of the trivial bundle $E = \mathbb{A}^n \times Y$. Thus we proved lemma. □

Proof of Claim 2.8. The claim follows immediately from lemma and the property (c) of the deformation to the normal cone construction. □

Proof of Claim 2.9. To prove this claim consider a diagram

$$(U, U - Y_U) \xleftarrow{\ e\ } (T, T - S) \xrightarrow{\ f\ } (Y_U \times \mathbf{A}^n, Y_U \times \mathbf{A}^n - Y_U \times \{0\})$$

with morphisms e and f satisfying the hypotheses of the excision property. Let $V \subset U$ be an open subset. Set $T = e^{-1}(V) \cap f^{-1}(Y_U \times \mathbf{A}^n)$ and $S = f^{-1}(Y_U)$. Then $S = f^{-1}(Y_U \times \{0\})$ and in the diagram

$$(V, V - Y_V) \xleftarrow{\ e_V\ } (T, T - S) \xrightarrow{\ f_V\ } (Y_V \times \mathbf{A}^n, Y_V \times \mathbf{A}^n - Y_V \times \{0\})$$

the morphisms e_V and f_V satisfy the hypotheses of the excision property as well. Thus the open subset V is good. □

Proof of Claim 2.10. This claim follows immediately from the properties (a) and (b) and the first claim comparing the Mayer–Vietoris sequence for $U \cup V$ with the one for $U_t \cup V_t$. □

Proof of Claim 2.11. This claim is proved in [32, Lemma ??]. □

Comment to the second assertion of theorem.

Recall that a Nisnevich neighborhood of a closed subset Y in X is an etale morphism $\pi\colon X' \to X$ such that for $Y' = \pi^{-1}(Y)$ the restriction map $\pi\colon Y' \to Y$

is an isomorphism. Clearly if π is a Nisnevich neighborhood of Y then for each closed subset Z in Y the map π is a Nisnevich neighborhood of the subset Z as well.

Recall as well that for any vector bundle $p: E \to X$ and any its section s and any closed subset $Z \subset X$ the two pull-back maps $p^A: A_Z(X) \to A_{p^{-1}(Z)}(E)$ and $s^A: A_{p^{-1}(Z)}(E) \to A_Z(X)$ are isomorphisms inverse of each other.

These two observations shows that the proof of the first assertion of theorem works as well for the second assertion of theorem.

2.4. RING COHOMOLOGY THEORIES

DEFINITION 2.12. Let $P = (X, U), Q = (Y, V) \in \mathcal{S}m\mathcal{O}p$. Set $P \times Q = (X \times Y, X \times V \cup U \times Y) \in \mathcal{S}m\mathcal{O}p$. This product is associative with the obvious associativity isomorphisms. The unit of this product is the variety pt.

This product is commutative with the obvious isomorphisms $P \times Q \cong Q \times P$.

DEFINITION 2.13. One says that a cohomology theory A is a ring cohomology theory if for every P, $Q \in \mathcal{S}m\mathcal{O}p$ there is given a natural bilinear morphism called cross-product

$$\times: A(P) \times A(Q) \to A(P \times Q)$$

which is functorial in both variables and satisfies the following properties

1. associativity: $(a \times b) \times c = a \times (b \times c) \in A(P \times Q \times R)$ for $a \in A(P)$, $b \in A(Q)$, $c \in A(R)$;
2. there is given an element $1 \in A(pt)$ such that for any pair $P \in \mathcal{S}m\mathcal{O}p$ and any $a \in A(P)$ one has $1 \times a = a = a \times 1 \in A(P)$;
3. partial Leibnitz rule: $\partial_{P \times Y}(a \times b) = \partial_P(a) \times b \in A(X \times Y, U \times Y)$ for a pair $P = (X, U) \in \mathcal{S}m\mathcal{O}p$, smooth variety Y and elements $a \in A(U)$, $b \in A(Y)$.

Given cross-products define cup-products $\cup: A_Z(X) \times A_{Z'}(X) \to A_{Z \cap Z'}(X)$ by

$$a \cup b = \Delta^A(a \times b), \tag{9}$$

where $\Delta: (X, U \cup V) \hookrightarrow (X \times X, X \times V \cup U \times X)$ is the diagonal. Clearly cup-products thus defined are bilinear and functorial in both variables. These cup-products are associative as well: $(a \cup b) \cup c = a \cup (b \cup c)$; the element $p^A(1) \in A(X)$, (here p is the projection $X \to pt$) is the unit for the cup-products $\cup: A_Z(X) \times A(X) \to A_Z(X)$ and $\cup: A(X) \times A_Z(X) \to A_Z(X)$; and a partial Leibnitz rule holds: $\partial(a \cup b) = \partial(a) \cup b$ for $a \in A(U)$, $b \in A(X)$.

Given cup-products one can construct cross-products by $a \times b = p_X^A(a) \cup p_Y^A(b)$ for $a \in A(X, U)$ and $b \in A(Y, V)$. Clearly these two constructions are inverse each to other. Thus having products of one kind we have products of the other kind and can use both products in the same time.

DEFINITION 2.14. A ring morphism $\varphi\colon (A, \partial_A, \times_A, 1_A) \to (B, \partial_B, \times_B, 1_B)$ of ring cohomology theories is a morphism $\varphi\colon (A, \partial_A) \to (B, \partial_B)$ of the underlying cohomology theories which takes the unit 1_A to the unit 1_B and commutes with the cross-products: $\varphi(1_A) = 1_B \in B(pt)$ and for every pairs $P, Q \in \mathcal{S}m\mathcal{O}p$ and every elements $a \in A(P), b \in A(Q)$ one has $\varphi_{P\times Q}(a \times b) = \varphi(a) \times \varphi(b) \in B(P \times Q)$.

DEFINITION 2.15. Let A be a ring cohomology theory and let X be a smooth variety. An element $a \in A(X)$ is called universally central if for any smooth variety $\tilde{X}$ and any morphism $f\colon \tilde{X} \to X$ the element $f^A(a)$ is central in $A(\tilde{X})$.

We will write below in the text $A^{uc}(X)$ for the subring of $A(X)$ consisting of all universally central elements and we set $\bar{A}^{uc} := A^{uc}(pt)$.

Remark 2.16. Note, that if the theory A takes values in the category of $\mathbb{Z}/2$-graded Abelian groups and grade-preserving homomorphisms, and is moreover a $\mathbb{Z}/2$-graded-commutative ring theory, i.e. for any $a \in A^p(P)$ and $b \in A^q(Q)$ one has the relation $a\times b = (-1)^{pq} b\times a$, then each even degree element is a universally central element.

One should remark as well that in the graded commutative case the second partial Leibnitz rule holds (if we assume that for every pair (X, U) the operator $\partial_{X,U}$ is a graded operator of degree $+1$). Namely, if $a \in A^p(U)$ and $b \in A^q(Y)$ and U is open in a smooth X, then the relation $\partial_{Y\times X,Y\times U}(b\times a) = (-1)^q b\times\partial_{X,U}(a)$ in $A(Y \times X, Y \times U)$.

If A is a ring cohomology theory, then for each pair $(X, U) \in \mathcal{S}m\mathcal{O}p$ the localization sequences from Section 2.2.3 are sequences of the $A(X)$-modules (partial Leibnitz rule). By the same reason for each open covering $X = U_1 \cup U_2$ the Mayer–Vietoris sequence from Section 2.2.4 is a sequence of the $A(X)$-modules. Thus the following two propositions hold.

PROPOSITION 2.17. *Let $f\colon (X, U) \to (X', U')$ be a morphism of pairs, let $\alpha \in A(X)$ and let $\alpha|_U = \alpha|_U \in A(U)$. Denote the composition operator* $(\cup\alpha \circ f^A\colon A(X') \to A(X))$ (*respectively* $(\cup\alpha_U \circ f^A\colon A(U') \to A(U))$ *and* $(\cup\alpha \circ f^A\colon A(X', U') \to A(X, U)))$ *by* $\cup\alpha$ (*respectively* $\cup\alpha_U$, *and* $\cup\alpha$). *Then these operators form a morphism of the localization sequences for the pairs* (X', U') *and* (X, U), *that is the diagram commutes*

$$\begin{array}{ccccccccc}
A(X) & \longrightarrow & A(U) & \xrightarrow{\partial_{X,U}} & A(X,U) & \longrightarrow & A(X) & \longrightarrow & A(U) \\
{\scriptstyle\cup\alpha}\uparrow & & {\scriptstyle\cup\alpha_U}\uparrow & & {\scriptstyle\cup\alpha}\uparrow & & {\scriptstyle\cup\alpha}\uparrow & & {\scriptstyle\cup\alpha_U}\uparrow \\
A(X') & \longrightarrow & A(U') & \xrightarrow{\partial_{X',U'}} & A(X',U') & \longrightarrow & A(X' & \longrightarrow & A(U')).
\end{array}$$

PROPOSITION 2.18. *Let $X = U_1 \cup U_2$ and let $X' = U_1' \cup U_2'$ be open coverings. Let $f\colon X \to X'$ be a morphism such that $f(U_i) \subset U_i'$ for $i = 1, 2$. Let $\alpha \in A(X)$ be* *an element, let* $\alpha_i = \alpha|_{U_i} \in A(U_i)$ *and let* $\alpha_{12} = \alpha|_{U_{12}} \in A(U_{12})$. *Denote the composition operator* $(\cup\alpha \circ f^A\colon A(X') \to A(X))$ (*respectively* $(\cup\alpha_i \circ f^A\colon A(U_i') \to A(U_i))$ *and* $(\cup\alpha_{12} \circ f^A\colon A(U_{12}') \to A(U_{12})))$ *by* $\cup\alpha$ (*respectively* $\cup\alpha_i$ *and* $\cup\alpha_{12}$). *Then these operators form a morphism of the Mayer–Vietoris sequences corresponding to the coverings* $X' = U_1' \cup U_2'$ *and* $X = U_1 \cup U_2$, *that is the diagram commutes*

$$\begin{array}{ccccccccc}
A(U_1)\oplus A(U_2) & \longrightarrow & A(U_{12}) & \xrightarrow{\partial} & A(X) & \longrightarrow & A(U_1)\oplus A(U_2) & \longrightarrow & A(U_12) \\
{\scriptstyle(\cup\alpha_1,\cup\alpha_2)}\uparrow & & {\scriptstyle\cup\alpha_{12}}\uparrow & & {\scriptstyle\cup\alpha}\uparrow & & {\scriptstyle(\cup\alpha_1,\cup\alpha_2)}\uparrow & & {\scriptstyle\cup\alpha_{12}}\uparrow \\
A(U_1')\oplus A(U_2') & \longrightarrow & A(U_{12}') & \xrightarrow{\partial} & A(X') & \longrightarrow & A(U_1')\oplus A(U_2') & \longrightarrow & A(U_{12}')).
\end{array}$$

The definition of a ring cohomology theory is equivalent to the following more technical but pretty useful one.

DEFINITION 2.19. A ring cohomology theory is a weak morphism

$$(A, \mu, e)\colon (\mathcal{S}m\mathcal{O}p, \times, pt) \to (\mathcal{A}b, \otimes, \mathbb{Z})$$

of the monoidal categories together with a functor transformation ∂ such that the pair (A, ∂) is a cohomology theory (Definition 2.1) and the boundary operator ∂ satisfies the partial Leibnitz rule saying that $\partial_{P\times Y}(\mu_{U,Y}(a \otimes b)) = \mu_{P,Y}(\partial_P(a) \otimes b) \in A(X \times Y, U \times Y)$.

(Under this variant of the notation the cross-product $c \times d \in A(P \times Q)$ of elements $c \in A(P)$ and $d \in A(Q)$ is the element $\mu_{P,Q}(a \otimes b) \in A(P \times Q)$).

One could replace in this form of the definition the monoidal category $(\mathcal{A}b, \otimes, \mathbb{Z})$ by any other Abelian monoidal category $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}})$ reformulating the partial Leibnitz rule as follows: for every pair $P \in \mathcal{S}m\mathcal{O}p$ and a smooth variety Y the relation holds

$$\partial_{P\times Y} \circ \mu_{U,Y} \circ (\mathrm{id}_U \otimes_{\mathcal{C}} \mathrm{id}_Y) = \mu_{P,Y} \circ (\partial_P \otimes_{\mathcal{C}} \mathrm{id}_{A(Y)}).$$

Once again we left such a replacement to a reader to avoid technicalities as much as it is possible.

2.5. EXAMPLES

Consider following examples.

2.5.1. *Etale Cohomology*

Let $A_Z^*(X) = \oplus_{q=-\infty}^{+\infty} H_Z^*(X, \mu_m^{\otimes q})$ be the etale cohomology theory, where m is an integer prime to $\mathrm{char}(k)$. The cup-products are described in [17, Chapter V, Section 1, 1.17].

2.5.2. *K-theory*

Let A be the algebraic K-theory from Section 2.1.8. So $A(X, U) = \oplus_{n=0}^{\infty} K_{-n}(X on Z)$, where $Z = X - U$. The idea of the definition of the products is given in [30].

2.5.3. *Motivic Cohomology*

Let $A_Z(X) = \oplus_{q=0}^{\infty} \oplus_{p=-\infty}^{\infty} H^p_{\mathcal{M},Z}(X, \mathbb{Z}_{\mathrm{tr}}(q))$ be the motivic cohomology [28]. Under the notation of Section 2.1.9 $A_Z(X)$ coincides with $\oplus_{q=0}^{\infty} A_Z^{\mathbb{Z}_{\mathrm{tr}}(q)}(X)$. Take $\partial = \oplus_{q=0}^{\infty} \partial^{\mathbb{Z}_{\mathrm{tr}}(q)}$. The products are defined in [28, the text just below Lemma 3.3] and are induced by the canonical pairings $\mathbb{Z}_{\mathrm{tr}}(r) \otimes_{\mathrm{tr}} \mathbb{Z}_{\mathrm{tr}}(s) \to \mathbb{Z}_{\mathrm{tr}}(r+s)$. The products are associative and graded commutative, the unit 1 of this product is the element $1 \in H^0_{\mathcal{M}}(pt, \mathbb{Z}_{\mathrm{tr}}(0)) = \mathbb{Z}$ [28].

2.5.4. *Semi-topological Complex and Real K-theories [6]*

If the ground field k is the field $\mathbb{R}$ of reals then the semi-topological K-theory of real algebraic varieties $K\mathbb{R}^{\mathrm{semi}}$ defined in [6] is a ring cohomology theory as it is proved in [6].

2.5.5. *Algebraic Cobordism Theory*

To introduce a ring structure on the algebraic cobordism theory (Section 2.1.11) it would be convenient to enrich MGL with a symmetric ring structure [13, Section 4]. For that we construct another T-spectrum $\mathbb{MGL}$ which is a commutative symmetric ring spectrum by the very construction and which is weekly equivalent to MGL as the T-spectrum. The desired T-spectra $\mathbb{MGL}$ is described in [23, 2.5.5]. A ring structure on the algebraic cobordism theory was introduced as well in [10].

2.5.6. *Singular Cohomology of the Real Point Sets*

Let $k = \mathbb{R}$ and let $A = A^{\mathrm{ev}} \oplus A^{\mathrm{odd}}$ with $A^{\mathrm{ev}}(X, U) = \oplus_0^{\infty} H^p((X(\mathbb{R}), U(\mathbb{R}); \mathbb{Z}/2)$ and $A^{\mathrm{odd}}(X, U) = 0$ (see Section 2.1.3). Take as a boundary ∂ the usual boundary map for the pair $(X(\mathbb{R}), U(\mathbb{R}))$. Clearly ∂ is grade-preserving with respect to the grading we choose on A. Now the cup product makes A a $\mathbb{Z}/2$-graded-commutative ring theory.

3. Orientations

In this section A is a ring cohomology theory. We introduce three following structures which A can be endowed with: an orientation, a Chern structure and a Thom structure. We show that there is a natural one-to-one correspondence between these structures (see Theorems 3.5, 3.35 and 3.36).

3.1. ORIENTATIONS ON A RING COHOMOLOGY THEORY

Let us recall that for a vector bundle E over a variety X we identify X with $z(X)$, where $z\colon X \to E$ is the zero section.

DEFINITION 3.1. An orientation on the theory A is a rule assigning to each smooth variety X, to each its closed subset Z and to each vector bundle E/X an operator

$$\mathrm{th}^E_Z\colon A_Z(X) \to A_Z(E)$$

which is a two-sided $A(X)$-module isomorphism and satisfies the following properties

1. invariance: for each vector bundle isomorphism $\varphi\colon E \to F$ the diagram commutes
$$\begin{array}{ccc} A_Z(X) & \xrightarrow{\mathrm{th}^E_Z} & A_Z(F) \\ \mathrm{id}\downarrow & & \downarrow\varphi^A \\ A_Z(X) & \xrightarrow{\mathrm{th}^E_Z} & A_Z(E) \end{array}$$
2. base change: for each morphism $f\colon (X', X'-Z') \to (X, X-Z)$ with closed subsets $Z \hookrightarrow X$ and $Z' \hookrightarrow X'$ and for each vector bundle E/X and for its pull-back E' over X' and for the projection $g\colon E' = E \times_X X' \to E$ the diagram commutes
$$\begin{array}{ccc} A_Z(X) & \xrightarrow{\mathrm{th}^E_Z} & A_Z(E) \\ f^A\downarrow & & \downarrow g^A \\ A_{Z'}(X') & \xrightarrow{\mathrm{th}^{E'}_{Z'}} & A_{Z'}(E') \end{array}$$
3. for each vector bundles $p\colon E \to X$ and $q\colon F \to X$ the following diagram commutes
$$\begin{array}{ccc} A_Z(X) & \xrightarrow{\mathrm{th}^E_Z} & A_Z(E) \\ \mathrm{th}^F_Z\downarrow & & \downarrow\mathrm{th}^{p^*(F)}_Z \\ A_Z(F) & \xrightarrow{\mathrm{th}^{q^*(E)}_Z} & A_Z(E \oplus F) \end{array}$$
and both compositions coincide with the operator $\mathrm{th}^{E\oplus F}_Z$.

The operators th^E_Z are called Thom isomorphisms. The theory A is called orientable if there exists an orientation of A. The theory A is called oriented if an orientation is chosen and fixed.

Next we are going to describe a number of data which allow to orient A.

3.2. CHERN AND THOM STRUCTURES ON A

In this section A is a ring cohomology theory. If X is a smooth variety we write $\mathbf{1}_X$ for the trivial rank 1 bundle over X. Often we will just write $\mathbf{1}$ for $\mathbf{1}_X$ if it is clear from a context what the variety X is.

DEFINITION 3.2. A Chern structure on A is an assignment $L \mapsto c(L)$ which associate to each smooth X and each line bundle L/X a universally central element $c(L) \in A(X)$ satisfying the following properties

1. functoriality:
   $c(L_1) = c(L_2)$ for isomorphic line bundles L_1 and L_2;
   $f^A(c(L)) = c(f^*(L))$ for each morphism $f\colon Y \to X$;
2. nondegeneracy: the operator $(1, \xi)\colon A(X) \oplus A(X) \to A(X \times \mathbf{P}^1)$ is an isomorphism where $\xi = c(\mathcal{O}(-1))$ and $\mathcal{O}(-1)$ is the tautological line bundle on $\mathbf{P}^1$;
3. vanishing: $c(\mathbf{1}_X) = 0 \in A(X)$ for any smooth variety X.

The element $c(L) \in A(X)$ is called Chern class of the line bundle L. (It will be proved below in Lemma 3.29 that the elements $c(L)$ are nilpotent).

Let E be a vector bundle over a smooth X and $m \in A_X(E)$ be an element. We will say below in the text that m is $A(X)$-*central*, if for any element $a \in A(X)$ one has the relations $m \cup a = a \cup m$ in $A_X(E)$ (we consider elements of $A(X)$ as elements of $A(E)$ by means of the pull-back operator induced by the projection $E \to X$). We will say that m is *universally* $A(X)$-*central* if for any morphism $f\colon X' \to X$ and the vector bundle $E' = X' \times_X E$ and its projection $F\colon E' \to E$ the element $F^A(m) \in A_{X'}(E')$ is $A(X')$-central.

Observe that for a universally $A(X)$-central element $m \in A_X(E)$ the element $z^A(i^A(m))$ in $A(X)$ is universally central in the sense of Definition 2.15 (here $i^A\colon A_X(E) \to A(E)$ is the support extension operator and $z\colon X \to E$ is the zero section of E).

DEFINITION 3.3. One says that A is endowed with a Thom structure if for each smooth variety X and each line bundle L/X it is chosen and fixed a universally $A(X)$-central element $\mathrm{th}(L) \in A_X(L)$ satisfying the following properties

1. functoriality: $\varphi^A(\mathrm{th}(L_2)) = \mathrm{th}(L_1)$ for each isomorphism $\varphi\colon L_1 \to L_2$ of line bundles;
   $f_L^A(\mathrm{th}(L)) = \mathrm{th}(L_Y)$ for each morphism $f\colon Y \to X$ and each line bundle L/X, where $L_Y = L \times_X Y$ is the pull-back line bundle over Y and $f_L\colon L_Y \to L$ is the projection to L;
2. nondegeneracy: the cup-product $\cup\mathrm{th}(1)\colon A(X) \to A_X(X \times \mathbf{A}^1)$ is an isomorphism (here X is identified with $X \times \{0\}$).

The element $\mathrm{th}(L) \in A_X(L)$ is called *the Thom class* of the line bundle L. Now we are going to describe a one-to-one correspondence between Chern and Thom structures on A.

LEMMA 3.4. *Assume A is endowed with a Chern structure $L \mapsto c(L)$. Let L be a line bundle over a smooth X and let $E = \mathbf{1} \oplus L$ and let $p\colon \mathbf{P}(E) \to X$ be the projection. Identify the group $A_{\mathbf{P(1)}}(\mathbf{P}(E))$ with a subgroup of $A(\mathbf{P}(E))$ via the support extension operator $A_{\mathbf{P(1)}}(\mathbf{P}(E)) \to A(\mathbf{P}(E))$ from the sequence* (8). *Then the element $c(\mathcal{O}_E(1) \otimes p^*L) \in A(\mathbf{P}(E))$ belongs to the subgroup $A_{\mathbf{P}(1)}(\mathbf{P}(E))$ of the group $A(\mathbf{P}(E))$. Below we will often write $\bar{\mathrm{th}}(L)$ for $c(\mathcal{O}_E(1) \otimes p^*L)$.*

Proof. The projection to the base X identifies the closed subvariety $\mathbf{P}(L)$ with the variety X. The restriction of the line bundle $\mathcal{O}_E(1)$ to $\mathbf{P}(L)$ is coincides with $L^\vee$. Thus the restriction of $\mathcal{O}_E(1) \otimes p^*L$ to $\mathbf{P}(L)$ is the trivial bundle. Now if $i_L\colon \mathbf{P}(L) \hookrightarrow \mathbf{P}(E)$ is the inclusion from (8) then

$$i_L^A(c(\mathcal{O}_E(1) \otimes p^*L)) = c(i_L^*(\mathcal{O}_E(1) \otimes p^*L)) = 0.$$

The exactness of the sequence (8) completes the proof. □

Now we are ready to describe the mentioned one-to-one correspondence. Assuming that A is endowed with a Thom structure $L \mapsto \mathrm{th}(L)$ endow A with a Chern structure as follows. For a line bundle L over a smooth X set

$$c(L) = [z^A \circ i^A](\mathrm{th}(L)) \in A(X), \tag{10}$$

where $i^A\colon A_X(L) \to A(L)$ is the support extension operator (see Definition 2.1) and $z^A\colon A(L) \to A(X)$ is the operator induced by the zero section $z\colon X \to L$.

Assuming that A is endowed with a Chern structure $L \mapsto c(L)$ endow A with a Thom structure as follows. For a line bundle L over a smooth X consider the vector bundle $E = \mathbf{1} \oplus L$, the projection $p\colon \mathbf{P}(E) \to X$, the natural inclusion $e\colon L \hookrightarrow \mathbf{P}(E)$ and the pull-back $e^A\colon A_{\mathbf{P(1)}}(\mathbf{P}(E)) \to A_X(L)$. The element $\bar{\mathrm{th}}(L) = c(\mathcal{O}_E(1) \otimes p^*L) \in A(\mathbf{P}(E))$ belongs to the subgroup $A_{\mathbf{P(1)}}(\mathbf{P}(E))$ by lemma above. Now set

$$\mathrm{th}(L) = e^A(c(\mathcal{O}_E(1) \otimes p^*L)) \in A(L, L^0) = A_X(L). \tag{11}$$

THEOREM 3.5. *For any ring cohomology theory A the following assertions hold.*

1. *If A is endowed with a Thom structure $L \mapsto \mathrm{th}(L)$ then the assignment $L \mapsto c(L)$ given by* (10) *endows A with a Chern structure.*
2. *If A is endowed with a Chern structure $L \mapsto c(L)$ then the assignment $L \mapsto \mathrm{th}(L)$ given by* (11) *endows A with a Thom structure.*
3. *The constructions described in the items* 2 *and in* 1 *are inverse of each other: namely if $L \mapsto c(L)$ is a Chern structure on A and $L \mapsto \mathrm{th}(L)$ is a Thom structure on A, then the relation* (10) *holds for all line bundles if and only if the relation* (11) *holds for all line bundle.*

Let $L \mapsto c(L)$ be a Chern structure on A and let $L \mapsto \mathrm{th}(L)$ be a Thom structure on A. In the case when (11) *holds for all line bundles* (*or* (10) *holds for all line bundles which is the same*) *we say that the Chern structure and the Thom structure on A correspond to each other.*

The item 1 describes the arrow γ from Section 1. The item 2 describes a unique arrow inverse to the arrow γ.

Proof. We start with the following lemma.

LEMMA 3.6. *Let $L \mapsto c(L)$ be a Chern structure on A. Let $\mathcal{O}(-1)$ be the tautological line bundle on the projective line $\mathbf{P}^1$ and let $\mathcal{O}(1)$ be the dual bundle and let $\xi = c(\mathcal{O}(-1))$, and $\zeta = \mathcal{O}(1)$. Then $\xi^2 = 0 = \zeta^2$ and $c(\mathcal{O}(1)) = -c(\mathcal{O}(-1))$.*

Proof of lemma. Since $\mathcal{O}(-1)|_{\{0\}}$ is the trivial bundle one has $\xi|_{\{0\}} = 0$. Thus $\xi \in A_{\{0\}}(\mathbf{P}^1)$ and $\xi \in A_{\{\infty\}}(\mathbf{P}^1)$. Therefore the element $\xi^2 \in A(\mathbf{P}^1)$ is in the image of $A_{\{0\}\cap\{\infty\}}(\mathbf{P}^1) = A_{\emptyset}(\mathbf{P}^1)$. This last group vanishes by the property Section 2.2.1 and thus $\xi^2 = 0$. Similarly $\zeta^2 = 0$.

The $A(pt)$-module $A(\mathbf{P}^1 \times \mathbf{P}^1)$ is a free module with the free bases 1, $\xi \otimes 1$, $1 \otimes \xi$ and $\xi \otimes \xi$ by the property of the Chern classes. Consider an element $\alpha = c(p_1^*(\mathcal{O}(-1) \otimes p_2^*(\mathcal{O}(1))))$. Write it in the form $\alpha = a_{00} 1 \otimes 1 + a_{10} \xi \otimes 1 + a_{01} 1 \otimes \xi + a_{11} \xi \otimes \xi$, where a_{ij} are elements in $A(pt)$. Restricting the element α to $\{0\} \times \{0\}$, to $\mathbf{P}^1 \times \{0\}$ and to the diagonal $\Delta(\mathbf{P}^1)$ one gets the following relations: $a_{00} = 0$, $a_{00} + a_{10}\xi = \xi$ in $A(\mathbf{P}^1)$ and $a_{00} + (a_{10} + a_{01})\xi = 0$ in $A(\mathbf{P}^1)$. Thus $a_{10} = 1$ and $a_{10} + a_{01} = 0$. Therefore $a_{01} = -a_{10} = -1$. The chain of the relations $\zeta = \alpha_{\{0\} \times \mathbf{P}^1} = a_{00} + a_{01}\xi = -\xi$ completes the proof of lemma. □

Now we are ready to prove the assertion (2) of theorem. The functoriality of the assignment $L \mapsto \mathrm{th}(L)$ is obvious. Now to prove that the element $\mathrm{th}(L)$ is universally $A(X)$-central it suffices to prove that the element $\mathrm{th}(L)$ is $A(X)$-central. Consider the element $\bar{\mathrm{th}}(L) = c(\mathcal{O}_E(1) \otimes p^*L)) \in A_{\mathbf{P(1)}}(\mathbf{P}(E))$. This element is $A(\mathbf{P}(E))$-central because it is the Chern class. Since $\mathrm{th}(L) = e^A(\bar{\mathrm{th}}(L))$ and the pull-back map $A(\mathbf{P}(E)) \to A(L)$ is surjective the element $\mathrm{th}(L)$ is $A(X)$-central.

It remains to prove the non-degeneracy property of the element $\mathrm{th}(1)$. For that consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{X\times\{0\}}(X \times \mathbf{P}^1) & \longrightarrow & A(X \times \mathbf{P}^1) & \longrightarrow & A(X \times \mathbf{A}^1) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \cup\bar{\mathrm{th}}(1)} & & \uparrow{\scriptstyle (\cup\bar{\mathrm{th}}(1),\cup 1)} & & \uparrow{\scriptstyle pr^A} & & \\
0 & \longrightarrow & A(X) & \longrightarrow & A(X) \oplus A(X) & \longrightarrow & A(X) & \longrightarrow & 0.
\end{array}$$

The non-degeneracy property of the Chern class and the relation $\bar{\mathrm{th}}(1) = -c(\mathcal{O}(-1)) = -\xi$ show that the middle vertical arrow is an isomorphism. The right vertical arrow is an isomorphism by the homotopy invariance property. Therefore the left vertical arrow is an isomorphism as well.

Now consider the commutative diagram

$$\begin{array}{ccc} A(X) & \xrightarrow{\cup \bar{\mathrm{th}}(1)} & A_{X\times\{0\}}(X\times \mathbf{P}^1) \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle e^A} \\ A(X) & \xrightarrow{\cup \mathrm{th}(1)} & A_{X\times\{0\}}(X\times \mathbf{A}^1). \end{array}$$

The map e^A is an isomorphism by the excision property. Therefore the bottom arrow is an isomorphism as well. The non-degeneracy property of the class th(1) is proved.

To prove the assertion (1) of theorem we need in some preliminaries.

NOTATION 3.7. Let M be a line bundle over a smooth variety X and let e^A: $A_{\mathbf{P(1)}}(\mathbf{P}(\mathbf{1}\oplus M)) \to A_X(M)$ be the excision isomorphism induced by the open inclusion e: $M \hookrightarrow \mathbf{P}(\mathbf{1}\oplus M)$. For an element $\alpha \in A_X(M)$ set

$$\bar{\alpha} = (e^A)^{-1}(\alpha) \in A_{\mathbf{P(1)}}(\mathbf{P}(\mathbf{1}\oplus M)).$$

Since the support extension map $A_{\mathbf{P(1)}}(\mathbf{P}(\mathbf{1}\oplus M)) \to A(\mathbf{P}(\mathbf{1}\oplus M))$ is injective (8) we will often write $\bar{\alpha}$ for the image of this element in $A(\mathbf{P}(\mathbf{1}\oplus M))$. If $\alpha = \mathrm{th}(M)$ is the Thom class of M then we will often write $\bar{\mathrm{th}}(M)$ for the element $\bar{\alpha}$.

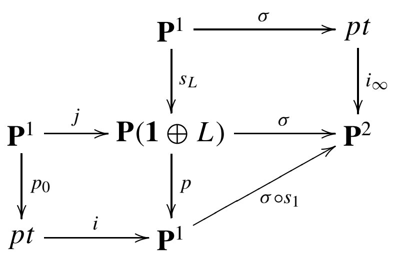
The following two observations will be useful for the proof as well

- if φ: $X_1 \to X$ is a morphism of smooth varieties and $M_1 = \varphi^*(M)$ is the line bundle over X_1 and Φ: $\mathbf{P}_{X_1}(\mathbf{1}\oplus M_1) \to \mathbf{P}_X(\mathbf{1}\oplus M) = \mathbf{P}(\mathbf{1}\oplus M)$ is the induced morphism of the projective bundles then for $\alpha_1 = \varphi^A(\alpha)$ one has the relation $\bar{\alpha}_1 = \Phi^A(\bar{\alpha})$.
- if s: $X \to \mathbf{P}(\mathbf{1}\oplus M)$ is the section identifying X with $\mathbf{P}(\mathbf{1})$ then one has $s^A(\bar{\alpha}) = z^A(i^A(\alpha))$, where z is the zero section of M and i^A: $A_X(M) \to A(M)$ is the support extension operator.

Now under Notation 3.7 one has the following lemma.

LEMMA 3.8. *Let an assignment* $L \mapsto \mathrm{th}(L)$ *be the Thom structure on* A. *Let* $L \mapsto c(L)$ *be the assignment given by the formula* (10). *Then for the line bundle* $\mathcal{O}(1)$ *on* $\mathbf{P}^1$ *one has the relation* $\bar{\mathrm{th}}(1) = c(\mathcal{O}(1))$ *in* $A(\mathbf{P}^1)$.

Proof of lemma. Let $\infty \in \mathbf{P}^2$ be a rational point and let σ: $\mathbf{P}' \to \mathbf{P}^2$ be the blow-up of the projective plane $\mathbf{P}^2$ at the point ∞. The linear projection $\mathbf{P}^2 - \infty \to \mathbf{P}^1$ extends canonically to a morphism p: $\mathbf{P}' \to \mathbf{P}^1$. Using this morphism the variety $\mathbf{P}'$ is naturally identified with the projective bundle $\mathbf{P}(\mathbf{1}\oplus L)$ over the projective line $\mathbf{P}^1$, where $L = \mathcal{O}(1)$. Under this identification the preimage $\sigma^{-1}(\infty)$ of the point ∞ coincides with the subvariety $\mathbf{P}(L)$ of the projective bundle $\mathbf{P}(\mathbf{1}\oplus L)$. The subvariety $\mathbf{P}(\mathbf{1}) \subset \mathbf{P}(\mathbf{1}\oplus L)$ is the image of a section s_1: $\mathbf{P}^1 \to \mathbf{P}(\mathbf{1}\oplus L)$ of the projection p. The image of $\mathbf{P}^1$ under the composite map $\sigma \circ s_1$ is a projective line l in $\mathbf{P}^2$ which avoids the point ∞. Let $x \in \mathbf{P}^1$ be a rational point and let $j\colon \mathbf{P}^1 = p^{-1}(x) \hookrightarrow \mathbf{P}(\mathbf{1} \oplus L)$ be the imbedding of the fiber into the total space. One can summarize these data in the following diagram



.

Now set $\alpha = \mathrm{th}(L) \in A_X(L)$, then $\bar{\alpha} = \bar{\mathrm{th}}(L) \in A_{\mathbf{P(1)}}(\mathbf{P}(\mathbf{1} \oplus L))$. In the commutative diagram of the pull-backs

$$\begin{array}{ccc} A_l(\mathbf{P}^2) & \xrightarrow{\sigma^A = u} & A_{\mathbf{P(1)}}(\mathbf{P}(\mathbf{1} \oplus L)) \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ A_l(\mathbf{P}^2 - \infty) & \xrightarrow{t} & A_{\mathbf{P(1)}}(\mathbf{P}(\mathbf{1} \oplus L) - \mathbf{P}(L)), \end{array}$$

the maps w, t and v are isomorphisms. In fact, w and v are isomorphisms by the excision property and t is isomorphism because the map σ identifies $\mathbf{P}(\mathbf{1} \oplus L) - \mathbf{P}(L)$ with $\mathbf{P}^2 - \infty$. Therefore the fourth arrow $u = \sigma^A$ is an isomorphism as well.

Therefore there exists an element $\beta \in A_l(\mathbf{P}^2)$ such that $\sigma^A(\beta) = \bar{\alpha}$. The mappings $\sigma \circ j, \sigma \circ s_1\colon \mathbf{P}^1 \to \mathbf{P}^2$ are two linear imbeddings of the projective line into $\mathbf{P}^2$. Therefore by the property (Section 2.10) one has the relation $(\sigma \circ j)^A(\beta) = (\sigma \circ s_1)^A(\beta)$ in $A(\mathbf{P}^1)$. Thus one gets the chain of relations in $A(\mathbf{P}^1)$

$$j^A(\bar{\alpha}) = (\sigma \circ j)^A(\beta) = (\sigma \circ s_1)^A(\beta) = s_1^A(\bar{\alpha}).$$

By the two observations mentioned just below Notation 3.7 one gets the relations $j^A(\bar{\alpha}) = \bar{\mathrm{th}}(1)$ and $s_1^A(\bar{\alpha}) = c(L) = c(\mathcal{O}(1))$. Thus $\bar{\mathrm{th}}(1) = c(\mathcal{O}(1))$ and lemma is proved. □

Now we are ready to prove assertion (1) of theorem. The functoriality of the class $L \mapsto c(L)$ given by the formula (10) is obvious. To prove that for the trivial line bundle $\mathbf{1}$ over a smooth variety X one has $c(1) = 0$ consider a section $s\colon X \to X \times \mathbf{A}^1$ of the trivial bundle 1 which takes a point $x \in X$ to the point $(x, 1)$. If z is the zero section of the same bundle then the pull-back mappings z^A and s^A coincides. In fact both are the inverse to the pull-back map $p^A\colon A(X) \to A(X \times \mathbf{A}^1)$ induced by the projection $p\colon X \times \mathbf{A}^1 \to X$. If $i^A\colon A_{X\times 0}(X \times \mathbf{A}^1) \to A(X \times \mathbf{A}^1)$ is the support extension map, then $c(1) = (z^A \circ i^A)(1) = (s^A \circ i^A)(1)$ and it remains to show that $s^A \circ i^A = 0$.

For that consider a commutative diagram

$$\begin{array}{ccc} A_{X\times\{0\}}(X\times\mathbf{A}^1) & \xrightarrow{i^A} & A(X\times\mathbf{A}^1) \\ {\scriptstyle j^A}\downarrow & & \downarrow{\scriptstyle s^A} \\ A_{\emptyset}(X\times(\mathbf{A}^1-\{0\})) & \longrightarrow & A(X), \end{array}$$

where the pull-back map j^A is induced by the inclusion $j\colon X\times(\mathbf{A}^1-\{0\})\hookrightarrow X\times\mathbf{A}^1$ and the bottom horizontal arrow is the pull-back induced by the inclusion $s\colon (X,\emptyset)\to(X\times(\mathbf{A}^1-\{0\}), X\times(\mathbf{A}^1-\{0\}))$. The group $A_{\emptyset}(X\times(\mathbf{A}^1-\{0\})$ vanishes by the vanishing property (Section 2.2.1). Thus $s^A\circ i^A=0$ which proves the relation $c(1)=0$.

It remains to prove the non-degeneracy property of the class $L\mapsto c(L)$. For that consider the assignment $L\mapsto c'(L)=c(L^{\vee})$. Clearly the class c' is functorial and satisfies the vanishing property. Moreover the map $(1,c'(\mathcal{O}(-1)))\colon A(X)\oplus A(X)\to A(X\times\mathbf{P}^1)$ is an isomorphism by the non-degeneracy property of the Thom class $L\mapsto \mathrm{th}(L)$ and the very last lemma. Thus the assignment $L\mapsto c'(L)$ is a Chern structure. Now the previous lemma shows that $c'(\mathcal{O}(-1))=-c'(\mathcal{O}(1))$. Thus $c(\mathcal{O}(1))=-c(\mathcal{O}(-1))$ and therefore the map $(1,c(\mathcal{O}(-1)))\colon A(X)\oplus A(X)\to A(X\times\mathbf{P}^1)$ is an isomorphism as well. The non-degeneracy property of the class $L\mapsto c(L)$ is proved and hence the assertion (1) of theorem is proved as well.

The third assertion of the theorem is proved just after Section 3.3 because the proof of the third assertion presented in this text uses Theorem 3.9. □

3.3. PROJECTIVE BUNDLE THEOREM

We are going to construct higher Chern classes for a ring cohomology theory A endowed with a Chern structure $L\mapsto c(L)$. Following the known Grothendieck's method one has to compute cohomology of a projective bundle.

THEOREM 3.9 (**Projective bundle cohomology**). *Let A be a ring cohomology theory endowed with a Chern structure $L\mapsto c(L)$ on A. Let X be a smooth variety and let E/X be a vector bundle with $rkE=n$. For $\xi_E=c(\mathcal{O}_E(-1))\in A(\mathbf{P}(E))$ we have an isomorphism*

$$(1,\xi_E,\dots,\xi_E^{n-1})\colon A(X)\oplus A(X)\cdots\oplus A(X)\to A(\mathbf{P}(E)),$$

where (and elsewhere) we denote the operator of $\cup$-product with a universally central element by the symbol of the element.

Moreover, for trivial E we have $\xi_E^n=0$. In addition, all the assertions hold if the element $\zeta_E=c(\mathcal{O}_E(1))\in A(\mathbf{P}(E))$ is used instead of ξ_E.

Proof. This variant of the proof is based on an oral exposition of Suslin. Let $\{0\}=[1:0:\cdots:0]\in\mathbf{P}^n$ be a point and let $\mathbf{A}^n$ be an affine subspace in $\mathbf{P}^n$ defined by the inequality $x_0\neq 0$. Let $\mathbf{P}^n_i$ be a hypersurface in $\mathbf{P}^n$ defined by $x_i=0$

and let $\mathbf{A}^n_i = \mathbf{P}^n_i \cap \mathbf{A}^n$. Let $p_i\colon \mathbf{A}^n \to \mathbf{A}^1$ be the projection on the ith axis and let $j_i\colon \mathbf{A}^1 \hookrightarrow \mathbf{A}^n$ be the ith axis. Finally let $\bar{j}_i\colon \mathbf{P}^1 \hookrightarrow \mathbf{P}^n$ be the closed imbedding extending the imbedding j_i.

Let $\mathrm{res}_i\colon A_{\mathbf{P}^n_i}(\mathbf{P}^n) \to A_{\mathbf{A}^n_i}(\mathbf{A}^n)$ be the pull-back map induced by the imbedding $\mathbf{A}^n \hookrightarrow \mathbf{P}^n$. Let res: $A_{\{0\}}(\mathbf{A}^n) \to A_{\{0\}}(\mathbf{P}^n)$ be the pull-back map induced by the same imbedding.

The element $\xi = c(\mathcal{O}(-1)) \in A(\mathbf{P}^1)$ vanishes being restricted to $\mathbf{P}^1 - \{0\}$. Thus the element $\bar{t} = \xi$ belongs to the subgroup $A_{\{0\}}(\mathbf{P}^1)$ of the group $A(\mathbf{P}^1)$. Set

$$t = j^A(\bar{t}) \in A_{\{0\}}(\mathbf{A}^1),$$

where j^A is the pull-back map $A_{\{0\}}(\mathbf{P}^1) \to A_{\{0\}}(\mathbf{A}^1)$. Set

$$\mathrm{th}_{(n)} = p_1^A(t) \cup p_2^A(t) \cup \cdots \cup p_1^A(t) \in A_{\{0\}}(\mathbf{A}^n).$$

Let $e_i\colon (\mathbf{P}^n, \emptyset) \hookrightarrow (\mathbf{P}^n, \mathbf{P}^n - \mathbf{P}^n_i)$ and let $e\colon (\mathbf{P}^n, \emptyset) \hookrightarrow (\mathbf{P}^n, \mathbf{P}^n - \emptyset)$ be the inclusions. The pull-back operators $e_i^A\colon A_{\mathbf{P}^n_i}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ and $e^A\colon A_{\{0\}}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ are just the support extension operators.

LEMMA 3.10. *Let Y be a smooth variety and let $Z \subset Y$ be a closed subset. Let* pr: $Y \times \mathbf{A}^1 \to \mathbf{A}^1$ *and p: $Y \times \mathbf{A}^1 \to Y$ be the projections. Then the composition operator*

$$(\cup\, \mathrm{pr}^A(t)) \circ p^A\colon A_Z(Y) \to A_{Z\times\{0\}}(Y \times \mathbf{A}^1)$$

is an isomorphism.

LEMMA 3.11. *The map* $\cup\, \mathrm{th}_{(n)}\colon A(pt) \to A_{\{0\}}(\mathbf{A}^n)$ *is an isomorphism.*

LEMMA 3.12. *Let $\xi_n = c(\mathcal{O}(-1)) \in A(\mathbf{P}^n)$ and $\zeta_n = c(\mathcal{O}(1)) \in A(\mathbf{P}^n)$. Then $\xi_n^{n+1} = 0$ and $\zeta_n^{n+1} = 0$.*

LEMMA 3.13. *The support extension map $A_{\mathbf{P}^n_i}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ is injective, the element $\xi_n = c(\mathcal{O}(-1))$ coincides with $e_i^A(\bar{t}_i)$ for an appropriative element $\bar{t}_i \in A_{\mathbf{P}^n_i}(\mathbf{P}^n)$ and the relation* $\mathrm{res}_i(\bar{t}_i) = p_i^A(t)$ *holds in $A_{\mathbf{A}^n_i}(\mathbf{A}^n)$.*

LEMMA 3.14. *The element $\xi^n \in A(\mathbf{P}^n)$ coincides with $e^A(\overline{\mathrm{th}}_{(n)})$ for an appropriative element* $\overline{\mathrm{th}}_{(n)} \in A_{\{0\}}(\mathbf{P}^n)$.

If the support extension operator $e^A\colon A_{\{0\}}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ is injective, then the relation $\mathrm{th}_{(n)} = \mathrm{res}(\overline{\mathrm{th}}_{(n)})$ *holds in $A_{\{0\}}(\mathbf{A}^n)$.*

Remark 3.15. The linear projection $\mathbf{P}^n - \{0\} \to \mathbf{P}^n_0 \to \mathbf{P}^n - \{0\}$ makes $\mathbf{P}^n - \{0\}$ a line bundle over $\mathbf{P}^n_0$ and the subvariety $\mathbf{P}^n_0$ is the zero section of this line bundle. By the homotopy invariance property the pull-back operator $A(\mathbf{P}^n - \{0\}) \to A(\mathbf{P}^n_0)$ is an isomorphism.

Given these five lemmas complete the proof of theorem as follows. The general case is reduced to the case of the trivial vector bundle E via the Mayer–Vietoris arguments using Proposition 2.18. If $E = \mathbf{1}^{k+1}$ then $\mathbf{P}(E) = X \times \mathbf{P}^k$. To use shorter notation we prove the theorem only for the case of the projective space $\mathbf{P}^k$ itself. We proceed the proof by the induction on the integer k. If $k = 1$, then the theorem holds by the very definition of the Chern structure on A. We will assume below that theorem holds for all integers $k < n$ and prove theorem for $k = n$. Consider the localization sequence for the pair $(\mathbf{P}^n, \mathbf{P}^n - \{0\})$

$$\cdots \to A_{\{0\}}(\mathbf{P}^n) \xrightarrow{\alpha} A(\mathbf{P}^n) \xrightarrow{\beta} A(\mathbf{P}^n - \{0\}) \to \cdots.$$

If $\xi_i \in A(\mathbf{P}^i)$ is the Chern class of the line bundle $\mathcal{O}(-1)$ on $\mathbf{P}^i$, then $\xi_n|_{\mathbf{P}^{n-1}} = \xi_{n-1}$ and $\xi_n^j|_{\mathbf{P}^{n-1}} = \xi_{n-1}^j$. By the inductive assumption the elements $1, \xi_{n-1}, \ldots, \xi_{n-1}^{n-1}$ form a free base of the $A(pt)$-module $A(\mathbf{P}^{n-1})$. Therefore the map $A(\mathbf{P}^n) \to A(\mathbf{P}_0^n)$ is a split surjection.

By Remark 3.15 the pull-back operator $A(\mathbf{P}^n - \{0\}) \to A(\mathbf{P}_0^n)$ is an isomorphism. Thus the pull-back operator $A(\mathbf{P}^n) \to A(\mathbf{P}^n - \{0\})$ is a split surjection as well. Now the localization sequence for the pair $(\mathbf{P}^n, \mathbf{P}^n - \{0\})$ shows that the support extension map $A_{\{0\}}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ is an injection. Therefore one gets a short exact sequence

$$0 \to A_{\{0\}}(\mathbf{P}^n) \xrightarrow{\alpha} A(\mathbf{P}^n) \xrightarrow{\beta'} A(\mathbf{P}_0^n) \to 0,$$

where β' is the pull-back map. One more consequence of the injectivity of the support extension operator $A_{\{0\}}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ is the relation

$$p_1^A(t) \cup p_2^A(t) \cup \cdots \cup p_1^A(t) = \mathrm{res}(\bar{\mathrm{th}}_n)$$

in $A_{\{0\}}(\mathbf{A}^n)$ which now holds by Lemma 3.14. An $A(pt)$-linear map $s\colon A(\mathbf{P}_0^n) \to A(\mathbf{P}^n)$ taking the element ξ_{n-1}^j to ξ_n^j $(j = 0, 1, \ldots, n-1)$ splits the surjection β'. The element $\xi_n^n \in A(\mathbf{P}^n)$ belongs to the subgroup $A_{\{0\}}(\mathbf{P}^n)$ because $\xi_{n-1}^n = 0$ in $A(\mathbf{P}^{n-1})$ by Lemma 3.12. It remains to show that the map $\cup\xi^n\colon A(pt) \to A_{\{0\}}(\mathbf{P}^n)$ is an isomorphism.

For that consider the diagram

$$\begin{array}{ccc}
A(pt) & \xrightarrow{\cup\bar{\mathrm{th}}_{(n)}} & A_{\{0\}}(\mathbf{P}^n) \\
\downarrow \scriptstyle{\mathrm{id}} & & \downarrow \scriptstyle{\mathrm{res}} \\
A(pt) & \xrightarrow{\cup\mathrm{th}_{(n)}} & A_{\{0\}}(\mathbf{A}^n).
\end{array}$$

It commutes by Lemma 3.14. The operator res is an isomorphism by the excision property. The operator $\cup\mathrm{th}_{(n)}$ is an isomorphism by Lemma 3.11. Thus the operator $\cup\bar{\mathrm{th}}_{(n)}$ is an isomorphism as well. To prove theorem it remains to prove Lemmas 3.10–3.14.

Remark 3.16. In certain texts proofs of the projective bundle theorem for some specific cohomology theories contain the following gap. It is verified that there is an isomorphism $A(pt) \cong A_{\{0\}}(\mathbf{P}^n)$, and it is missed to check that specifically the operator $\cup\xi^n\colon A(pt) \to A_{\{0\}}(\mathbf{P}^n)$ is an isomorphism.

Proof of Lemma 3.10. Let $\bar{pr}\colon Y \times \mathbf{P}^1 \to \mathbf{P}^1$ and $\bar{p}\colon Y \times \mathbf{P}^1 \to Y$ be the projections. We will write for short $\cup t$ for the operator $(\cup \mathrm{pr}^A(t)) \circ p^A$ and will write in this proof $\cup\xi$ for the operator $\cup\bar{\mathrm{pr}}^A(\xi) \circ \bar{p}^a\colon A_Z(Y) \to A_{Z\times\mathbf{P}^1}(Y \times \mathbf{P}^1)$ and write $\cup 1$ for the operator $\bar{p}^A\colon A_Z(Y) \to A_{Z\times\mathbf{P}^1}(Y \times \mathbf{P}^1)$. We begin with verifying that the operator

$$(\cup 1, \cup\xi)\colon A_Z(Y) \oplus A_Z(Y) \to A_{Z\times\mathbf{P}^1}(Y \times \mathbf{P}^1) \tag{12}$$

is an isomorphism. In fact, by Proposition 2.17 the diagram commutes (here U = Y−Z)

$$\begin{array}{ccccccc}
A(Y\times\mathbf{P}^1) & \longrightarrow & A(U\times\mathbf{P}^1) & \xrightarrow{\partial} & A_{Z\times\mathbf{P}^1}(Y\times\mathbf{P}^1) & \longrightarrow & A(Y\times\mathbf{P}^1)\\
{\scriptstyle(\cup1,\cup\xi)}\uparrow & & {\scriptstyle(\cup1,\cup\xi)}\uparrow & & {\scriptstyle(\cup1,\cup\xi)}\uparrow & & {\scriptstyle(\cup1,\cup\xi)}\uparrow\\
A(Y)\oplus A(Y) & \longrightarrow & A(U)\oplus A(U) & \xrightarrow{\partial\oplus\partial} & A_Z(Y)\oplus A_Z(Y) & \longrightarrow & A(Y)\oplus A(Y).
\end{array}$$

Since $\xi = c(\mathcal{O}(-1)) \in A(\mathbf{P}^1)$ the five-lemma proves that the operator (12) is an isomorphism.

The next step is to check that the operator

$$\cup\xi\colon A_Z(Y) \to A_{Z\times\{0\}}(Y \times \mathbf{P}^1) \tag{13}$$

is an isomorphism. For that consider the localization sequence

$$\cdots \to A_{Z\times\{0\}}(Y \times \mathbf{P}^1) \xrightarrow{\alpha} A_{Z\times\mathbf{P}^1}(Y \times \mathbf{P}^1) \xrightarrow{\beta} A_{Z\times\mathbf{A}^1}(Y \times \mathbf{P}^1 - \\ - Z \times \{0\}) \to \cdots$$

for the triple $(Y\times\mathbf{P}^1, Y\times\mathbf{P}^1 - Z\times\{0\}, Y\times\mathbf{P}^1 - Z\times\mathbf{P}^1)$. We claim that the operator β is always surjective (and thus the operator α is always injective and therefore the localization sequence splits in short exact sequences).

In fact, if $i\colon Y\times\mathbf{A}^1 \hookrightarrow Y\times\mathbf{P}^1 - Z\times\{0\}$ is the open inclusion and $q\colon Y\times\mathbf{P}^1 - Z\times\{0\}$ is the projection then $q \circ i = p\colon Y\times\mathbf{A}^1 \to Y$ and thus $i^A \circ q^A = p^A$. The pull-back operator $i^A\colon A_{Z\times\mathbf{A}^1}(Y\times\mathbf{P}^1 - Z\times\{0\}) \to A_{Z\times\mathbf{A}^1}(Y\times\mathbf{A}^1)$ is an isomorphism by the excision property and the pull-back operator $p^A\colon A_Z(Y) \to A_{Z\times\mathbf{A}^1}(Y\times\mathbf{A}^1)$ is an isomorphism by the strong homotopy invariance property. Thus $q^A\colon A_Z(Y) \to A_{Z\times\mathbf{A}^1}(Y\times\mathbf{P}^1 - Z\times\{0\})$ is an isomorphism. This proves the surjectivity of β and the injectivity of α.

We are ready to verify that the operator (13) is an isomorphism. For that consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{Z\times\{0\}}(Y\times\mathbf{P}^1) & \xrightarrow{\alpha} & A_{Z\times\mathbf{P}^1}(Y\times\mathbf{P}^1) & \xrightarrow{\beta} & A_{Z\times\mathbf{A}^1}(Y\times\mathbf{P}^1 - Z\times\{0\}) & \longrightarrow & 0\\
 & & {\scriptstyle\cup\xi}\uparrow & & {\scriptstyle(\cup\xi,\cup1)}\uparrow & & {\scriptstyle q^A}\uparrow & & \\
0 & \longrightarrow & A_Z(Y) & \xrightarrow{\mathrm{in}} & A_Z(Y)\oplus A_Z(Y) & \xrightarrow{\mathrm{pr}} & A_Z(Y) & \longrightarrow & 0,
\end{array}$$

where the operator in is the inclusion to the first summand and the operator pr is the projection on the second summand. The diagram commutes because the cup-product is functorial. The sequence on the top is short exact and the map q^A is an isomorphism as was just checked above. The operator $(\cup\xi, \cup 1)$ is an isomorphism as was checked as well above in this proof. Thus the operator (13) is an isomorphism as well.

The proof of the lemma is completed as follows. Consider the diagram

$$\begin{array}{ccc} A_Z(Y) & \xrightarrow{\cup\xi} & A_{Z\times\{0\}}(Y\times \mathbf{P}^1) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \gamma} \\ A_Z(Y) & \xrightarrow{\cup t} & A_{Z\times\{0\}}(Y\times \mathbf{A}^1), \end{array}$$

where γ is the pull-back operator induced by the inclusion $Y\times\mathbf{A}^1 \hookrightarrow Y\times\mathbf{P}^1$. The operator γ is an isomorphism by the excision property. Since the operator is an isomorphism the operator $\cup t$ is an isomorphism as well. Lemma 3.10 is proved. □

Proof of Lemma 3.11. For every integer i let $p_{i,i}\colon \mathbf{A}^i \to \mathbf{A}^1$ be the projection of the affine space $\mathbf{A}^i$ to its last coordinate. Using the induction by n it straightforward to check that the cup-product operator $\cup\mathrm{th}_{(n)}\colon A(pt) \to A_{\{0\}}(\mathbf{A}^n)$ coincides with the composition operator

$$A(pt) \xrightarrow{\cup t} A_{\{0\}}(\mathbf{A}^1) \xrightarrow{\cup p^A_{2,2}(t)} \cdots \xrightarrow{\cup p^A_{n,n}(t)} A_{\{0\}}(\mathbf{A}^n).$$

Each arrow in this sequence of arrows is an isomorphism by Lemma 3.10. The lemma follows. □

Proof of Lemma 3.12. For every integer $i = 0, 1, \ldots, n$ one has $\xi_n|_{\mathbf{P}^n-\mathbf{P}^n_i} = 0$ because the Chern class of a trivial line bundle vanishes. Thus ξ_n belongs to the image of the support extension operator $e^A_i\colon A_{\mathbf{P}^n_i}(\mathbf{P}^n) \to A(\mathbf{P}^n)$, say $\xi_n = e^A_i(\bar{t}_i)$ for appropriative element $\bar{t}_i \in A_{\mathbf{P}^n_i}(\mathbf{P}^n)$. Now the element ξ_n^{n+1} coincides with the image of the cup-product $\bar{t}_0 \cup \bar{t}_1 \cup \cdots \cup \bar{t}_n$ under the support extension map $A_{\mathbf{P}^n_0\cap\cdots\cap\mathbf{P}^n_n}(\mathbf{P}^n) \to A(\mathbf{P}^n)$. The group $A_{\mathbf{P}^n_0\cap\cdots\cap\mathbf{P}^n_n}(\mathbf{P}^n)$ vanishes because $\cap^n_0 \mathbf{P}^n_i = \emptyset$. Thus $\xi_n^{n+1} = 0$. Similarly one gets the relation $\zeta_n^{n+1} = 0$. The lemma is proved.

Proof of Lemma 3.13. The localization sequence for the pair $(\mathbf{P}^n, \mathbf{P}^n - \mathbf{P}^n_i)$ cuts into short exact sequences

$$0 \to A_{\mathbf{P}^n_n}(\mathbf{P}^n) \to A(\mathbf{P}^n) \to A(\mathbf{P}^n - \mathbf{P}^n_i) \to 0$$

because the composite map $A(pt) \xrightarrow{p^A} A(\mathbf{P}^n) \to A(\mathbf{P}^n - \mathbf{P}^n_i)$ is an isomorphism. This proves the first assertion of the lemma.

The fact that $\xi_n = e^A_i(\bar{t}_i)$ for the element $\bar{t}_i \in A_{\mathbf{P}^n_i}(\mathbf{P}^n_i)$ is proved in the proof of Lemma 3.12.

To prove the relation $\mathrm{res}_i(\bar{t}_i) = p_i^A(t)$ consider the commutative diagram

$$\begin{array}{ccc}
A(\mathbf{P}^n) & \xrightarrow{\bar{j}_i^A} & A(\mathbf{P}^1) \\
{\scriptstyle e_i^A}\uparrow & & \uparrow{\scriptstyle i^A} \\
A_{\mathbf{P}_i^n}(\mathbf{P}^n) & \xrightarrow{\bar{j}_i^A} & A_{\{0\}}(\mathbf{P}^1) \\
{\scriptstyle \mathrm{res}_i}\downarrow & & \downarrow{\scriptstyle j^A} \\
A_{\mathbf{A}_i^n}(\mathbf{A}^n) & \xrightarrow{j_i^A} & A_{\{0\}}(\mathbf{A}^1),
\end{array}$$

where the maps e_i^A and i^A are the support extension operators. The relation $\bar{j}_i^A(\xi_n) = \xi$ in $A(\mathbf{P}^1)$ and the injectivity of the map i^A prove the relation $\bar{j}_i^A(\bar{t}_i) = \bar{t}$ in the group $A_{\{0\}}(\mathbf{P}^1)$. Since $t = j^A(\bar{t})$ in $A_{\{0\}}(\mathbf{A}^1)$ hence one gets the relation $j_i^A(\mathrm{res}_i(\bar{t}_i)) = t$ in $A_{\{0\}}(\mathbf{A}^1)$. The pull-back homomorphisms $j_i^A\colon A_{\mathbf{A}_i^n}(\mathbf{A}^n) \to A_{\{0\}}(\mathbf{A}^1)$ and $p_i^A\colon A_{\{0\}}(\mathbf{A}^1) \to A_{\mathbf{A}_i^n}(\mathbf{A}^n)$ are inverse to each other isomorphisms by the homotopy invariance property. This proves the desired relation $\mathrm{res}_i(\bar{t}_i) = p_i^A(t)$ in $A_{\mathbf{A}_i^n}(\mathbf{A}^n)$.

Proof of Lemma 3.14. Lemma 3.12 (applied to $\mathbf{P}^{n-1}$) shows that the element $\xi_n^n|_{\mathbf{P}_0^n}$ vanishes. By Remark 3.15 the pull-back operator $A(\mathbf{P}^n - \{0\}) \to A(\mathbf{P}_0^n)$ is an isomorphism. Thus the element $\xi_n^n|_{\mathbf{P}^n-\{0\}}$ vanishes as well. Therefore $\xi_n^n = e^A(\bar{\mathrm{th}}_{(n)})$ for an appropriative element $\bar{\mathrm{th}}_{(n)} \in A_{\{0\}}(\mathbf{P}^n)$. This proves the first assertion of the lemma.

To prove the last assertion of the lemma consider the commutative diagram

$$\begin{array}{ccc}
\prod_{i=1}^n A(\mathbf{P}^n) & \xrightarrow{\cup} & A(\mathbf{P}^n) \\
{\scriptstyle \prod e_i^A}\uparrow & & \uparrow{\scriptstyle e^A} \\
\prod_{i=1}^n A_{\mathbf{P}_i^n}(\mathbf{P}^n) & \xrightarrow{\cup} & A_{\{0\}}(\mathbf{P}^n) \\
{\scriptstyle \prod \mathrm{res}_i}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\
\prod_{i=1}^n A_{\mathbf{A}_i^n}(\mathbf{A}^n) & \xrightarrow{\cup} & A_{\{0\}}(\mathbf{A}^n),
\end{array}$$

where the maps e_i^A and e^A are the support extension maps and the horizontal arrows are the cup-products. The commutativity of the upper square of this diagram and the injectivity of the map e^A prove the relation $\bar{t}_1 \cup \bar{t}_2 \cup \cdots \cup \bar{t}_n = \bar{\mathrm{th}}_{(n)}$ in the group $A_{\{0\}}(\mathbf{P}^n)$. Now the commutativity of the bottom square and the relations $\mathrm{res}_i(\bar{t}_i) = p_i^A(t)$ prove the desired relation $p_1^A(t) \cup p_2^A(t) \cup \cdots \cup p_n^A(t) = \mathrm{res}(\bar{\mathrm{th}}_{(n)})$ in the group $A_{\{0\}}(\mathbf{A}^n)$. □

Now Lemmas 3.10–3.14 are proved. The proof of Theorem 3.9 is completed. □

COROLLARY 3.17 (**Projective bundle cohomology with supports**). *Under the hypotheses of Theorem* 3.9 *the map*

$$(1, \xi_E, \dots, \xi_E^{n-1}) : A_Z(X) \oplus A_Z(X) \cdots \oplus A_Z(X) \to A_{\mathbf{P}(E_Z)}(\mathbf{P}(E))$$

is an isomorphism where $E_Z = E|_Z$ is the restriction of the vector bundle E to Z.

The short exact sequence (14) written-down below is useful as well. Namely, let X be a smooth variety and let M and N be two vector bundles over X. Let $\bar{i}_M: \mathbf{P}(M) \hookrightarrow \mathbf{P}(M \oplus N)$ and $\bar{i}_N: \mathbf{P}(N) \hookrightarrow \mathbf{P}(M \oplus N)$ be the closed imbeddings induced by the direct summands M and N, respectively. Let $p: \mathbf{P}(M \oplus N) \to X$ be the projection. Let $j_M^A: A_{\mathbf{P}(M)}(\mathbf{P}(M \oplus N)) \to A(\mathbf{P}(M \oplus N))$ be the support extension operator and let $\bar{i}_N^A: A(\mathbf{P}(M \oplus N)) \to A(\mathbf{P}(N))$ be the pull-back operator.

COROLLARY 3.18. *With these notation under the hypotheses of Theorem* 3.9 *the sequence*

$$0 \to A_{\mathbf{P}(M)}(\mathbf{P}(M \oplus N)) \xrightarrow{j_M^A} A(\mathbf{P}(M \oplus N)) \xrightarrow{\bar{i}_E^A} A(\mathbf{P}(N)) \to 0 \tag{14}$$

is short exact.

To prove this consider $U = \mathbf{P}(M \oplus N) - \mathbf{P}(M)$ with the open inclusion $j: U \hookrightarrow \mathbf{P}(M \oplus N)$ and observe that U becomes a vector bundle over X by means of the linear projection $q: U \to \mathbf{P}(N)$. The obvious inclusion $i_N: \mathbf{P}(N) \hookrightarrow U$ is just the zero section of this vector bundle, $\bar{i}_N = j \circ i_N$ and the pull-back operator $i_N^A: A(U) \to A(\mathbf{P}(N))$ is an isomorphism (the inverse to the one q^A).

Now consider the pair $(\mathbf{P}(M \oplus N), U)$. By the localization property (Definition 2.1) the following sequence

$$\cdots \to A_{\mathbf{P}(M)}(\mathbf{P}(M \oplus N)) \xrightarrow{j_M^A} A(\mathbf{P}(M \oplus N)) \xrightarrow{j^A} A(U) \to \cdots$$

is exact. We claim that this sequence splits in short exact sequences with the surjective j^A and the injective j_M^A. To prove this claim observe that one has the relation

$$\xi_N = \bar{i}_N^A(\xi_{M \oplus N})$$

which holds because the restriction of the line bundle $\mathcal{O}_{M \oplus N}(-1)$ to $\mathbf{P}(N)$ is $\mathcal{O}_N(-1)$. Thus $\xi_N \in \bar{i}_N^A(A(\mathbf{P}(M \oplus N)))$ and by the projective bundle theorem (Theorem 3.9) the operator $\bar{i}_N^A: A(\mathbf{P}(M \oplus N)) \to A(\mathbf{P}(N))$ is surjective. The operator i_N^A is an isomorphism, $\bar{i}_N = j \circ i_N$ and thus $j^A: A(\mathbf{P}(M \oplus N)) \to A(U)$ is surjective and the support extension operator j_M^A is injective. Now the sequence (14) is short exact because the operator i_N^A is an isomorphism and $\bar{i}_N = j \circ i_N$. The corollary is proved.

The last corollary and Lemma 2.4 prove the following corollary.

COROLLARY 3.19. *Under the hypotheses of Theorem* 3.9 *and the notation of Lemma* 2.4 *let* $V'_t = X'_t - i_t(Y \times \mathbf{A}^1)$ *and let* $j_t\colon V'_t \hookrightarrow X'_t$ *be the open inclusion. Then*

$$\mathrm{Ker}(j_0^A) \cap \mathrm{Ker}(j_t^A) = (0).$$

In the other words the operator $(j_0^A, j_t^A)\colon A(X'_t) \to A(\mathbf{P}(\mathbf{1} \oplus N)) \oplus A(V'_t)$ *is a monomorphism.*

3.4. END OF THE PROOF OF THEOREM 3.5

The third assertion of Theorem 3.5 is proved in this section.

Assume we are given with a Chern structure $L \mapsto c(L)$ on A and let $L \mapsto \mathrm{th}(L)$ be the Thom structure given by (11). We will now check that for each line bundle L over a smooth variety X one has $z^A(i^A(\mathrm{th}(L))) = c(L)$. For that consider the commutative diagram

$$\begin{array}{ccc} A_{\mathbf{P(1)}}(\mathbf{P}(\mathbf{1} \oplus L)) & \xrightarrow{\bar{i}^A} & A(\mathbf{P}(\mathbf{1} \oplus L)) \\ \big\downarrow{\scriptstyle e^A} & & \big\downarrow{\scriptstyle e^A} \\ A_X(L) & \xrightarrow{i^A} & A(L) \\ & & \big\downarrow{\scriptstyle z^A} \\ & & A(X) \end{array}$$

The chain of relations (here $\bar{z} = e \circ z$)

$$z^A(i^A(\mathrm{th}(L))) = z^A(e^A(c(\mathcal{O}(1) \otimes p^*(L)))) = c(\bar{z}^*(\mathcal{O}(1) \otimes L)) = c(L)$$

proves the desired relation.

In the rest of the proof Notation 3.7 are used. Now suppose we are given with a Thom structure $L \mapsto \mathrm{th}(L)$ on A and let $L \mapsto c(L)$ be the Chern structure on A given by the formula (10). For a line bundle L over a smooth X consider the vector bundle $E = 1 \oplus L$, the projection $p\colon \mathbf{P}(E) \to X$, the natural inclusion $e\colon L \hookrightarrow \mathbf{P}(E)$ and the pull-back $e^A\colon A_{\mathbf{P(1)}}(\mathbf{P}(E)) \to A_X(L)$. We have to check the relation (11). Since the operator e^A is an isomorphism it suffices to check the relation in $A(\mathbf{P}(\mathbf{1} \oplus L))$

$$\overline{\mathrm{th}}(L) = c(\mathcal{O}(1) \otimes q^*(L)) \tag{15}$$

To do this we need in some preliminary lemmas.

LEMMA 3.20. *The elements* $\overline{\mathrm{th}}(L)$ *and* $c(\mathcal{O}_E(1) \otimes p^*(L))$ *are central in* $A(\mathbf{P}(E))$. *Moreover both elements belongs to the ideal* $A_{\mathbf{P(1)}}(\mathbf{P}(E))$.

Proof. Let $j\colon \mathbf{P}(E) - \mathbf{P}(\mathbf{1}) \hookrightarrow \mathbf{P}(\mathbf{E})$ be the inclusion. The restriction of the projection $p\colon \mathbf{P}(E) \to X$ to $\mathbf{P}(E) - \mathbf{P}(\mathbf{1})$ makes the last variety in a line bundle over

X. The inclusion $s_L\colon X \hookrightarrow \mathbf{P}(E) - \mathbf{P}(\mathbf{1})$ identifying X with the subvariety $\mathbf{P}(L)$ in $\mathbf{P}(E)$ is the zero section of the mentioned line bundle. By the strong homotopy property of the pretheory A the pull-back operator $s_L^A\colon A(\mathbf{P}(E) - \mathbf{P}(\mathbf{1})) \to A(X)$ is an isomorphism. The line bundle $s_L^*(\mathcal{O}_E(1) \otimes p^*(L))$ coincides with the line bundle $L^\vee \otimes L$ and therefore it is the trivial line bundle. Thus the Chern class $c(s_L^*(\mathcal{O}_E(1) \otimes p^*(L))$ vanishes and the element $s_L^A(c(\mathcal{O}_E(1) \otimes p^*(L)))$ vanishes as well. Therefore $j^A(c(\mathcal{O}_E(1) \otimes p^*(L))) = 0$ in $A(\mathbf{P}(E) - \mathbf{P}(\mathbf{1}))$, which proves the inclusion $c(\mathcal{O}_E(1) \otimes p^*(L)) \in A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E))$.

The class $\bar{\text{th}}(L)$ is in the subgroup $A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E))$ by the very definition of the class $\bar{\text{th}}(L)$.

The element $c(\mathcal{O}_E(1) \otimes p^*(L))$ is central in $A(\mathbf{P}(E))$ because it is a Chern class. To prove that the element $\bar{\text{th}}(L)$ is central recall that for every smooth variety X and every line bundle L over X the element $\text{th}(L) \in A_X(L)$ is $A(X)$-central. Now for every element $a \in A(\mathbf{P}(E))$ one has a chain of relations in $A_X(L)$

$$e^A(\bar{\text{th}}(L) \cup a) = \text{th}(L) \cup e^A(a) = e^A(a) \cup \text{th}(L) = e^A(a \cup \bar{\text{th}}(L)).$$

Since the support extension operator $e^A\colon A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E)) \to A_X(L)$ is an isomorphism one gets the relation $\bar{\text{th}}(L) \cup a = a \cup \bar{\text{th}}(L)$ in $A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E))$. Thus the element $\bar{\text{th}}(L) \in A(\mathbf{P}(E))$ is central. The lemma is proved. □

LEMMA 3.21. *The operator* $(\cup \bar{\text{th}}(L))\colon A(X) \to A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E))$ *is an isomorphism.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} A(X) & \xrightarrow{\cup \bar{\text{th}}(L)} & A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E)) \\ {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle e^A} \\ A(X) & \xrightarrow{\cup \text{th}(L)} & A_X(L). \end{array}$$

The operator e^A is an isomorphism by the excision property. The operator $\cup\,\text{th}(L)$ is an isomorphism because $\text{th}(L)$ is the Thom class. The lemma follows. □

LEMMA 3.22. *The diagram commutes*

$$\begin{array}{ccc} A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E)) & \xrightarrow{s^A} & A(X) \\ {\scriptstyle \cup \bar{\text{th}}(L)}\uparrow & & \uparrow{\scriptstyle \cup c(L)} \\ A(X) & \xrightarrow{\text{id}} & A(X), \end{array}$$

and $s^A(\bar{\text{th}}(L)) = s^A(c(\mathcal{O}_E(1) \otimes p^*(L)))$.

Proof. Clearly $z^A \circ e^A = s^A\colon A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E)) \to A(X)$, where $z\colon X \to L$ is the zero section of L. Thus $s^A(\bar{\text{th}}(L)) = z^A(\text{th}(L)) = c(L)$ by the very definition of $c(L)$. The commutativity of the diagram is checked. It remains to check that

$c(L) = s^A(c(\mathcal{O}_E(1) \otimes p^*(L)))$. This is obvious because the line bundle $s^*(\mathcal{O}_E(1))$ is trivial and the line bundle $s^*(p^*(L))$ coincides with the line bundle L. □

CLAIM 3.23. *For any variety X and any line bundle L over X there exists a finite-dimensional vector space V and a diagram of the form*

$$X \xleftarrow{p} X' \xrightarrow{f} \mathbf{P}(V) \tag{16}$$

in which X' is a torsor under a vector bundle over X and the morphism f is such that the line bundles $p^*(L)$ and $f^*(\mathcal{O}_V(-1))$ are isomorphic.

Proof of the claim. To construct the diagram (16) recall that by the Jouanalou trick [14] there is a torsor X'/X under a vector bundle over X such that X' is an affine variety. Now take the projection $p\colon X' \to X$ and consider the pull-back $p^*(L)$ of the line bundle L. Since the variety X' is affine the line bundle $p^*(L)$ can be induced from a projective space via a morphism $f\colon X \to \mathbf{P}(V)$. The claim is proved. □

Proof of the relation (15). Take $X = \mathbf{P}^\infty$ and $L = \mathcal{O}_{\mathbf{P}^\infty}(1)$ and consider the commutative diagram from the last lemma. By the projective bundle theorem (Theorem 3.9) the ring $A(pt)[[t]]$ of formal power series in one variable is identified with the ring $A(\mathbf{P}^\infty)$ identifying the variable t with the Chern class $c(L)$. Thus the operator $\cup c(L)\colon A(X) \to A(X)$ is injective. The operator $\cup \bar{\mathrm{t}}\mathrm{h}(L)$ is an isomorphism by Lemma 3.21. Hence the operator $s^A\colon A_{\mathbf{P(1)}}(\mathbf{P}(E)) \to A(X)$ is injective. Now the relation $c(\mathcal{O}_E(1) \otimes p^*(L)) = \bar{\mathrm{t}}\mathrm{h}(L)$ in $A(X)$ holds by the last lemma. The relation (15) is proved in the considered case. Clearly this implies the relation (15) in the case $X = \mathbf{P}(V)$ and $L = \mathcal{O}_V(1)$ for any finite-dimensional k-vector space V. The general case of the relation (15) will be reduced now to this particular case.

Let X be a variety and let L be a line bundle over X. By Claim 3.23 there exists a diagram of the form (16) such that the pull-back operator $p^A\colon A(X) \to A(X')$ is an isomorphism and the line bundles $L' = p^*(L)$ and $f^*(\mathcal{O}_V(1))$ are isomorphic.

Set $E' = \mathbf{1} \oplus L'$ and $E_V = \mathbf{1} \oplus \mathcal{O}_V(1)$ and let $p'\colon \mathbf{P}(E') \to X'$ and $p_V\colon \mathbf{P}(E_V) \to \mathbf{P}(V)$ be the projections. A choice of a line bundle isomorphism $L' \to f^*(\mathcal{O}_V(1))$ gives rise to the following Cartesian diagram

$$\begin{array}{ccccc} \mathbf{P}(E) & \xleftarrow{P} & \mathbf{P}(E') & \xrightarrow{F} & \mathbf{P}(E_V) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p_V} \\ X & \xleftarrow{p} & X' & \xrightarrow{f} & \mathbf{P}(V). \end{array}$$

Clearly one has relations

$$c(\mathcal{O}_{E'}(1) \otimes (p')^*(L')) = F^A(c(\mathcal{O}_{E_V}(1) \otimes (p_V)^*(L_V))),$$

and

$$c(\mathcal{O}_{E'}(1) \otimes (p')^*(L')) = P^A(c(\mathcal{O}_E(1) \otimes p^*(L))).$$

As we already know $c(\mathcal{O}_{E_V}(1) \otimes (p_V)^*(L_V)) = \bar{\text{th}}(\mathcal{O}_V(1))$ in $A(\mathbf{P}(E_V))$. Now one has the chain of relations in $\mathbf{P}(E)$

$$\begin{aligned}&P^A(c(\mathcal{O}_E(1) \otimes p^*(L))) = c(\mathcal{O}_{E'}(1) \otimes (p')^*(L')) \\ &\quad = F^A(c(\mathcal{O}_{E_V}(1) \otimes (p_V)^*(L_V))) \\ &\quad = F^A(\bar{\text{th}}(\mathcal{O}_V(1))) = \bar{\text{th}}(f^*(\mathcal{O}_V(1))) = \bar{\text{th}}(L') = P^A(\bar{\text{th}}(L)).\end{aligned}$$

Since the pull-back operator P^A is an isomorphism the desired relation follows. The theorem is proved. □

3.5. SPLITTING PRINCIPLE

Let A be a ring cohomology theory endowed with a Chern structure $L \mapsto c(L)$ on A. Here a variant of splitting principle is given which will be used in the text below. It will be convenient to fix certain notation. Let $p\colon Y \to X$ and $f\colon X' \to X$ be morphisms. Then we will write Y' for the scheme $X' \times_X Y$ and write p' for the projection $X' \times_X Y \to X'$ and f' for the projection $X' \times_X Y \to Y$.

LEMMA 3.24. *Let E be a rank n vector bundle over a smooth variety X. Then there exists a smooth morphism $r\colon T \to X$ such that the vector bundle $r^*(E)$ is a direct sum of line bundles and for each closed subset Z of X and for $S = r^{-1}(Z)$ the pull-back map $r^A\colon A_Z(X) \to A_S(T)$ is a split injection and moreover for a smooth variety X' and any morphism $f\colon X' \to X$ the pull-back map $(r')^A$: $A(X') \to A(T')$ is a split injection.*

Proof. Let E be a rank n vector bundle over a smooth variety X and let p: $\mathbf{P}(E) \to X$ be the associated projective bundle over X and let $\mathcal{O}_E(-1)$. Then there is the canonical short exact sequences of vector bundles on $\mathbf{P}(E)$ with the rank $n-1$ vector bundle E'

$$0 \to \mathcal{O}_E(-1) \to p^*(E) \to E' \to 0,$$

and the pull-back map $p^A\colon A_Z(X) \to A_{p^{-1}(Z)}(\mathbf{P}(E))$ is a split injection by the projective bundle theorem.

Repeating this construction several times one gets a smooth variety Y, a morphism $q\colon Y \to X$ and a filtration $(0) \subset E_1 \subset E_2 \subset \cdots \subset E_n = q^*(E)$ of the vector bundle $q^*(E)$ such that all the quotients E_i/E_{i-1} are line bundles. Moreover the pull-back map $q^A\colon A_Z(X) \to A_{q^{-1}(Z)}(Y)$ is a split injection and for a smooth variety X' and any morphism $f\colon X' \to X$ the pull-back map $(q')^A\colon A(X') \to A(Y')$ is a split injection as well.

CLAIM 3.25. *Let S be a smooth variety and let F_1, F_2 be two vector bundles over S and let $\alpha\colon F_1 \to F_2$ be a vector bundle epimorphism and let $K = \ker(\alpha)$. Then there is an affine bundle $g\colon T \to S$ such that the epimorphism $g^*(\alpha)$ splits.*

Assuming for a moment this claim complete the proof of lemma as follows. Take $S = Y$ and consider the filtration $(0) \subset E_1 \subset E_2 \subset \cdots \subset E_n = q^*(E)$ on the vec-

tor bundle $q^*(E)$. Applying several times claim one gets an affine bundle $g\colon T \to S$ such that one has a direct sum decomposition $g^*(q^*(E)) = \oplus_{i=1}^n (g^*(E_i/E_{i-1}))$ of the vector bundle $(q \circ g)^*(E)$. Show that the morphism $r = q \circ g\colon T \to X$ have the desired property.

For each smooth variety S' an each morphism $S' \to S$ the pull-back map $(g')^A\colon A(S') \to A(T')$ is an isomorphism because T' is an affine bundle over S'. Now if X' is a smooth variety and $f\colon X' \to X$ is a morphism, then Y' is smooth over X' and therefore $S' = Y'$ is smooth as well. The pull-back map $(q')^A\colon A(X') \to A(Y')$ is a split injection by the projective bundle cohomology. Thus the composite map $(q' \circ g')\colon A(X') \to A(T')$ is a split injection as well.

Proof of claim. Let $\mathrm{Hom}(F_2, F_1)$ be the scheme representing the sheaf $\mathcal{H}om(F_2, F_1)$ and let $\phi\colon \mathrm{Hom}(F_2, F_1) \to \mathrm{Hom}(F_2, F_2)$ be the morphism corresponding to the morphism $\mathcal{H}om(F_2, F_1) \to \mathcal{H}om(F_2, F_2)$ induced by the vector bundle map $\alpha\colon F_1 \to F_2$. Let $\mathrm{id}\colon S \to \mathrm{Hom}(F_2, F_2)$ be the section of the projection $\mathrm{Hom}(F_2, F_2) \to S$ corresponding to the identity map $F_2 \to F_2$. Let $\mathrm{Sect}(\alpha) = \phi^{-1}(\mathrm{id}(S))$ be a closed subscheme of the scheme $\mathrm{Hom}(F_2, F_1)$ and let $g\colon T = \mathrm{Sect}(\alpha) \to S$ be the projection. The scheme T represents the sheaf of sections of the sheaf epimorphism α. Thus there exists a canonical section $s\colon g^*(F_2) \to g^*(F_1)$ of the epimorphism $g^*(\alpha)\colon g^*(F_1) \to g^*(F_2)$. This section gives rise by a standard way to a vector bundle isomorphism $g^*(F_1) \cong g^*(F_2) \oplus g^*(K)$.

To prove claim it remains to observe that the variety $T = \mathrm{Sect}(\alpha)$ is a torsor under the vector bundle $\mathrm{Hom}(F_2, K)$. The claim is proved. □□

3.6. CHERN CLASSES

Let A be a ring cohomology theory.

DEFINITION 3.26. A Chern classes theory on A is an assignment which associate to each smooth variety X and each vector bundle E on X certain elements $c_i(E) \in A(X)$ $(i = 0, 1, \dots)$ which are universally central and satisfy the following properties

1. $c_0(E) = 1$:
   the restriction of the assignment $L \mapsto c_1(L)$ to line bundles is a Chern structure on A.
2. functoriality:
   $c_i(E) = c_i(E')$ for isomorphic vector bundles E and E';
   $f^A(c_i(E)) = c_i(f^*(E))$ for each morphism $f\colon Y \to X$.
3. Cartan formula:
   $c_r(E) = c_0(E_1) \cup c_r(E_2) + \cdots + c_r(E_1) \cup c_0(E_2)$ for each short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of vector bundles.
4. Vanishing property:
   $c_m(E) = 0$ for $m > rk(E)$.

THEOREM 3.27. *Let A be endowed with a Chern structure $L \mapsto c(L)$. Then there exists a unique Chern classes theory on A such that for each line bundle L one has $c_1(L) = c(L)$. Moreover the Chern classes $c_i(E)$ are nilpotent for $i > 0$.*

Proof. First prove the uniqueness assertion. If there are two assignments $E/X \mapsto c'_i(E)$ and $E/X \mapsto c''_i(E)$ satisfying the required properties. Then they coincide on line bundles by the properties 1 and 4. Therefore they coincide on direct sums of line bundles by the Cartan formula 3. Thus they coincide on all vector bundles by the splitting principle (Lemma 3.24).

It remains to construct a Chern classes theory. We follow here the well-known construction of Grothendieck [12]. Let X be a smooth variety and E/X be a vector bundle with $\mathrm{rk}E = n$. Set $\xi = c(\mathcal{O}_E(-1))$. By Theorem 3.9 there are unique elements $c_i(E) \in A(X)$ such that

$$\xi^n - c_1(E)\xi^{n-1} + \cdots + (-1)^n c_n(E) = 0. \tag{17}$$

Set $c_0(E) = 1$ and $c_m(E) = 0$ if $m > n$.

CLAIM 3.28. *Classes $c_i(E)$ satisfy the theorem.*

The rest of the proof is devoted to the proof of this claim. The property $c_0(E) = 1$ holds by the very definition. To prove the property $c_1(L) = c(L)$ for a line bundle L observe that $\mathbf{P}(L) = X$ and $\mathcal{O}_L(-1) = L$ over X. Thus $\xi = c(L)$ in $A(X)$ and the relation (17) shows that $c_1(L) = c(L)$.

LEMMA 3.29. *For each line bundle L over a smooth X the class $c(L) \in A(X)$ is nilpotent.*

To prove this lemma recall that by Claim 3.23 one can find a diagram of the form (16) with a torsor under a vector bundle $p\colon X' \to X$ and a morphism $f\colon X' \to \mathbf{P}(V)$ such that the line bundles $L' = p^*(L)$ and $f^*(\mathcal{O}_V(1))$ over X' are isomorphic.

The class $c(\mathcal{O}_V(1)) \in A(\mathbf{P}(V))$ is nilpotent by Lemma 3.12. Thus the class $c(L') = f^A(c(\mathcal{O}_V(1)))$ is nilpotent as well. The pull-back map $p^A\colon A(X) \to A(X')$ is an isomorphism by the strong homotopy invariance (Section 2.2.6). Therefore the class $c(L) \in A(X)$ is nilpotent as well. The lemma is proved.

Now prove the functoriality of the classes c_i. A vector bundle isomorphism $\phi\colon E \to E'$ induces an isomorphism $\Phi\colon \mathbf{P}(E) \to \mathbf{P}(E')$ of the projective bundles and a line bundle isomorphism $\Phi^*(\mathcal{O}_{E'}(-1)) \to \mathcal{O}_E(-1)$ over $\mathbf{P}(E)$. Therefore $\Phi^A(c(\mathcal{O}_{E'}(-1))) = c(\mathcal{O}_E(-1))$ in $A(\mathbf{P}(E))$. Now the relations $c_i(E) = c_i(E')$ follows immediately from the projective bundle cohomology and the relation (17). The property $f^A(c_i(E)) = c_i(f^*(E))$ is proved similarly.

For the rest of the proof we need the following claim.

CLAIM 3.30. *For a rank r vector bundle F set* $c_t(F) = 1+c_1(F)t+\cdots+c_n(F)t^r$. *Let T be a smooth variety and let* $F = \oplus_{i=1}^r L_i$ *for certain line bundles* L_i *over* T. *Then one has*

$$c_t(F) = \prod_{i=1}^r c_t(L_i).$$

In particular the elements $c_i(F)$ *are universally central and nilpotent.* (*The nilpotence of the class* $c_1(L)$ *is proved just above.*)

Assuming for a moment Claim 3.30 complete the proof of Claim 3.28 as follows. By the splitting principle (Lemma 3.24) there exists a smooth variety T and a morphism $r\colon T \to X$ such that each the vector bundle $r^*(E_i)$ is a sum of line bundles and the pull-back map $r^A\colon A(X) \to A(T)$ is injective.

The Claim 3.30 and the injectivity of the map r^A show the Cartan formula $c_t(E) = c_t(E_1)c_t(E_2)$. Furthermore the Claim 3.30 shows that the elements $r^A(c_i(E)) \in A(T)$ are universally central. In particular for a smooth variety X' and a morphism $f\colon X' \to X$ and for $T' = X' \times_X T$ the element $(f')^A(r^A(c_i(E)))$ is central in $A(T')$. By the same splitting principle the pull-back map $(r')^A\colon A(X') \to A(T')$ is injective. Now the relation $(f')^A(r^A(c_i(E))) = (r')^A(f^A(c_i(E)))$ and the injectivity of the map $(r')^A\colon A(X') \to A(T')$ show that the element $f^A(c_i(E))$ is central in $A(X')$. Thus the elements $c_i(E)$ are universally central.

Finally the Claim 3.30 shows that the elements $r^A(c_i(E))$ are nilpotent. The injectivity of the map r^A proves the nilpotence of the elements $c_i(E) \in A(X)$.

It remains to prove the Claim 3.30.

If $\xi = c(\mathcal{O}_F(-1))$, where $\mathcal{O}_F(-1)$ is the tautological line bundle on $\mathbf{P}(F)$ then it suffices to prove the relation $\prod(\xi - c_1(L_i)) = 0$ in $A(\mathbf{P}(F))$. To prove the very last relation set $F^i = L_1 \oplus \cdots \oplus \bar{L}_i \oplus \cdots \oplus L_n$, where the bar means that the corresponding summand has to be omitted. Since $\mathcal{O}_F(-1)|_{\mathbf{P}(L_i)} = L_i$ over X the element $\xi - c_1(L_i)$ vanishes being restricted to $\mathbf{P}(L_i)$. Therefore $\xi - c_1(L_i)$ belongs to the subgroup $A_{\mathbf{P}(F^i)}(\mathbf{P}(F))$ of the group $A(\mathbf{P}(F))$ (see (14)). Thus the cup-product $\prod_{i=1}^n(\xi - c_1(L_i))$ belongs to the subgroup $A_{\cap\mathbf{P}(F^i)}(\mathbf{P}(F))$ of the group $A(\mathbf{P}(F))$. Since the intersection $\cap_{i=1}^n \mathbf{P}(F^i)$ is empty the group $A_{\cap\mathbf{P}(F^i)}(\mathbf{P}(F))$ vanishes by the vanishing property. Hence indeed the relation $\prod_{i=1}^n(\xi - c_1(L_i)) = 0$ holds in $A(\mathbf{P}(F))$ and $\prod_{i=1}^n(1 + c_1(L_i)t) = c_t(F)$ in $A(X)$.

Finally since each of the elements $c_1(L_i)$ is universally central and nilpotent hence each of the elements $c_j(E)$ is universally central and nilpotent as well. Claim 3.30 is proved. □

PROPOSITION 3.31. *Let X be a smooth variety and let E be a vector bundle over X of the constant rank n. Let* $p\colon \mathbf{P}(E) \to X$ *be the projection. Then the element* $c_n(\mathcal{O}_E(1) \otimes p^*(E)) \in A(\mathbf{P}(E))$ *vanishes.*

Proof. Define a rank $n-1$ vector bundle Q over $\mathbf{P}(E)$ by the short exact sequence $0 \to \mathcal{O}_E(-1) \to p^*(E) \to Q \to 0$. Tensoring this short exact sequence with the line bundle $\mathcal{O}_E(1)$ one gets a short exact sequence $0 \to \mathcal{O} \to$

$\mathcal{O}_E(1) \otimes p^*(E) \to \mathcal{O}_E(1) \otimes Q \to 0$. Now the Cartan formula for the Chern classes gives the relation $c_n(\mathcal{O}_E(1) \otimes p^*(E)) = c_1(\mathcal{O})c_{n-1}(\mathcal{O}_E(1) \otimes Q)$. Thus $c_n(\mathcal{O}_E(1) \otimes p^*(E)) = 0$. □

3.7. ORIENTING A THEORY

In this section A is a ring cohomology theory. Two theorems in this section shows how one can construct an orientation using a Chern structure (or a Thom structure) on A and how one can construct a Chern structure (or a Thom structure) using an orientation.

Before to state theorems it is convenient to fix a notion of Thom classes theory, which is equivalent to the notion of orientation but it is defined in terms of elements rather than in terms of homomorphisms.

The definition of $A(X)$-central elements in $A_X(E)$ (for a vector bundle E over a smooth variety X) is given just below the definition of a Chern structure.

DEFINITION 3.32. A Thom classes theory on A is an assignment which associate to each smooth variety X and to each vector bundle E over X an element $\mathrm{th}(E) \in A_X(E)$ satisfying the following properties

(1) $\mathrm{th}(E)$ is $A(X)$-central;
(2) $\varphi^A(\mathrm{th}(F)) = \mathrm{th}(E)$ for each vector bundle isomorphism $\varphi\colon E \to F$;
(3) $f^A(\mathrm{th}(E)) = \mathrm{th}(f^*(E))$ for each morphism $f\colon Y \to X$ with a smooth variety Y;
(4) the operator $A(X) \to A_X(E)$, $a \to \mathrm{th}(E) \cup a$ is an isomorphism;
(5) multiplicativity property: for the projections $q_i\colon E_1 \oplus E_2 \to E_i$ (i $= 1, 2$) one has

$$q_1^* \,\mathrm{th}(E_1) \cup q_2^* \,\mathrm{th}(E_2) = \mathrm{th}(E_1 \oplus E_2) \in A_X(E_1 \oplus E_2). \tag{18}$$

The element $\mathrm{th}(E)$ is called the Thom class of the vector bundle E.

LEMMA 3.33. *If ω is an orientation on the theory A then the assignment $E \mapsto \mathrm{th}_X^E(1) \in A_X(E)$ is a Thom classes theory on A. We write $\mathrm{th}_X(E)$ for the element $\mathrm{th}_X^E(1) \in A_X(E)$.*

If an assignment $E/X \mapsto \mathrm{th}(E) \in A_X(E)$ is a Thom classes theory on A, then the family of homomorphisms $\cup\,\mathrm{th}(E)\colon A_Z(X) \to A_Z(E)$ form an orientation on A.

The two mentioned correspondences between orientations and Thom classes theories are inverse to each other.

Proof. It is obvious. □

LEMMA 3.34. *If an assignment $E/X \mapsto A_X(E)$ is a Thom classes theory on A, then its restriction to line bundles is a Thom structure on A.*

If two Thom classes theories coincide on each line bundle then they coincide.

Proof. The first assertion is obvious. To prove the second assertion consider two Thom classes theories $E \mapsto \mathrm{th}(E) \in A_X(E)$ and $E \mapsto \mathrm{th}'(E) \in A_X(E)$

which coincide on line bundles. To prove that for a vector bundle E one has the relation $\mathrm{th}(E) = \mathrm{th}'(E)$ one may assume by the splitting principle (Lemma 3.24) that $E = \oplus L_i$ is a direct sum of line bundles. Let $q_i \colon E \to L_i$ be the projection to the ith summund. Now the chain of relations

$$\mathrm{th}(E) = \cup q_i^A(\mathrm{th}(L_i)) = \cup q_i^A(\mathrm{th}'(L_i)) = \mathrm{th}'(E)$$

completes the proof of the assertion. □

THEOREM 3.35. *Given a Chern structure* $L \mapsto c(L)$ *on* A (*or the corresponding by Theorem* 3.5 *Thom structure* $L \mapsto \mathrm{th}(L)$ *on* A) *there exists an orientation* $(X, Z, E) \mapsto \mathrm{th}_Z^E$ *on* A *such that the following properties hold*

1. *for each smooth variety* X *and each line bundle* L/X *one has* $\mathrm{th}(L) = \mathrm{th}_X^L(1)$;
2. *for each smooth* X *and each line bundle* L/X *one has* $z^A \circ i^A \circ \mathrm{th}_X^L(a) = c(L) \cup a$ *where* $a \in A(X)$ *is any element,* $i^A \colon A_X(L) \to A(L)$ *is the support extension operator for the pair* $(L, L - X)$, $z \colon X \to L$ *is the zero section.*

Moreover the required orientation is uniquely determined both by the property (1) *and by the property* (2).

This theorem describes the arrow δ and the composition $\delta \circ \gamma$ from Section 1.

THEOREM 3.36. *If* $(X, Z, E) \mapsto \mathrm{th}_Z^E$ *is an orientation on* A *then the assignment* $L \mapsto z^A \circ i^A \circ \mathrm{th}_X^L(1)$ *is a Chern structure on* A, *the assignment* $L \mapsto \mathrm{th}_X^L(1)$ *is a Thom structure on* A *and so constructed Chern and Thom structures correspond to each other.*

Moreover the construction of an orientation by means of a Chern (*or a Thom*) *structure given by Theorem* 3.35 *and the construction of a Chern and a Thom structure by means of an orientation are inverse of each other.*

This theorem describes the arrow ρ and the composition $\delta \circ \gamma$ from Section 1. Moreover it states that the arrow ρ and the composition $\delta \circ \gamma$ are inverse to each other, and it states that the composition $\gamma \circ \rho$ and the arrow δ are inverse to each other.

Proof of Theorem 3.35. To construct an orientation on A it suffices (see Lemma 3.33) to construct a Thom classes theory $E \mapsto \mathrm{th}(E) \in A_X(E)$.

Let E/X be a rank n vector bundle and let $F = E \oplus 1$. Let $p \colon \mathbf{P}(F) \to X$ be the projection. The support extension operator $A_{\mathbf{P(1)}}(\mathbf{P}(F)) \to A(\mathbf{P}(F))$ is injective because the sequence (14) is exact. The same exact sequence and Proposition 3.31 show that the element $c_n(\mathcal{O}_F(1) \otimes p^*E) \in A(\mathbf{P}(F))$ belongs to the subgroup $A_{\mathbf{P(1)}}(\mathbf{P}(F))$. Set

$$\bar{\mathrm{th}}(E) = c_n(\mathcal{O}_F(1) \otimes p^*E) \in A_{\mathbf{P(1)}}(\mathbf{P}(F)), \tag{19}$$

and define the element $\mathrm{th}(E) \in A_X(E)$ as follows

$$\mathrm{th}(E) = e^A(\bar{\mathrm{th}}(E)) = e^A(c_n(\mathcal{O}_F(1) \otimes p^*E)) \in A_X(E). \tag{20}$$

To show that the assignment $E \mapsto \mathrm{th}(E) \in A_X(E)$ is a Thom classes theory it remains to check the properties (1)–(5) from Definition 3.32.

The second and the third property follows immediately from the functoriality of the Chern classes (Theorem 3.27).

The element $\bar{\mathrm{th}}(L) \in A(\mathbf{P}(F))$ is central because it is a Chern class. Since the pull-back map $e^A\colon A(\mathbf{P}(F)) \to A(E)$ is surjective the element $\mathrm{th}(L) = e^A(\bar{\mathrm{th}}(L))$ is $A(X)$-central. For a smooth variety Y and a morphism $f\colon Y \to X$ one has $\mathrm{th}(f^*(E)) = f^A(\mathrm{th}(E))$ in $A_Y(f^*(E))$. Thus the element $\mathrm{th}(E)$ is universally $A(X)$-central. This proves the property (1).

To prove the fourth property consider the commutative diagram

$$\begin{array}{ccc} A_{\mathbf{P(1)}}(\mathbf{P}(F)) & \xrightarrow{e^A} & A_X(E) \\ \uparrow{\scriptstyle \cup\bar{\mathrm{th}}(E)} & & \uparrow{\scriptstyle \cup\,\mathrm{th}(E)} \\ A(X) & \xrightarrow{\mathrm{id}} & A(X). \end{array}$$

The map e^A is an isomorphism by the excision property. Thus the right vertical arrow is an isomorphism if the cup-product with the class $\bar{\mathrm{th}}(E)$ is an isomorphism. For that consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{\mathbf{P(1)}}(\mathbf{P}(F)) & \longrightarrow & A(\mathbf{P}(F)) & \longrightarrow & A(\mathbf{P}(E)) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \cup\bar{\mathrm{th}}(E)} & & \uparrow{\scriptstyle (\cup\bar{\mathrm{th}}(E),\alpha)} & & \uparrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & A(X) & \longrightarrow & A(X)\oplus A(X)^n & \longrightarrow & A(X)^n & \longrightarrow & 0, \end{array}$$

where $\beta = (\cup 1, \cup\zeta_E, \ldots, \cup\zeta_E^{n-1})$, $\zeta_E = c(\mathcal{O}_E(1)) \in A(\mathbf{P}(E))$ and $\alpha = (\cup 1, \cup\zeta_F, \ldots, \cup\zeta_F^{n-1})$, $\xi_F = c(\mathcal{O}_F(1)) \in \mathbf{P}(F)$.

The map β is an isomorphism by the projective bundle theorem. Thus to prove that the left vertical arrow is an isomorphism it suffices to check that the map $(\cup\bar{\mathrm{th}}(E), \alpha)$ is an isomorphism. Using the Mayer–Vietoris property and Proposition 2.18 one may assume that the bundle E is the trivial rank n bundle. In this case one has $\bar{\mathrm{th}}(E) = c_n(\mathcal{O}_F(1)^n) = \zeta_F^n$. Thus the map $(\bar{\mathrm{th}}(E), \alpha)$ coincides in this case with the map $(\cup 1, \cup\zeta_F, \ldots, \cup\zeta_F^n)$ and it is an isomorphism by the projective bundle theorem. The property (4) is proved.

Basically the property (5) follows from the Cartan formula for Chern classes. But to give a detailed prove one needs certain preliminaries.

Let $E = E_1 \oplus E_2$ be a vector bundle over a smooth variety X and let $F_r = E_r \oplus 1$ $(r = 1, 2)$ and let $F = E \oplus 1$ and let $p\colon \mathbf{P}(F) \to X$ be the projection. Let $q_r\colon E \to E_i$ be the projection and let $i_r\colon E_r \hookrightarrow E$ be the imbedding. We will identify E with the open subset $\mathbf{P}(F) - \mathbf{P}(E)$ of $\mathbf{P}(F)$ and identify E_i with the open subset $\mathbf{P}(F_r) - \mathbf{P}(E_i)$ of $\mathbf{P}(F_r)$. Let $\mathbf{P}(F_i)$ be the subvariety in $\mathbf{P}(F)$ defined by the direct summund F_r of F. Let the closed imbedding $\bar{i}_r\colon \mathbf{P}(F_i) \hookrightarrow \mathbf{P}(F)$ be the one extending the imbedding i_r. We will write $p\colon \mathbf{P}(F) \to X$ for the projection to X.

Let $\mathrm{res}_r\colon A_{\mathbf{P}(F_r)}(\mathbf{P}(F)) \to A_{E_r}(E)$ be the pull-back map induced by the imbedding $E \hookrightarrow \mathbf{P}(F)$. Let $\mathrm{res}\colon A_{\mathbf{P(1)}}(\mathbf{P}(F)) \to A_X(E)$ be the pull-back map induced by the same imbedding. The support extension operators

$$A_{\mathbf{P}(F_r)}(\mathbf{P}(F)) \to A(\mathbf{P}(F))$$

are injective for $(r = 1, 2)$ because they are the operators from the short exact sequences of the form (14). The same exact sequences and Proposition 3.31 show that the elements

$$c_{n_1}(\mathcal{O}_F(1) \otimes p^*(E_1)) \in A(\mathbf{P}(F));\ \ c_{n_2}(\mathcal{O}_F(1) \otimes p^*(E_2)) \in A(\mathbf{P}(F))$$

belongs to the subgroups $A_{\mathbf{P}(F_2)}(\mathbf{P}(F))$ and $A_{\mathbf{P}(F_1)}(\mathbf{P}(F))$, respectively. Consider elements $x_1 = c_{n_2}(\mathcal{O}_F(1) \otimes p^*(E_2)) \in A_{\mathbf{P}(F_1)}(\mathbf{P}(F))$, $x_2 = c_{n_1}(\mathcal{O}_F(1) \otimes p^*(E_1)) \in A_{\mathbf{P}(F_2)}(\mathbf{P}(F))$. We claim that they satisfy the following relations

$$q_1^A(\mathrm{th}(E_1)) = \mathrm{res}_1(x_2);\ \ q_2^A(\mathrm{th}(E_2)) = \mathrm{res}_2(x_1). \tag{21}$$

In fact, the commutative diagram

$$\begin{CD}
A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(F_2)) @<{\bar{j}_2^A}<< A_{\mathbf{P}(F_1)}(\mathbf{P}(F)) \\
@V{e_2^A}VV @VV{\mathrm{res}_2}V \\
A_X(E_2) @<{j_2^A}<< A_{E_1}(E),
\end{CD}$$

and the relation $\bar{j}_2^A(x_1) = \bar{\mathrm{th}}(E_2)$ in $A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(F_2))$ show that $j_2^A(\mathrm{res}_2(x_1)) = \mathrm{th}(E_2)$. Now the relation $q_2^A \circ j_2^A = \mathrm{id}$ proves the relation $\mathrm{res}_2(x_1) = q_2^A(\mathrm{th}(E_2))$. The second of the two relations (21) is proved. The first one is proved similarly. So the relations (21) are proved.

Consider one more commutative diagram

$$\begin{CD}
A(\mathbf{P}(F)) \times A(\mathbf{P}(F)) @>{\cup}>> A(\mathbf{P}(F)) \\
@A{I_1^A \times I_2^A}AA @AA{I^A}A \\
A_{\mathbf{P}(F_1)}(\mathbf{P}(F)) \times A_{\mathbf{P}(F_2)}(\mathbf{P}(F)) @>{\cup}>> A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(F)) \\
@V{\mathrm{res}_1 \times \mathrm{res}_2}VV @VV{\mathrm{res}}V \\
A_{E_1}(E) \times A_{E_2}(E) @>{\cup}>> A_X(E).
\end{CD}$$

The commutativity of the upper square of this diagram proves the relation $x_1 \cup x_2 = \bar{\mathrm{th}}(E)$ in $A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E))$ because

$$c_{n_2}(\mathcal{O}_F(1) \otimes p^*(E_2)) \cup c_{n_1}(\mathcal{O}_F(1) \otimes p^*(E_1)) = c_n(\mathcal{O}_F(1) \otimes p^*(E))$$

in $A(\mathbf{P}(F))$. Now the chain of relations

$$\begin{aligned}\mathrm{th}(E) &= \mathrm{res}(\bar{\mathrm{th}}(E)) = \mathrm{res}(x_1 \cup x_2) = \mathrm{res}_1(x_1) \cup \mathrm{res}_2(x_2) \\ &= q_1^A(\mathrm{th}(E_1)) \cup q_2^A(\mathrm{th}(E_2))\end{aligned}$$

prove the property (5). Thus the assignment $E \mapsto \mathrm{th}(E) \in A_X(E)$ is indeed a Thom classes theory on A.

We still have to check that the orientation corresponding to this Thom classes theory by Lemma 3.33 satisfies the requirements 1 and 2 of Theorem 3.35.

The property $\mathrm{th}(L) = \mathrm{th}_X^L(1)$ holds because the map th_X^L is defined as the cup-product with the class $\mathrm{th}(L)$.

The requirement 2 is satisfied by the following reasons. The composite map $z^A \circ i^A \circ \mathrm{th}_X^L\colon A(X) \to A(X)$ is a two-sided $A(X)$-module map. It takes the unit 1 to the class $c(L)$ by Theorem 3.5. The requirement is checked.

To complete the proof of theorem it remains to prove the uniqueness of the orientation. To prove the uniqueness of the orientation satisfying the property 1 take two orientations ω and ω' on A satisfying the property 1. Then the assignments $E \mapsto \mathrm{th}(E) = \mathrm{th}_X^E(1)$ and $E \mapsto \mathrm{th}'(E) = \mathrm{th}_X'^E \in A_X(E)$ are two Thom classes theories on A by Lemma 3.33. To check that they coincide it suffices by Lemma 3.34 to check that their restrictions to line bundles coincide. This is the case by the requirement 1. Thus $\omega = \omega'$.

Now prove the uniqueness of the orientation satisfying the requirement 2. Let $L \mapsto c(L)$ be a Chern structure and let ω and ω' be two orientations satisfying the requirement 2. We will show that $\omega = \omega'$.

It suffices to check that the corresponding Thom classes theories $E \mapsto \mathrm{th}_\omega(E)$ and $E \mapsto \mathrm{th}_{\omega'}(E)$ coincide (see Lemma 3.33). By Lemma 3.34 the restriction of these Thom classes theories to line bundles are Thom structures on A. By the same lemma the Thom classes theories coincide if the mentioned Thom structures on A coincide. To prove that the two Thom structures on A coincide it suffices by Theorem 3.5 to check that the two corresponding Chern structures $L \mapsto c_\omega(L)$ and $L \mapsto c_{\omega'}(L)$ coincide. This holds because $c_\omega(L) = c(L)$ and $c_{\omega'}(L) = c(L)$ by the requirement 2. The proof of the relation $\omega = \omega'$ is completed. □

Proof of Theorem 3.36. The assignment $E \mapsto \mathrm{th}_X^E(1)$ is a Thom classes theory by Lemma 3.33. Its restriction to line bundles is a Thom structure on A by Lemma 3.34. The first assertion is proved.

The assignment $L \mapsto c(L) = z^A(i^A(\mathrm{th}_X^L(1))) = z^A(i^A(\mathrm{th}(L)))$ is a Chern structure by Theorem 3.5. The Chern structure $L \mapsto c(L)$ and the Thom structure $L \mapsto \mathrm{th}(L)$ correspond to each other by the same Theorem 3.5. The first part of theorem is proved.

Now verify that the correspondences between orientations and Thom structures in the two theorems are inverse to each other.

Now let $L \mapsto \mathrm{th}(L)$ be a Thom structure and let ω be the corresponding by Theorem 3.35 orientation and let $L \mapsto \mathrm{th}_X^L(1) = \mathrm{th}'(L)$ be the Thom structure corresponding to ω by Theorem 3.36. We have to check that for each line bundle L one has $\mathrm{th}'(L) = \mathrm{th}(L)$.

If $L \mapsto c(L)$ is the Chern structure corresponding to the Thom structure $L \mapsto \mathrm{th}(L)$, then $\mathrm{th}'(L) = c_1(\mathcal{O}(1) \otimes p^*(L))$ by the very construction of the orientation ω.

From the other side the assignment $L \mapsto c_1(\mathcal{O}(1)) \otimes p^*(L)$ is exactly the Thom structure corresponding to the Chern structure $L \mapsto c(L)$. Thus $\mathrm{th}'(L) = \mathrm{th}(L)$ by Theorem 3.5.

Let ω be an orientation and let $L \to \mathrm{th}_\omega(L)$ be the corresponding Thom structure and let ω' be the orientation corresponding to the Thom structure $L \to \mathrm{th}_\omega(L)$. We have to check that $\omega' = \omega$.

It was proved just above that the Thom structure $L \mapsto \mathrm{th}_{\omega'}(L)$ corresponding to ω' coincide with the Thom structure $L \mapsto \mathrm{th}_\omega(L)$. Now by Lemma 3.34 the Thom classes theory $E \mapsto \mathrm{th}_{\omega'}(E)$ corresponding to ω' coincides with the Thom classes theory $E \mapsto \mathrm{th}_\omega(E)$ corresponding to ω. Thus $\omega' = \omega$ by Lemma 3.33.

It is verified simultaneously that the correspondences between orientations and Chern structures on A described in the two theorems are inverse to each other.

The theorem is proved. □

3.8. EXAMPLES

3.8.1.

Let A be the algebraic K-theory (Section 2.1.8). The rule $L \to [1] - [L^\vee]$ endows A with a Chern structure (the property (4) follows from [25, Section 8, Theorem 2.1]) and thus orients A.

It is interesting to observe that the corresponding Chern class c_n of a rank n vector bundle E is exactly the known class $\lambda_{-1}(E^\vee) = [1] - [E^\vee] + [\wedge^2 E^\vee] + \cdots + (-1)^n[\wedge^n E^\vee]$.

3.8.2.

Let A be the etale cohomology theory $A^*_Z(X) = \oplus_{q=-\infty}^{+\infty} H^*_Z(X, \mu_m^{\otimes q})$, where m is an integer prime to $\mathrm{char}(k)$. Consider the short exact sequence of the etale sheaves $0 \to \mu_m \to \mathbb{G} \stackrel{\times m}{\to} \mathbb{G} \to 0$ and denote by $\partial\colon H^1(X, \mathbb{G}_m) \to H^2(X, \mu_m)$ the boundary map. For a line bundle L over a smooth variety X let $[L] \in H^1(X, \mathbb{G}_m)$ be its isomorphism class. It is known [17] that the rule $L \mapsto \partial([L])$ endows A with a Chern structure. Thus A is oriented.

3.8.3.

Let A be the motivic cohomology [28]: $A^p_Z(X) = \oplus_{q=0}^{\infty} H^p_Z(X, \mathbb{Z}(q))$. Recall that $H^2_{\mathcal{M}}(X, \mathbb{Z}(1)) = CH^1(X)$ for a smooth X [28]. For a line bundle L over a smooth variety X let $D(L) \in CH^1(X)$ be the associated class divisor. The rule $L \mapsto D(L)$ endows A with a Chern structure in the characteristic zero [28, Corollary 4.12.1] (now it is known in any characteristic). Thus A is oriented.

3.8.4.
Let A be the K-cohomology [25, Section 7, 5.8]: $A^p_Z(X) = \oplus_{q=0}^{\infty} H^p_Z(X, \mathcal{K}_q)$, where $\mathcal{K}$ is the sheaf of K-groups. Recall that the sheaf $\mathcal{K}_1$ coincides with the sheaf $\mathcal{O}^*$ of invertible functions. For a line bundle L over a smooth variety X let $[L] \in H^1(X, \mathcal{K}_1) = H^1(X, \mathcal{O}^*)$ be the isomorphism class of L. The rule $L \mapsto [L]$ endows A with a Chern structure [11, Theorem 8.10] and thus orients A.

3.8.5.
Let $k = \mathbb{R}$ and A be the $\mathbb{Z}/2$-graded-commutative ring theory from Section 2.5.6. For a line bundle L consider the real line bundle $L(\mathbb{R})$ over the topological space $X(\mathbb{R})$ and set $c_1(L) = w_1(L(\mathbb{R})) \in H^1(X(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}) \subset A^{\mathrm{ev}}(X)$ (the first Stiefel–Whitney class). Since $\mathbf{P}^n(\mathbb{R}) = \mathbb{R}P^n$ is the real projective space the rule $L \mapsto c_1(L)$ endows A with a Chern structure and thus orients A.

3.8.6. *Semi-topological Complex and Real K-theories [6]*
If the ground field k is the field $\mathbb{R}$ of reals then the semi-topological K-theory of real algebraic varieties $K\mathbb{R}^{\mathrm{semi}}$ defined in [6] is an oriented theory as it is proved in [6]. For a real variety X it interpolates between the algebraic K-theory of X and Atiyah's real K-theory of the associated real space of complex points, $X(\mathbb{C})$.

3.8.7. *Orienting the Algebraic Cobordism Theory*
In this example the notation of Section 2.5.5 are used.

The identity morphism $\mathbb{MGL}_1$ to itself gives rise in the standard manner to an element $[\mathrm{id}_1] \in \mathbb{MGL}^{2,1}(\mathbb{MGL}_1)$. By the very definition $\mathbb{MGL}_1 = \mathrm{Th}(\mathcal{T}(1))$ and $\mathcal{T}(1)$ is the tautological line bundle $\mathcal{O}(-1)$ over the space $G(1) = \mathbf{P}(V) = \mathbf{P}^{\infty}$. Now set

$$\mathrm{th} = [\mathrm{id}_1] \in \mathbb{MGL}^{2,1}(\mathbb{MGL}_1) = \mathbb{MGL}^{2,1}_{\mathbf{P}^{\infty}}(\mathcal{O}(-1)).$$

Consider the fiber $\mathbb{A}^1$ of $\mathcal{T}(1)$ over the point $g_1 \in \mathbf{P}(V)$.

The restriction of the element th to the Thom space $\mathrm{Th}(\mathbb{A}^1) = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ coincides with the T-suspension $\sigma \in \mathbb{MGL}^{2,1}(\mathrm{Th}(\mathbb{A}^1)) = \mathbb{MGL}^{2,1}_{\{0\}}(\mathbb{A}^1)$ of the unite $1 \in \mathbb{MGL}^{0,0}(pt)$. Thus the element th orients the algebraic cobordism theory $\mathbb{MGL}$ due to (??).

3.9. THE FORMAL GROUP LAW F_{ω}

Let ω be an orientation of A. Thus A is endowed with the Chern structure which correspond to ω (see Theorems 3.35 and 3.36). Following Novikov, Mischenko [19] and Quillen [24] we associate a formal group law F_{ω} with ω. This formal group law is defined over the ring $\bar{A}^{uc} := A(pt)^{uc}$ and gives an expression of the first Chern class of $L_1 \otimes L_2$ in terms of the first Chern classes of line bundles L_1, L_2.

Using Theorem 3.9 identify the formal power series in one variable $\bar{A}[[u]]$ with the ring $A(\mathbf{P}^\infty)$ identifying u with $c_1(\mathcal{O}(1)) \in A(\mathbf{P}^\infty)$. The two 'projections' $p_i\colon \mathbf{P}^\infty \times \mathbf{P}^\infty \to \mathbf{P}^\infty$ induce two pull-back maps $p_i^A\colon A(\mathbf{P}^\infty) \to A(\mathbf{P}^\infty \times \mathbf{P}^\infty)$. Using Theorem 3.9 again identify $A(\mathbf{P}^\infty \times \mathbf{P}^\infty)$ with $\bar{A}[[u_1, u_2]]$, where $u_i = p_i^*(u) = c_1(p_i^*(\mathcal{O}(1))$. Set

$$F_\omega(u_1, u_2) = c_1(p_1^*(\mathcal{O}(1) \otimes p_2^*(\mathcal{O}(1))) \in \bar{A}[[u_1, u_2]]. \tag{22}$$

Since the first Chern class is a universally central the element F_ω belongs to $\bar{A}^{uc}[[u_1, u_2]]$.

PROPOSITION 3.37. *For any $X \in \mathcal{S}m$ and line bundles L_1/X, L_2/X one has the following relation in $A(X)$*

$$c_1(L_1 \otimes L_2) = F_\omega(c_1(L_1), c_1(L_2)).$$

Here the right-hand side is well defined since the first Chern classes are universally central and nilpotent (*Theorem* 3.27).

Proof. Using the Jouanolou device one may assume (compare with the proof of Lemma 3.29) that $L_i = f_i^*(\mathcal{O}(1))$ for a maps $f_i\colon X \to \mathbf{P}^N$. Let $f = (f_1, f_2)\colon X \to \mathbf{P}^N \times \mathbf{P}^N$. The chain of relations

$$\begin{aligned} c_1(L_1 \otimes L_2) &= f^A(c_1(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1)))) \\ &= f^A(F_\omega(u_1, u_2)) = F_\omega(c_1(L_1), c_1(L_2)) \end{aligned}$$

completes the proof. □

PROPOSITION 3.38. *The formal power series $F_\omega \in \bar{A}^{uc}[[u_1, u_2]]$ is a commutative formal group law* [8] *with the 'inverse element' $I_\omega(u) = c_1(\mathcal{O}(-1)) \in A^{uc}(\mathbf{P}^\infty) = \bar{A}^{uc}[[u]]$.*

Proof. One has to verify that the formal power series $F_\omega \in \bar{A}^{uc}[[u_1, u_2]]$ satisfies the following properties

- normalization: $F_\omega(u_1, u_2) \equiv u_1 + u_2$ modulo the degree 2;
- associativity: $F_\omega(F_\omega(u_1, u_2), u_3) = F_\omega(u_1, F_\omega(u_2, u_3))$;
- commutativity: $F_\omega(u_1, u_2) = F_\omega(u_2, u_1)$;
- 'inverse element': $F_\omega(u, I_\omega(u)) = 0$ for $I_\omega(u) = c_1(\mathcal{O}(-1)) \in A^{uc}(\mathbb{P}^\infty) = \bar{A}^{uc}[[u]]$.

Proposition 3.37 shows that the associativity and the commutativity follows immediately from the corresponding properties of the tensor products of line bundles.

To prove the normalization property consider the element $\alpha = c_1(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1))) \in A(\mathbf{P}^1 \times \mathbf{P}^1)$. The $A(pt)$-module $A(\mathbf{P}^1 \times \mathbf{P}^1)$ is a free module with the free bases 1, $\zeta \otimes 1$, $1 \otimes \zeta$ and $\zeta \otimes \zeta$ by the projective bundle theorem.

Write the element α in the form $\alpha = a_{00} 1 \otimes 1 + a_{10}\zeta \otimes 1 + a_{01} 1 \otimes \zeta + a_{11}\zeta \otimes \zeta$, where a_{ij} are elements in $A(pt)$. By the projective bundle theorem $A(\mathbf{P}^1 \times \mathbf{P}^1) = A[[u_1, u_2]]/(u_1^2, u_2^2)$ and thus it suffices to prove that $a_{00} = 0$ and $a_{10} = 1 = a_{01}$.

Restricting the element α to $\{0\} \times \{0\}$, to $\mathbf{P}^1 \times \{0\}$ and to $\{0\} \times \mathbf{P}^1$ and using the relation $\zeta|_{\{0\}} = 0$ one gets the relations: $a_{00} = 0$, $a_{00} + a_{10}\zeta = \zeta$ in $A(\mathbf{P}^1)$ and $a_{00} + a_{01}\zeta = \zeta$ in $A(\mathbf{P}^1)$. Thus $a_{10} = 1$ and $a_{01} = 1$ and the normalization property is proved.

The relation $F_\omega(u, I_\omega(u)) = 0$ follows from Proposition 3.37 applied to the line bundles $L_1 = \mathcal{O}(1)$ and $L_2 = \mathcal{O}(-1)$ on $\mathbf{P}^\infty$. □

DEFINITION 3.39. The formal group law F_ω is called the formal group law associated with A endowed with the orientation ω. Its the 'inverse element' is the series I_ω.

Let $E^-_\omega\colon \bar{A}[[u]] \to A(\mathbf{P}^\infty)$ be an isomorphism taking the variable u to the element $\xi_A = c^A_1(\mathcal{O}(-1)) \in A(\mathbf{P}^\infty)$. The formal power series

$$F^-_\omega(u_1, u_2) = (E^-_\omega \otimes E^-_\omega)^{-1}[c_1(p^*_1(\mathcal{O}(-1)) \otimes \otimes p^*_2(\mathcal{O}(-1)))] \in A(pt)^{uc}[[u_1, u_2]] \tag{23}$$

satisfies exactly the same property as the series $F_\omega(u_1, u_2)$ above. Namely, for any $X \in \mathcal{S}m$ and line bundles L_1/X, L_2/X one has the following relation in $A(X)$

$$c_1(L_1 \otimes L_2) = F^-_\omega(c_1(L_1), c_1(L_2)).$$

Taking $X = \mathbf{P}^\infty \times \mathbf{P}^\infty$ and line bundles $L_i = p^*_i(\mathcal{O}(-1))$ for $i = 1, 2$ one gets

$$F^-_\omega(u_1, u_2) = F_\omega(u_1, u_2).$$

So there is no difference which line bundle is used $\mathcal{O}(-1)$ or $\mathcal{O}(1)$. The formal group law $F_\omega(u_1, u_2)$ is the same in both cases. We usually for the purposes of definitions will use the tautological line bundle $\mathcal{O}(-1)$. However for certain computations it will be convenient to use the line bundle $\mathcal{O}(1)$.

3.9.1. *Examples*

- If $A = H^*_{\mathcal{M}}(-, \mathbb{Z}(*))$ with the first Chern class c^H_1 then one has the relation $c^H_1(L_1 \otimes L_2) = c^H_1(L_1) + c^H_1(L_2)$.
- If $A = K$-theory with the first Chern class defined by $c^K_1(L) = [1] - [L^\vee]$ then one has the relation $c^K_1(L_1 \otimes L_2) = c^K_1(L_1) + c^K_1(L_2) - c^K_1(L_1) \cdot c^K_1(L_2)$.
- Let $k = \mathbb{C}$ and $A = \Omega$ the complex cobordism theory equipped with the Chern structure $L \mapsto cf(L)$ given by the Conner–Floyed class cf [5, pp. 48–52], then the formal group law F_Ω is the universal commutative formal group law by [24]. Its ring of coefficients $\Omega(pt)$ is canonically isomorphic to the Lazard ring L [24].

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