PROBLEMS for Lecture 5

5.1. Three planes P_1, P_2, P_3 passing through the z-axis of Euclidean space $\mathbb{R}^3$ are given. The angles between P_1 and P_2, P_2 and P_3 are α and β, respectively.

(a) Under what conditions on α and β will the group generated by reflections with respect to the three planes be finite?

(b) If these conditions are satisfied, how can one find the fundamental domain of this action?

5.2. Three straight lines L_1, L_2, L_3 in the Euclidean plane form a triangle with interior angles α, β, and γ.

(a) Under what conditions on α, β, γ will the group generated by reflections with respect to the three lines be discrete?

(b) If these conditions are satisfied, how can one find the fundamental domain of this action?

5.3. Consider the six lines $L_1, \ldots, L_6$ containing the six sides of a regular plane hexagon and denote by G the group generated by reflections with respect to these lines. Does this group determine a Coxeter geometry?

5.4. Let F be a Coxeter triangle, s_1, s_2, s_3 be the reflections with respect to its sides, and G_F the corresponding transformation group.

(a) Give a geometric description and a description by means of words in the alphabet s_1, s_2, s_3 of all the elements of G_F that leave a chosen vertex of F fixed.

(b) Give a geometric description and a description by means of words in the alphabet s_1, s_2, s_3 of all the elements of G_F which are parallel translations.

Consider the three cases of different Coxeter triangles separately.

5.5. Draw the Coxeter schemes of

(a) all the Coxeter triangles;

(b) all the three-dimensional Coxeter polyhedra.

5.6. Prove that all the edges at each vertex of any three-dimensional Coxeter polyhedron lie on three straight lines passing through that vertex.

5.7. Let $(F : G_F)$ be a Coxeter geometry of arbitrary dimension. Prove that

(a) if $s \in G_F$ is the reflection in a hyperplane P, then, for any $g \in G_F$, gsg^{-1} is the reflection in the hyperplane gP;

(b) any reflection from the group G_F is conjugate to the reflection in one of the faces of the polyhedron F,

5.8. Describe some four-dimensional Coxeter polyhedron other than the four-dimensional cube.

5.9. (a) Does the transformation group generated by the reflections in the faces of regular tetrahedron define a Coxeter geometry?

(b) Same question for the cube.

(c) Same question for the octahedron.

(d) Same question for the dodecahedron.