2019-3

Lecture 3 REFLECTIONS & COXETER GEOMETRIES

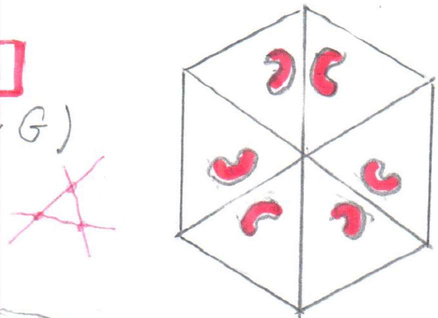
§1. The kaleidoscope ...

... is a children's toy $\boxed{\text{кал.: картинки}}$

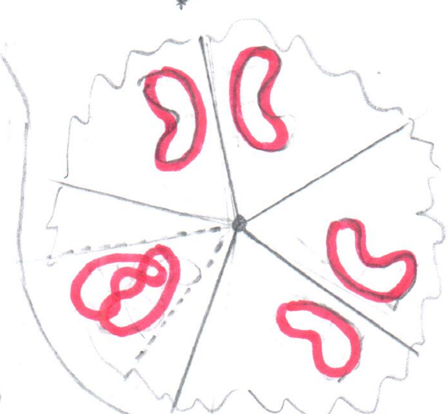
and an example of a Coxeter geometry $(X:G)$

$X=\mathbb{R}^2$, $G=$ group, generated by the reflections

with fundamental domain ▲



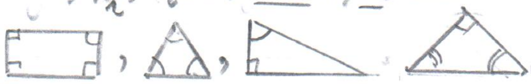
§2. Two-dimensional Coxeter geometries



Let $P\subset\mathbb{R}^2$ be a (convex) polygon and G_P be the group generated by the reflections in the sides of P s.t. P is a fundamental domain of G_P, i.e., • $g\neq id \Rightarrow Pg\cap P=\emptyset$

• $\bigcup_{g\in G}\overline{P}g=\mathbb{R}^2$. Then $(\mathbb{R}^2:G_P)$ is called the Coxeter geometry corresponding to P and P is a Coxeter polygon, provided all angles $=\pi/k_i$, $k_i\geq 3$. Example ▲

Theorem 1. There are four Coxeter polygons:



Proof. (1) All the angles are $\leq\pi/2$. (2) #vertices $=V\leq 4$ $(\Leftarrow(1))$

Case I: $V=4\Rightarrow P$ is a rectangle.

Case II: $V=3\Rightarrow \pi/k+\pi/l+\pi/m=\pi\Rightarrow \frac{1}{k}+\frac{1}{l}+\frac{1}{m}=1$, $2\leq k\leq l\leq m$

$m=2$ ✗; $m=3\Rightarrow P=\triangle$; $m=4\Rightarrow$ ◺ $\pi/4$; $m=5$ ✗;

$m=6\Rightarrow P=$ ◺ $\pi/6$; $m\geq 7\Rightarrow$ ✗

We obtain the four two-dimensional Coxeter geometries $\boxed{\text{Рис 5.2}}$ $(\mathbb{R}^2:G_P)$

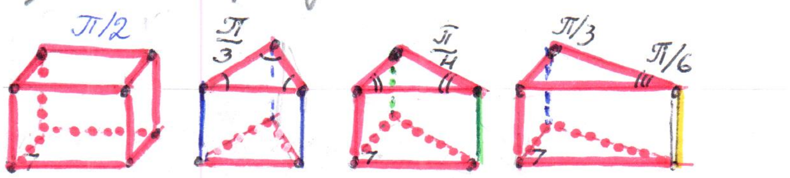
The group G_P acts transitively on P in all 4 cases

§3. Three-dimensional Coxeter geometries

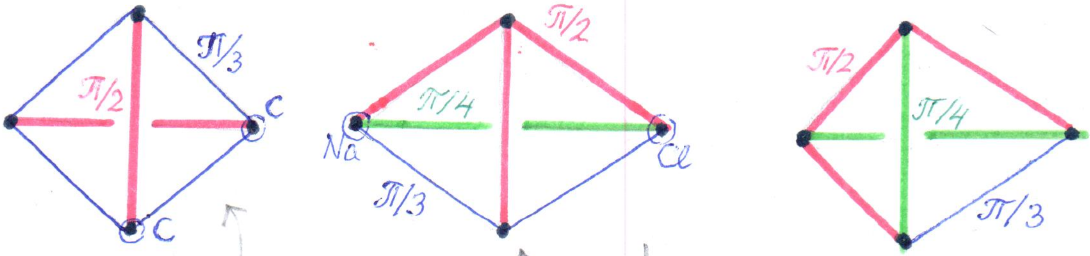
Let $Q\subset\mathbb{R}^3$ be a convex polyhedron all of whose angles between faces are of the form $\alpha_i=\pi/k_i$, $k_i\geq 3$ and let G_Q be the group generated by the reflections in the planes of the faces such that Q is a fund. domain of G_Q. Then Q is the 3-dimensional Coxeter polyhedron and $(\mathbb{R}^3:G_Q)$ is the Coxeter geometry corresponding to Q. Expl ☐ $=Q$

Theorem 2. There are $7(=4+3)$ Coxeter polyhedra.

Here are the 4 simple ones:



§3 (cont'd) The other three are: $\boxed{\text{3 тетраэдра}}$



They are diamonds and salt
Proof: linear algebra, Gramm matrix.

In higher dimensions — Coxeter polytopes

§4. Coxeter diagrams (схемы)

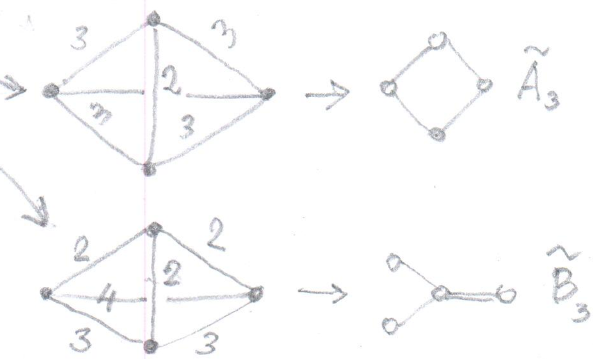
A Coxeter graph is a graph (with integer weights on edges) corresponding to a Coxeter polytope.

vertex = face, $[v, v']$ edge if v, v' have a common side;

m written on $[v, v']$ if angle between v and v' is π/m

Examples Diamond:

Salt



A Coxeter diagram is a modification of a Coxeter graph: instead of writing m, we draw $m-2$ lines