1. Let $\omega = (\omega_{ij})$ be a constant nondegenerate anti-symmetric $2n \times 2n$ matrix; we define the Weyl algebra as

$$W_\hbar(\mathbb{R}^n, \omega) = \mathbb{R}[\hbar]\langle y_1, \dots, y_{2n}\rangle / ([y_i, y_j] = \hbar\omega_{ij})$$

where $\mathbb{R}\langle y_1, \dots, y_{2n}\rangle$ stand for noncommutative polynomials and $\hbar$ is a formal varialble. Let $\sigma : \mathbb{R}[\hbar][y_1, \dots, y_{2n}] \to W_\hbar(\mathbb{R}^n, \omega)$ be the $\hbar$-linear map from the space of polynomials in variables $y_1, \dots, y_{2n}$ with coefficients in $\mathbb{R}[\hbar]$ into the Weyl algebra, given by

$$\sigma(y_{i_1} \dots y_{i_k}) = \frac{\hbar^k}{k!} \sum_{s \in S_k} y_{i_{s(1)}} \dots y_{i_{s(k)}}.$$

   (a) Prove, that σ is linear isomorphism.

   (b) Find an expression for the $\star$-product in $\mathbb{R}[\hbar][y_1, \dots, y_{2n}]$ induced by

$$f \star g = \sigma^{-1}(\sigma(f) \cdot \sigma(g)).$$

2. Use the integral formula

$$a \star b = \frac{1}{(2\hbar\pi)^{2n}} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} e^{\frac{1}{i\hbar}\omega(z-z', z-z'')} a(z')b(z'')dz'dz''$$

to prove the associativity of the Moyal $\star$-product in symplectic case.

3. Prove that differential 2-form ω on a compact $2n$-dimensional manifold is nondegenerate iff the form ω^n is non-vanishing in all points.

4. Show that all symplectic manifolds are orientable.

5. Let ξ be a vector field on a symplectic manifold M such that $\mathcal{L}_\xi \omega = 0$. Show that locally there exists a 1-form α on M such that $d\alpha = 0$ and $\xi = \pi^\sharp(\alpha)$ (here $\pi = \omega^{-1}$). Such fields are called *locally Hamiltonian* or *Poisson*.

6. * Show that if $H^1_{dR}(M) = 0$, where M is symplectic manifold, then every Poisson field on M is in fact Hamiltonian.

7. Show that the locally Hamiltonian fields on a symplectic manifold form a Lie subalgebra; moreover, for any two locally Hamiltonian fields ξ, η their commutator is in fact Hamiltonian with Hamiltonian function equal to $\omega(\xi, \eta)$.

8. Prove that the Schouten bracket, defined by explicit formula for decomposable polyvectrors (i.e. for polyvectors equal to wedge-product of vector fields) is well defined and verifies Jacobi identity.

9. Prove Lichnerowicz's formula:

$$\langle \omega, [P, Q] \rangle = (-1)^{(p-1)(q-1)} \langle d i_Q \omega, P \rangle - \langle d i_P \omega, Q \rangle + (-1)^p \langle d\omega, P \wedge Q \rangle,$$

where $\omega \in \Omega^{p+q-1}(M)$, $P \in \Lambda^p(M)$, $Q \in \Lambda^q(M)$, $\langle , \rangle$ is the natural pairing between forms and polyvector fields and $i_P \omega$ is the convolution of form and polyvector.

10. *Bernoulli numbers* B_n are usually defined by the formula:

$$\frac{t}{e^t - 1} = \sum_n B_n \frac{t^n}{n!}.$$

Prove recursive formula $\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$.

11. Prove that the bracket, which sends $a, b \in Gr_{\mathscr{F}}\mathcal{U}\mathfrak{g}$ to the image in $Gr_{\mathscr{F}}\mathcal{U}\mathfrak{g}$ of the commutator of $\bar{a}, \bar{b} \in \mathcal{U}\mathfrak{g}$ (where a and b are images of $\bar{a}$ and $\bar{b}$ under the natural projection $\mathcal{U}\mathfrak{g} \to Gr_{\mathscr{F}}\mathcal{U}\mathfrak{g}$) is well-defined and check the conditions of Poisson brackets for it.

12. * Open a book on homological algebra and read about the basic constructions, such as the exact sequences, chain maps, chain homotopies, 5-lemma, spectral sequences, etc.

13. * Compute the Lichnerowicz-Poisson cohomology of $\mathfrak{g}^*$ with usual Poisson structure.

14. * Compute the Lichnerowicz-Poisson cohomology of $\mathbb{R}^2$ with the Poisson structure, given by equation $\{x, y\} = x^2 + y^2$.