Lecture 2

The Conway polynomial

In this lecture, we define the Alexander–Conway polynomial, which is an isotopy invariant of knots and links, by means of three simple axioms due to John Conway. These axioms include the so-called Conway skein relation, an unusual geometric-combinatorics relation, which opened the way to the modern theory of knot invariants.

We then learn to calculate the values of the Alexander–Conway polynomial for concrete knots and links and see how well that polynomial distinguishes nonisotopic knots and nonisotopic links. It turns out that the Alexander–Conway polynomial is a strong, but not a complete invariant.

We conclude the lecture by sketching a proof of the fact that the Alexander–Conway polynomial actually exists, i.e., that there is a unique map ∇ of the set of (isotopy classes of) links to the ring $\mathbb{Z}[x]$ of one-variable polynomials with integer coefficients such that ∇ satisfies the three axioms.

Conway proved that such a polynomial $\nabla_L(x)$ exists and is unique by showing that the three axioms are satisfied by the polynomial $A_L(t) \in \mathbb{Z}[t, t^{-1}]$ originally defined by J.W. Alexander if one changes the variable according to the rule $x \leftrightarrow \sqrt{t} - 1/\sqrt{t}$. Alexander defined his polynomial $A_L(t)$ as an element of $\mathbb{Z}[t, t^{-1}]$ by a concrete rather sophisticated topological construction involving 1-homology groups and cyclic coverings. In this lecture, and in this course, we do not explain Conway's proof nor Alexander's definition – they are not elementary.

2.1. Axiomatic definition

If we want to prove that two knot (link) diagrams represent the same knot (link), it suffices to find a series of Δ-moves (or Reidemeister moves) taking one to the other. But what must we do to prove that two knot (link) diagrams represent different knots (links)? We must use an invariant. Here we shall use the Conway polynomial (Conway's version of the Alexander polynomial), defined as follows.

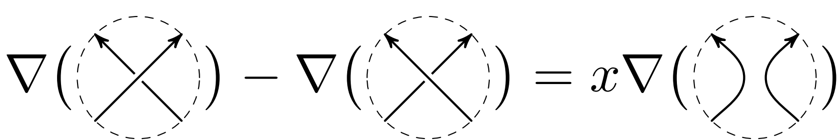
To each oriented link (in particular knot) diagram L, a polynomial with integer coefficients in the variable x, called the *Conway polynomial* of the link L and denoted by $\nabla(L)$ or $\nabla_L(x)$ is assigned; this assignment must satisfy to the three following conditions (*Conway axioms*):

(I) *Invariance*: If L is equivalent to L', then $\nabla(L) = \nabla(L')$.

(II) *Normalization*: $\nabla(\bigcirc) = 1$ for the unknot $\bigcirc$.

(III) *Skein relation*:

$$\nabla(L^+) - \nabla(L^-) = x \cdot \nabla(L^\circ)$$

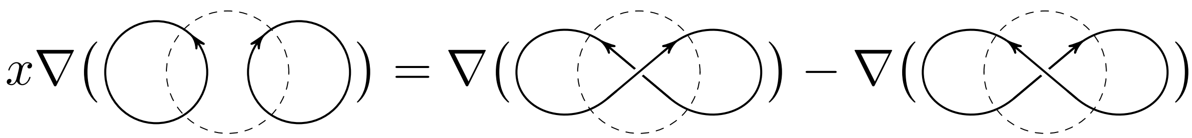


The above relation should be understood as follows: we are given three links L^+, L^-, L° that are identical outside the small disks bounded by the three dashed circles, inside which they are as shown in the picture, and their Conway polynomials satisfy the displayed relation.

It turns out that these three axioms are quite sufficient for calculating the values of the Conway polynomial of concrete knots and links.

2.2. Calculations

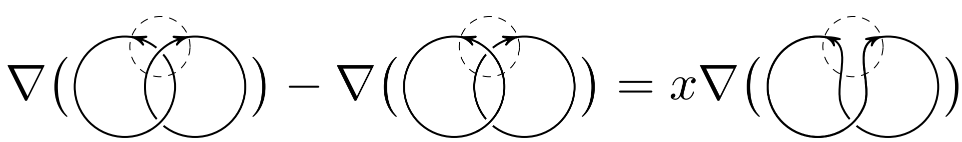
Let us calculate the Conway polynomial of the two-component trivial link ◯◯. Using (III), and then (I) and (II) twice, we obtain



$$x \cdot (?) \quad = \quad 1 \quad - \quad 1$$

Thus $\nabla($◯◯$) = 0$.

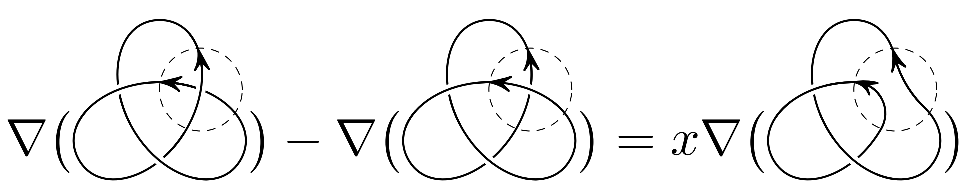
Now let us calculate the Conway polynomial of the right Hopf link, i.e., the Hopf link with oppositely oriented circles. We have



$$? \quad - \quad 0 \quad = \quad x \cdot 1$$

Thus $\nabla(\text{right Hopf link}) = x$.

Finally, let us calculate the Conway polynomial of the right trefoil. We have



$$? \quad - \quad 1 \quad = \quad x \cdot x$$

Thus $\nabla(\text{right trefoil}) = 1 + x^2$.

What do those calculations show? They show that the Hopf link cannot be unlinked (i.e., is not isotopic to the trivial two-component link) and that the right trefoil cannot be unknotted (i.e., is not isotopic to the unknot) *provided* that we know that an assignment $L \mapsto \nabla(L)$ satisfying axioms (I). (II), (III) exists and is unique. This will be proved in Sec. 2.3, and now we shall continue similar calculations assuming that this is the case.

The next link whose Conway polynomial we will find is the left Hopf link (Exercise 2.1), then we calculate that of the left trefoil (Exercise 2.2), obtaining $\nabla(\text{left trefoil}) = 1 + x^2$ and then of the eight knot (Exercise 2.3), $\nabla(\text{eight knot}) = 1 - x^2$.

These calculations show that the eight knot is not a trefoil, nor is it the unknot, and that the Conway polynomial does not distinguish the right trefoil from the left one. Actually, the two trefoils are not isotopic to each other – we shall prove this by using the Jones polynomial (Lecture 4). Thus we see that the Conway polynomial is not a complete invariant.

2.3. Existence of the Conway polynomial

We shall need need the following lemma, which will also be useful in other contexts, in particular in Lectures 3 and 4.

Lemma 2.1. *Any link diagram can be transformed into a trivial link diagram by an appropriate series of crossing changes.*

Proof. We shall prove the lemma for knots and leave the (easy) generalization to arbitrary links to the reader. Suppose we are given a knot diagram K. Let us choose an arbitrary point P on K and move along the knot in the direction of orientation until we reach the first crossing point 1; if 1 is an underpass for us, we do not make a crossing change at 1; if 1 is an overpass for us, we do make a crossing change at 1. We continue our motion until we come to a new crossing point 2 (it may happen that we will first come through 1 again if the knot has a little loop near 1, as in the Example in Fig. 2.1 (a)); if 2 is an underpass for us, we do not make a crossing change at 1, otherwise we do. We then go on to the third crossing point 3 and make (or do not make) a crossing change there according to the same rule as before, and continue further in the same way until we return to P. As the result, we obtain a new knot diagram K'. For the example in Fig. 2.1 (a), it is shown in Fig. 2.1 (b).

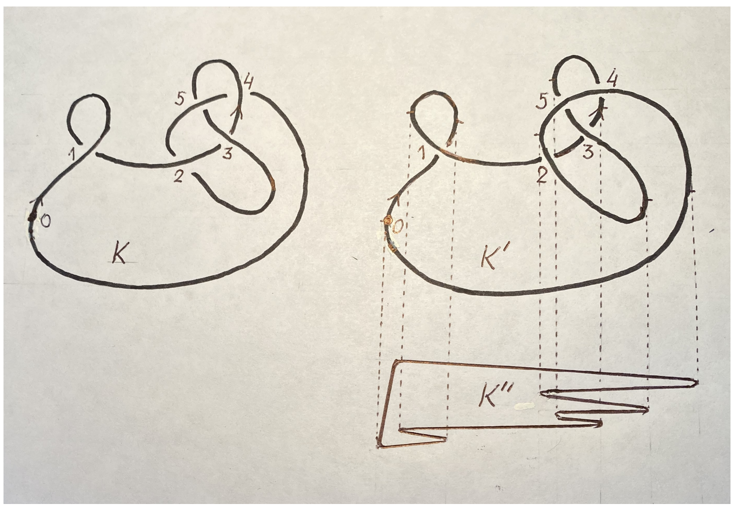


Figure 2.1. Trivializing a knot diagram by crossing changes

Now let us prove that K' is a diagram of the unknot. To do that we shall trace out a knot K'' in space (near the horizontal plane "almost containing" K') that will be obviously equivalent to K'. To do that, we start at P and move along and vertically above the curve K', uniformly rising upward very very slowly and as we go around K' until we come back to some point P' near P, and then move down to close up the curve at P.

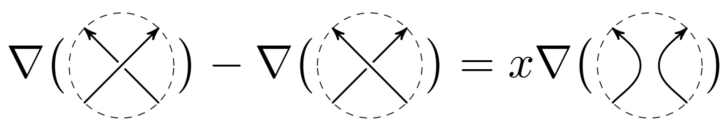
To prove that K'' (and hence K'!) is the unknot, it suffices to look at K'' from a point in the horizontal plane (from the "eye" in Fig.2.1(b)): we will then see a winding closed curve without self intersections. This proves the lemma.

Theorem 2.1. *There exists a unique assignment* $\nabla : \mathcal{L} \to \mathbb{Z}[x]$ *that takes any oriented link diagram* $L \in \mathcal{L}$ *to a polynomial in* x *with integer coefficients so that the following three axioms hold:*

I. *Invariance:* $\nabla(L) = \nabla(L')$ *if* L *is ambient isotopic to* L'.

II. *Normalization:* $\nabla\big(\bigcirc\big) = 1$, *where* $\bigcirc$ *denotes the unknot.*

III. *Conway skein relation:* $\nabla(L^+) - \nabla(L^-) = x\,\nabla(L^\circ)$, *or*



where L^+, L^-, L° *are identical link diagrams outside a small disk, inside which they are as pictured inside the dotted circles.*

Proof. First, assuming existence of an assignment $\nabla : \mathcal{L} \to \mathbb{Z}[x]$ staisfying axioms I, II, III, we will show by induction on n that for any link diagram L with $\leq n$ crossings, the polynomial $\nabla(L)$ is uniquely determined by the three axioms.

The *base of induction* is satisfied, because a link diagram L without crossings is either the unknot (in which case its polynomial is equal to 1 by axioms I and II) or a trivial m-component link with $m \geq 2$, in which case its polynomial is zero, as can be proved similarily to the case $m = 2$ considered above).

We assume (*induction hypothesis*) that for any link diagram L with $\leq n - 1$ crossing points $\nabla(L)$ is uniquely determined by the three axioms. Let us consider an arbitrary link diagram with n crossings and choose one of the crossings. Denote this link diagram by L^+ if the chosen crossing is positive and by L^- if it is negative. In either case, by the skein relation, we have

$$\nabla(L^+) - \nabla(L^-) = x\,\nabla(L^\circ), \tag{1}$$

where L^o denotes the corresponding link with $n-1$ crossings. Since we know the right-hand side (by the induction hypothesis) it remains to show that one of the two terms on the left-yand side is uniquely determined by the axioms.

To do that, we will show, for a fixed value of n, by induction on the unknotting number k (i.e., the smallest k such that the link can be trivialized by k crossing changes) that, for any link diagram L with n crossings and unknotting number $k \leq n$, the polynomial $\nabla(L)$ is uniquely determined by the axioms. When $k = 0$, this is true because the assertion is the same as the base of induction ($n = 0$) considered above. Let us assume that it is true for $k-1$ and prove it for k.

Consider a link diagram with n crossings and unknotting number k. Let us consider two cases. In the *first case*, let $k = 0$. Then the link L under consideration must be trivial; we then claim that $\nabla(L) = 1$ if L has one component and $\nabla(L) = 0$ if it has more than one. We have the following statement:

Fact. *If a trivial oriented link L has one component, then its Alexander polynomial, as well as its Conway polynomial, is equal to 1, and if it has more than one, its Alexander polynomial, as well as its Conway polynomial, is equal to 0.*

We do not prove this statement because it is based on the nonelementary definition of the Alexander polynomial (involving homology theory and a nontrivial topological construction) and the fact that the Conway polynomial can be defined via the Alexander polynomial by a change of variable.

Now let us consider the *second case*, i.e. let $k > 0$.

Then there is a crossing in the diagram at which the crossing change produces a link diagram with unknotting number $n-1$. If that crossing is positive, we denote our diagram by $\tilde{L}^+$, and by $\tilde{L}^-$ if it is negative. Then, in any case, by the skein relation, we have

$$\nabla(\tilde{L}^+) - \nabla(\tilde{L}^-) = x\,\nabla(\tilde{L}^\circ), \tag{2}$$

where $\tilde{L}^\circ$ is the corresponding link diagram with $n-1$ crossings. We know the value of the right-hand side (by the induction hypothesis for n) and so it suffices to show that one of the two terms on the left-hand side is determined by the axioms. But this follows from the induction hypothesis (for k this time!), since one of these terms (not the one containing the chosen link diagram, but the other one!) must have unknotting number $n-1$ (because of our special choice of crossing).

This concludes the induction on k and therefore the induction on n, concluding our proof of the theorem.

2.4. Chirality, orientation-reversal and multiplicativity

Theorem 2.2. (a) *The knot* $\overleftarrow{K}$ *obtained from an oriented knot* K *by reversing its orientation has the same Conway polynomial*:

$$\nabla(\overleftarrow{K}) = \nabla(K)$$

(b) *The mirror image* K^* *of an oriented knot* K *has the same Conway polynomial*:

$$\nabla(K^*) = \nabla(K).$$

The proof of (a) and (b) are the object of Exercises 2.8 and 2.9.

2.5. Knot tables

Because of the unique decomposition theorem into primes, in order to classify knots, it suffices to classify prime knots. This has been done for knots with 16 crossings or less, first by hand (for small crossing numbers), and then by computer. For a small number (≤ 10) of crossings, it is customary to classify knots by listing them (as pictures) in a *knot table* in increasing order of their crossing number. This is done in Rolfsen's classical (and beautiful) knot table (you can look it up in the internet). Here we present only a small table (7 crossings or less) in Fig.2.2.

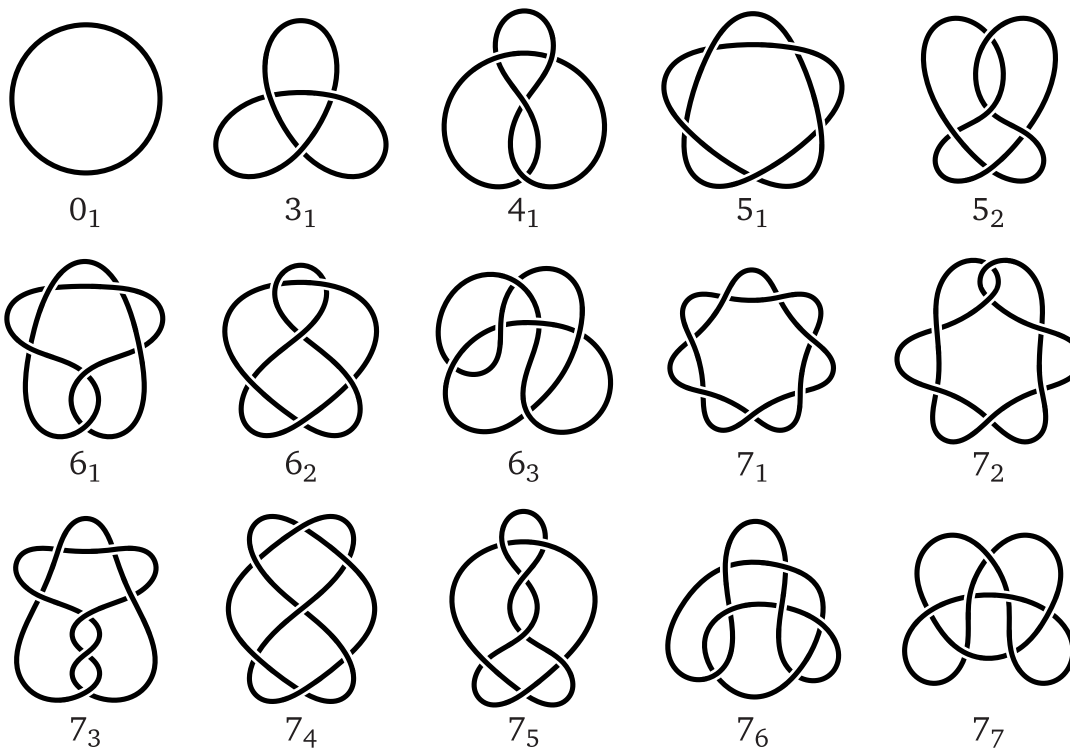


Figure 2.2. Knot table for knots with ≤ 7 crossings

Traditionally, only one picture of a knot that does not coincide with its mirror image is shown in the table. Thus the table in Fig.2.2 shows the left trefoil, but not the right trefoil.

2.6. Exercises

2.1. Compute the Conway polynomial of the left Hopf link.

2.2. Compute the Conway polynomial of the left trefoil. Is it the same as that of the right trefoil? Can you conclude that the right and left trefoils are isotopic knots?

2.3. Show that the figure eight knot is not isotopic to the trefoil by computing its Conway polynomial.

2.4. Is the granny knot isotopic to one of the trefoils? To the figure eight knot? To the unknot?

2.5. Using Reidemeister moves, show that reversing the orientation transforms the knot diagram of the right trefoil into an isotopic knot.

2.6. Show that the Conway polynomial of a link diagram consisting of two components located in different half planes is equal to zero.

2.7. Show that the Conway polynomial of the trivial m-component knot is zero.

2.8. Show that the Conway polynomial of a knot K does not change if we replace K by its inverse knot $\overleftarrow{K}$ (i.e., the knot obtained from K by reversing its orientation).

2.9. Show that the Conway polynomial of a knot K does not change if we replace K by its mirror image K^*.

2.10. Does the analog of Theorem 2.2 (a), (b) hold for links?