Lecture 4

The Jones Polynomial

The Jones polynomial was invented by Vaughan Jones in 1985. It is a powerful knot and link invariant. Unlike the Conway polynomial, it distiguishes a knot K from its mirror image K^* whenever K is not isotopic to K^*. However, it is not a complete invariant.

Jones' original definition was based on deep topological and algebraic constructions and facts, namely the Markov theorem on the closure of braids and the Ocneano trace in the Temperley-Lieb algebra. Our exposition, however, is elementary – follows the work of Louis Kauffman and is based on the Kauffman bracket.

4.1. Definition via the Kauffman bracket

Let L be an oriented link or knot. Any link diagram has a finite number of crossing points. In the definition of the Conway polynomial, we distinguish positive and negative crossings (shown in Fig. 4.1 (a) and (b), respectively).

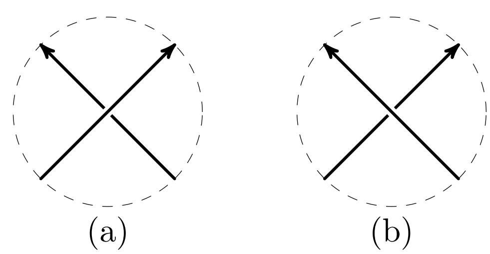


Figure 4.1. Positive and negative crossings

Using this distinction here, let us number the crossings points of L and set $\varepsilon(P_i) = +1$ if the ith crossing point P_i is positive and $\varepsilon(P_i) = -1$ if it is negative, and define the *writhe* $w(L)$ of an oriented link diagram L by setting

$$w(L) := \sum_{\{\text{crossings points } P_i\}} \varepsilon(P_i)$$

Lemma 4.1. *The writhe $w(\cdot)$ is Ω_2- and Ω_3-invariant. Under the Ω_1 move, it changes by -1 (resp. $+1$) when the disappearing crossing point is positive (resp. negative).*

The proof is the object of Exercise 4.6.

Given an oriented link diagram L, we denote by $|L|$ the same link, but without the orientation. Recall that in the previous lecture, we defined and learned to calculate the Kauffman bracket $\langle |L| \rangle$. We can now define the (preliminary version $J(\cdot)$ of) the *Jones polynomial* by setting

$$J(L) := (-a)^{-3w(L)} \langle |L| \rangle.$$

Thus the Jones polynomial assigns to any oriented link L a Laurent polynomial $J(L) \in \mathbb{Z}[a, a^{-1}]$.

Theorem 4.1. *The Jones polynomial $J(\cdot)$ (in its preliminary version) is an ambient isotopy invariant.*

Sketch of the proof. It suffices to check that $J(\cdot)$ is invariant w.r.t. the three Reidemeister moves. The fact that it is Ω_2- and Ω_3-invariant immediately follows from Lemma 4.1 and item (V) of Theorem 3.1. Its Ω_1-invariance is less obvious, and is the object of Exercise 4.3.

Remark 4.1. To pass from the preliminary version J of the Jones polynomial to its final (=usual) version V, it suffices to make the change of variables $a = q^{-1/4}$. We will do this a little later, after we have performed some calculations with the preliminary version in order to establish the main properties of the Jones polynomial. (These calculations are more conveniently performed with the variable a than with the variable q.)

4.2. Main properties of $J(\,\cdot\,)$

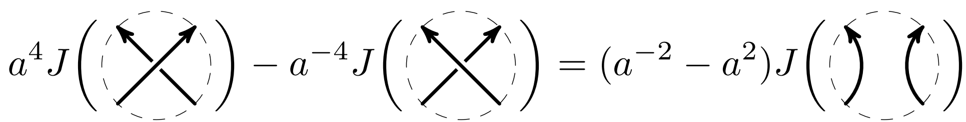
($\mathbf{I}_J$) *Normalization*: $J(\bigcirc) = 1$.

This immediately follows from item (I) of Theorem 5.1.

($\mathbf{II}_J$) *Skein relation for the Jones polynomial*:

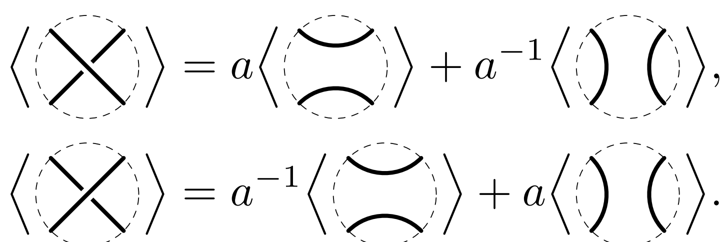
$$a^4 J(L^+) - a^{-4} J(L^-) = \big(a^{-2} - a^2\big) J(L^\circ),$$

or in standard symbolic form

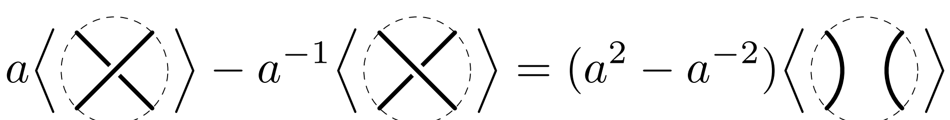


where L^+, L^-, L° are three oriented links, identical outside three little disks, and are as shown inside the disks.

To prove this, we begin by writing out the skein relation for the Kauffman bracket twice



then multiply the first equality by $-a^{-1}$, the second by a, and add the resulting equations, obtaining



Now let us denote by L^+, L^-, L° the oriented links obtained from the nonoriented links in the above relation by orienting them as shown in Fig. 4.2.

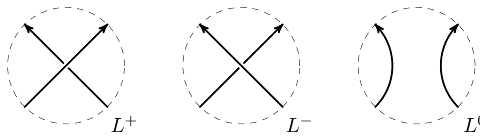


Figure 4.2. The oriented links L^+, L^-, L°

We then have

$$a\langle|L^+|\rangle - a^{-1}\langle|L^-|\rangle = \big(a^2 - a^{-2}\big)\langle|L^\circ|\rangle.$$

By definition of the writhe, we have $w(L^\pm) = w(L^\circ) \pm 1$, and recalling the definition of the polynomial $J(\cdot)$, we obtain

$$a(-a^3)J(L^+) - a^{-1}(-a)^{-3}J(L^-) = (a^2 - a^{-2})J(L^\circ),$$

which is exactly the required relation.

(III$_J$) *Adding an unknot*:

$$J\big(L \sqcup \bigcirc\big) = -\big(a^2 - a^{-2}\big)J(L).$$

The proof of this statement is the object of Exercise 4.4.

4.3. Axioms for the Jones polynomial

We now make the substitution $a = q^{-1/4}$, obtaining the *Jones polynomial* in its usual form

$$V(L) :\ = J(L)\big|_{a=q^{-1/4}}.$$

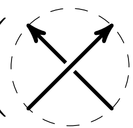
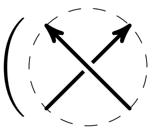
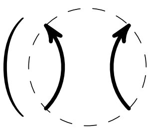
The main properties of the Jones polynomial $V(\cdot)$ immediately follow from the corresponding properties of the polynomial $J(\cdot)$ proved above.

As we shall soon see, these properties may be regarded as axioms for the Jones polynomial.

(0) *Invariance*: *The Jones polynomial* $J(L) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ *of any oriented link (in particular, of any oriented knot) L is an ambient isotopy invariant.*

(I) *Normalization*: $J(\bigcirc) = 1$.

(II) *Skein relation for the Jones polynomial*:

$qV\Big($  $\Big) - q^{-1}V\Big($  $\Big) = \Big(\frac{1}{\sqrt{q}} - \sqrt{q}\Big)V\Big($  $\Big)$

(III) *Adding an unknot*:

$$V\big(L \sqcup \bigcirc\big) = -\Big(\frac{1}{\sqrt{q}} + \sqrt{q}\Big)V(L).$$

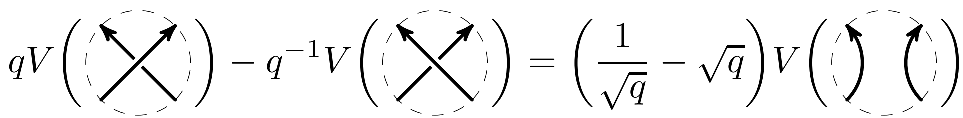
Theorem 4.2. *The Jones polynomial $J(\cdot)$ is an ambient isotopy invariant of oriented links satisfying axioms* (I), (II), (III) *and is uniquely determined by these axioms.*

Proof. The *existence* of the polynomial $J(L)$ satisfying axioms (I)-(III) was proved above. Let us prove *uniqueness*.

We shall prove this by induction on the number k of crossings. If $k = 0$, then L is a trivial link, say with m components. In that case, its Jones polynomial can be computed by successively applying property (III_J). Its actual value is not important for the proof – it is the object of Exercise 4.5.

Assuming uniqueness for $k < n$, let us prove it for $k = n$. We shall prove this for a fixed n by induction on the number l of crossing changes needed to trivialize the given link diagram L (such a finite number exists by Lemma 2.1). If $k = 0$, then L is a trivial link, and we know what $J(L)$ is equal to. So we assume that uniqueness has been proved for links that can be trivialized by l crossing changes. Let us prove it for $l+1$.

Applying axiom (II) to any crossing point of the given link L, we can write



Of the two summands on the left-hand side, at least one can be trivialized by l crossing changes, and so its value is well defined by the induction hypothesis on l. The link diagram appearing in the right-hand side has $n-1$ crossing points, and so the value of the right-hand side is known inductively. Therefore, we can calculate the unknown term in the left-hand side and thus obtain the value of the Jones polynomial of any link diagram with n crossings that can be trivialized by $l+1$ crossing changes. Thus induction on k and l concludes the proof of Theorem 4.2.

4.4. The knot semigroup

There is a natural binary operation in the set of (isotopy classes of) knots known as the *connected sum*, denoted by # and defined as shown in Fig. 4.3.

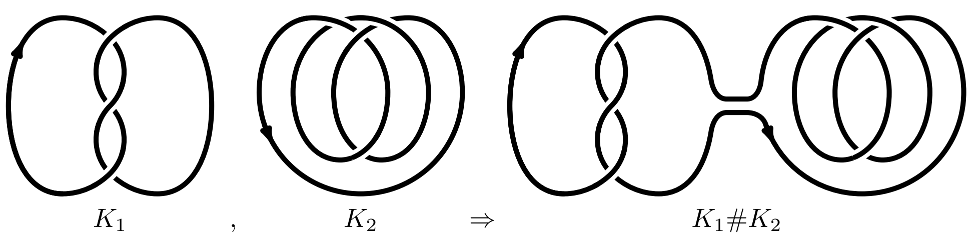


Figure 4.3. Connected sum of oriented knots

It can be shown (by a mildly difficult geometric argument) that the connected sum operation is well defined. A knot K is called *prime* if

$$K = K_1 \# K_2 \implies K_1 = \bigcirc \text{ or } K_2 = \bigcirc.$$

Theorem 4.3. *The connected sum operation supplies the set of isotopy classes of knots with the structure of commutative semigroup without inverse elements:*

$$K \neq \bigcirc, K\#K' = \bigcirc \implies K = K' = \bigcirc,$$

and with unique prime decomposition property:

$$K \neq \bigcirc \implies \exists!\{P_1, \ldots, P_n\}, \quad K = P_1\# \ldots \#P_n,$$

where all the P_i are prime.

4.5. Multiplicativity

The Jones polynomial behaves very nicely w.r.t. the connected sum operation for knots, and there is nice formula for the Jones polynomial for the disjoint union of two links (by the disjoint union of two links, one means the link obtained by placing the given two links in different half spaces and taking their union). Indeed, we have the following theorem.

Theorem 4.4. (a) *With respect to the connected sum operation for knots, the Jones polynomial is multiplicative, i.e.,*

$$V\big(K_1\#K_2\big) = V\big(K_1\big) \cdot V\big(K_2\big).$$

(b) *With respect to the disjoint union of links, the Jones polynomial behaves as follows:*

$$V\big(L_1 \sqcup L_2\big) = \left(\frac{1}{\sqrt{q}} - \sqrt{q}\right) V\big(L_1\big) \cdot V\big(L_2\big).$$

The proof of this theorem is the object of Exercise 4.7.

Remark 4.2. There is no well-defined connected sum operation for multicomponent links, because there is no prefered way to choose the components that we are to connect. Nevertheless, we will use the notation $L_1 \# L_2$ to indicate *any one of the possible ways to connect the two links* L_1 *and* L_2 (by specifying one component in each link). In that notation, we have

$$V\big(L_1 \# L_2\big) = V\big(L_1\big) \cdot V\big(L_2\big).$$

The proof of this formula is the object of Exercise 4.6.

4.6. Chirality and reversibility

Theorem 4.5. (a) *Reversing the orientation of all the components of a link diagrams does not change its Jones polynomial.*

(b) *The Jones polynomial of the mirror image of a link diagram* L *is obtained from the Jones polynomial of* L *by substituting* q^{-1} *for* q.

The proofs of the two assertions of the theorem are the object of Exercises 4.13 and 4.14, respectively.

4.7. Is the Jones polynomial a complete invariant?

The Jones polynomial, unlike the Conway polynomial, distinguishes the left trefoil from the right trefoil, as the reader can verify by doing Exercise 4.2. Is it a complete invariant, i.e., do we have the implication

$$V(L = V(L')) \implies L \sim L'\,?$$

The answer is “no” – a simple counterexample, obtained by taking connected sums of links in different ways, appears in Exercise 4.9.

Vaughan Jones conjectured that his polynomial is a complete invariant for *prime knots*, but his conjecture was very quickly refuted by several knot theorists. A counterexample appears in Fig.4.8.

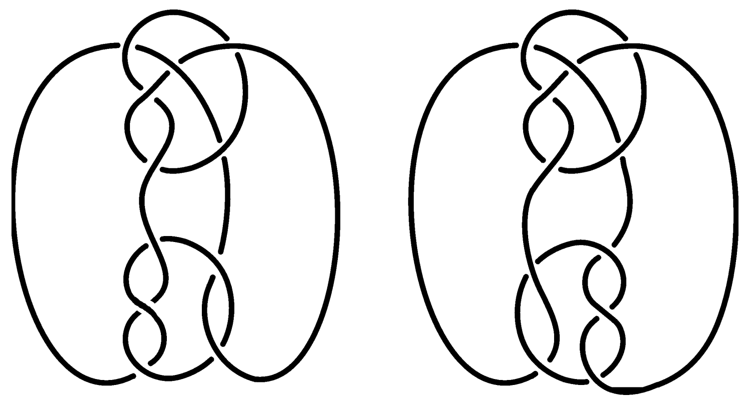


Figure 4.8. Nonequivalent knots with same Jones polynomial

The proof of the fact that the two knots have the same Jones polynomial is the object of Exercise 4.8. The fact that the two knots are not equivalent will not be proved in this course.

4.8. Is V a Laurent polynomial in q?

The answer is given by the following theorem:

Theorem 4.5. (a) *If the number of components of an oriented link L is odd (in particular if it is a knot), then $V(L) \in \mathbb{Z}[q, q^{-1}]$, i.e., $V(L)$ contains only terms of the form q^k, $k \in \mathbb{Z}$.*

(b) *If the number of components of an oriented link L is even, then $V(L)$ contains only terms of the form $q^{(2k+1)/2}$, $k \in \mathbb{Z}$.*

Proof. We know from Exercise 4.5 that the Jones polynomial of the trivial m-component oriented link is given by

$$\big(-q^{-1/2} - q^{1/2}\big)^{m-1} = q^{(m-1)/2}\big(-q^{-1} - 1\big)^{m-1},$$

so that both statements of the theorem hold for trivial m-component links. We also know (from the proof of Theorem 4.2) that $V(L)$ can be computed from V of the trivial link by successively applying the Jones skein relation (II). Let us denote by m_+, m_-, m_0 the number of components of L^+, L^-, L^0, respectively. To prove the theorem, it suffices to show that

(1) the numbers m_+ and m_- have the same parity;

(2) the numbers m_+ and m_0 have opposite parities.

Statement (1) is obvious (actually $m_+ = m_-$). To prove (2), it suffices to show that $m_+ = m_0 \pm 1$. The proof of that equality is the subject of Exercise 4.10.

4.9. Knot tables revisited

As we mentioned in Lecture 2, because of the unique decomposition theorem into primes, in order to classify knots, it suffices to classify prime knots. For knots with a small number of crossings, say ≤ 9, prime knots (represented by their knot diagrams) are listed in knot tables. In these tables, only one picture of a knot that does not coincide with its mirror image is usually shown. The Jones polynomial has been calculated for all these knots, and it turned out that the Jones polynomials of any pair of different prime knots with ≤ 9 crossings are different, which means that

the Jones polynomial classifies prime knots with ≤ 9 *crossings*,

so that Jones' conjecture is true for these prime knots.

For larger values of the crossing number, this is no longer true, as we saw above (Fig. 4.8). The number of knots with given crossing number c grows exponentially with c, in particular, there are 3 prime knots with 6 crossings, 7 prime knots with 7 crossings (as can be seen in the knot table in Fig. 4.6), 166 with 10 crossings, and 1 388 705 with 16 crossings.

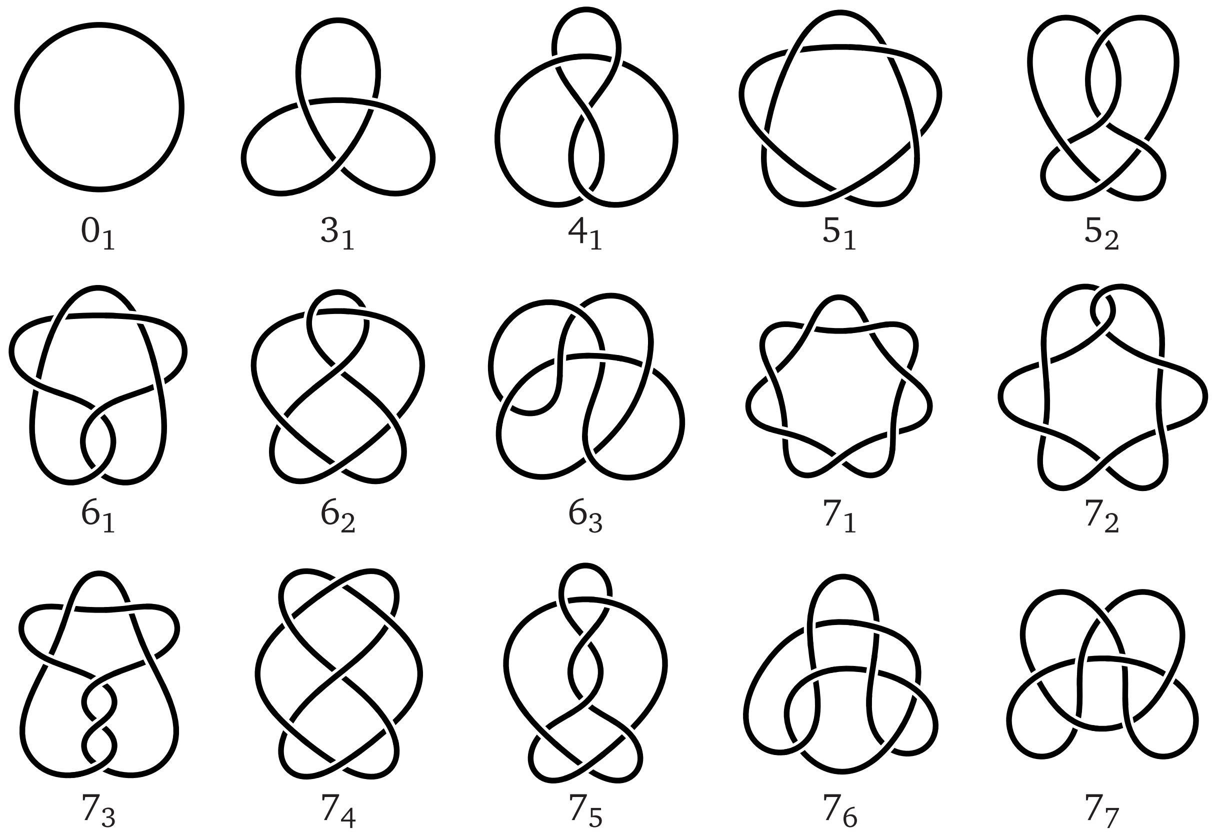


Figure 4.6. Knot table for knots with ≤ 7 crossings

For this reason, for a large number of crossings, the computer is used to produce knot tables. In them, knots are not presented as pictures, but by a special numerical encoding known as the Dowker–Thislethwaithe code of the knot.

To generate the Dowker–Thistlethwaite code, move along the knot using an arbitrary starting point and direction. Label each of the n crossings with the numbers $1, \ldots, 2n$ in order of traversal (each crossing is visited and labelled twice), with the following modification: if the label is an even number and the strand followed is an overcrossing, then change the sign on the label to be a negative. When finished, each crossing will be labelled a pair of integers, one even and one odd. The Dowker—Thistlethwaite notation notation is the sequence of even integer labels associated with the labels $1, 3, \ldots, 2n-1$ in turn.

For example, a knot diagram may have crossings labelled with the pairs $(1, 6), (3, -12), (5, 2), (7, 8), (9, -4), (11, -10)$. Then the Dowker–Thistlethwaite code for this labelling will be the sequence: $6, -12, 2, 8, -4, -10$ (or any of its cyclic permutations).

It can be shown that any prime knot is uniquely determined by its Dowker—Thistlethwaite code. The current software that produces knot tables is called “Knotscape” and is due to Thislethwaite, Weeks, and Hoste.

There are also tables of prime links (their definition is the object of Exercise 4.11) for links with 13 crossings compiled by Thislethwaite. The reader can try to construct prime link tables for links with a small number of crossings (Exercise 4.12).

4.9. Exercises

4.1. Prove Lemma 3.1 in the general case of multicomponent links.

4.2. Compute the (preliminary versions of) Jones polynomials of the two trefoils. Are they isotopic?

4.3. Prove the Ω_1-invariance of the Jones polynomial $J(\cdot)$ (preliminary version).

4.4. Prove property (III_J) of the Jones polynomial J.

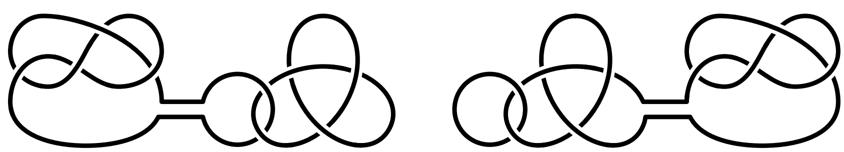
4.5. Compute the Jones polynomial of the trivial m-component link and show that it equals $\big(-q^{-1/2} - q^{1/2}\big)^{m-1}$.

4.6. Prove the formula in Remark 4.2.

4.7. Prove Theorem 4.2. (*Hint*: use the definition of the Jones polynomial via the Kauffman bracket and the state sum definition of the Kauffman bracket.)

4.8. Show that the two knots in Fig.5.4 have the same Jones polynomial. *Hint:* You need not explicitly compute the Jones polynomials of the two knots – it suffices to order the crossing points to which you apply the skein relation so that it yields the same result at each step.

4.9. Show that the two links in the figure below have identical Jones polynomials, but are not ambient isotopic.



4.10. Show that $m_+ = m_0 \pm 1$ in the proof of Theorem 6.5.

4.11. Define the notion of prime link and give examples (if any) of prime links with 3 and 4 crossings.

4.12. Compile a table of prime links with ≤ 5 crossings.

4.13. Prove that the Jones polynomial of the mirror image L^* of a link diagram L is obtained from the Jones polynomial of L by replacing q by q^{-1}.

4.14. Prove that the Jones polynomial of an oriented link is unchanged if the orientations of all its components are reversed.

4.15. (a) Compute the Jones polynomial of the eight knot in two ways: by using the definition (via the Kauffman bracket) and by using the axioms of the Jones polynomial (including the Jones skein relation).

4.16. Draw a picture of the knot with Dowker—Thistlethwaite code $6, -12, 2, 8, -4, -10$.

4.17. Write down the Dowker—Thistlethwaite code of the eight knot.