REPORT ON THE YOUNG MATHEMATICAL CONTEST – IUM FELLOWSHIP 2019

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Results

Scientific results of 2019

(1) Classical Schur-Weyl duality theorem states that there is a duality (equivalence) between the category of $\mathbb{S}_n$ representations and polynomial representations of the linear group GL_m of degree n (whenever $m \geq n$). We prove the current anlogue of this famous theorem. That is, we explain an equivalence of the category of graded finitely generated $\mathbb{S}_n \ltimes \mathbb{C}[t_1, \dots, t_n]$-modules and the Serre subcategory of the $\mathfrak{gl}_m[t]$-modules that are polynomial GL_m-modules of degree n. In particular, we prove that both categories are Highest weight categories. The proof is based on a decomposition of the bimodule $V[t]^{\otimes n}$ into the multiplicity free sum of cyclic modules. Moreover, we also prove the analogue of the Howe duality that deals with the decomposition of the module $S(V \otimes W \otimes \mathbb{C}[t])$ and the current analogue of the Peter-Weyl theorem.

(2) The manifold $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$ of real points of the moduli spaces of stable rational curves with $n+1$-marked points is known to be a $K(\pi, 1)$ space, whose fundamental group is called *pure cacti group*. We prove that the rational cohomology of these spaces (as well as of the pure cacti groups) are quadratic Koszul algebras for all n. In order to prove the Koszulness that remains open for almost 15 years we recognize the structure of an operad that assembles $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$ for all n. Moreover, we describe different algebraic models of the corresponding operads. We apply similar methods in order to prove that the space of n-ary operations of any Hopf operad generated by a single (skew)-symmetric element is also a quadratic and Koszul algebra. We use the theory of Gróbner bases for operads in order to find monomial bases for these algebras.

(3) The structure of quadratic Poisson structure on the set of functions on a given (flat) manifold X. In particular, this structure can be considered as a representation of a quadratic properad and even as a representation of a quadratic wheeled properad if the underlying space X is finitedimensional. We prove that the corresponding properad is Koszul and showed that the derivation complex is quasiisomorphic to the associated graded complex of the graph complex with respect to certain filtration. In particular, they are bounded from below by the graph cohomology that are known to be infinite.

(4) We continue to work with the generalization of the PBW property for a morphism of operads. In particular, we prove different PBW theorems dealing with universal envelope of algebras over Koszul operads. Moreover,

we formulate a notion of a derived PBW property that happens in the case of A_∞ univeral envlope of an L_∞-algebra. We explain why all known definitions of evlopes are A_∞-quasiisomorphic

Comparison with the proposal

I would like to thank Young Math Contest for support that increases the quality of my working conditions and allows me to achive some progress in all directions mentioned in my application for this contest. In particular, as promised, I proved

- the coformality of the framed little discs operad and described several combinatorial models of these operads that helps us to explains the deformation theory of these operads in arXiv:1705.08108.
- almost all known conjectures concerning the pure Cacti groups and on the cohomology of the real locus of the moduli spaces of stable rational curves with marked points in arxiv:1905.04499. Moreover, I described a combinatorial model for this operad and showed that the subcomplex of a graph complex of even loop order is responsible for the deformation theory of the latter operad.
- the relationship of the Grönthendieck-Teichmüller Lie algebra and deformations of quadratic Poisson structures that leads to additional understanding of existing and new cocycles in graph complex (upcoming preprint with S.Merkulov).

Papers

[1] "Quadratic Algebras arising from Hopf operads generated by a single element"
`math.arXiv`:1907.05573 submitted to Letters in Mathematical Physics

The operads of Poisson and Gerstenhaber algebras are generated by a single binary element if we consider them as Hopf operads (i.e. as operads in the category of cocommutative coalgebras). In this note we discuss in details the Hopf operads generated by a single element of arbitrary arity. We explain why the dual space to the space of n-ary operations in this operads are quadratic and Koszul algebras. We give the detailed description of generators, relations and a certain monomial basis in these algebras.

[2] with Thomas Willwacher
`math.arXiv`:1905.04499 submitted to Journal of EMS

We give a description of the operad formed by the real locus of the moduli space of stable genus zero curves with marked points $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$ in terms of a homotopy quotient of an operad of associative algebras. We use this model to find different Hopf models of the algebraic operad of Chains and homologies of $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$. In particular, we show that the operad $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$ is not formal. The manifolds $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$ are known to be Eilenberg-MacLane spaces for the so called pure Cacti groups. As an application of the operadic constructions we prove that for each n the cohomology ring $H^\bullet(\overline{\mathcal{M}_{0,n+1}}(\mathbb{R}), \mathbb{Q})$ is a Koszul algebra and that the manifold $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$ is not formal but is a rational $K(\pi, 1)$ space. We give the description of the Lie algebras associated with the lower central series filtration of the pure Cacti groups.

[3] with Evgeny Feigin and Ievgen Makedonskyi
`math.arXiv`:1906.03290 submitted to Compositio

The classical Peter-Weyl theorem describes the structure of the space of functions on a semi-simple algebraic group. On the level of characters (in type A) this

boils down to the Cauchy identity for the products of Schur polynomials. We formulate and prove the analogue of the Peter-Weyl theorem for the current groups. In particular, in type A the corresponding characters identity is governed by the Cauchy identity for the products of q-Whittaker functions. We also formulate and prove a version of the Schur-Weyl theorem for current groups. The link between the Peter-Weyl and Schur-Weyl theorems is provided by the (current version of) Howe duality.

[4] with Pedro Tamaroff

to appear

We define derived Poincaré–Birkhoff–Witt maps of dg operads —or derived PBW maps, for brevity. These extend the definition of PBW maps of operads of P. Tamaroff and V. Dotsenko, and show that the map from the homotopy Lie operad to the homotopy associative operad is derived PBW.

As an application, we show that three different known definitions of universal envelope of an L_∞-algebra invented by V.Baranovsky, J.Moreno-Fernandez and M.Markl-T.Lada are all A_∞-quasiisomorphic.

Scientific conferences and seminar talks

[1] Workshop "Props, graph complexes and moduli spaces" held at the University of Luxembourg http://math.uni.lu/propmoduli/programme.html

talk "Cacti groups, real locus of Deligne-Mumford compactification of $\overline{M_{0,n}}$".

[2] Conference "Graph complexes in algebraic geometry and topology" held at the university of Manchester: http://ibykus.sdf.org/graphc/index.html

talk "Real locus of $\overline{M_{0,n}}$, cacti groups and single generated Hopf operads"

Teaching

[1] Basic algebra for I year bachelor students, Independent university of Moscow, fall 2019, 2hours lecturs + 2 hours exercises class per week Program include:

- Commutative algebraic structures: rings, fields, abelian groups;
- Vector spaces, System of linear equations, linear maps;
- Groups, group action, orbits, stabilizers, normal subgroups;
- p-Groups, Sylov's theorems;
- Jordan-Hölder decomposition, simple, solvable and nilpotent groups;
- Dual vector space, bilinear forms, orthogonal basis;
- multilinear forms and determinants.

[2] Introduction to Commutative Algebra, NRU Higher School of Economics, III-IV year bachelor students and master students, Spring 2019, 2 hours lectures + 2 hours seminars per week.

Program includes the following subjects

- Rings, algebras, ideals and modules
- Noetherian rings
- Unique factorization domains
- Rings and modules of fractions
- Integral dependence and Noether's normalization theorem
- The going-up and going-down theorems
- Limits, colimits and tensor product
- Flat and projective modules
- Hilbert Nullstellensatz
- The spectrum of the ring

- Krull dimension and transcendence degree
- Primary decomposition
- Discrete valuation rings and Dedekind domains
- Dimension theory for noetherian rings
- Hilbert series

[3] Additional chapters of Algebra, NRU Higher School of Economics, II-III year bachelor students, Spring 2019, 2 hours seminars per week (lectures are given by L.Rybnikov).
Program includes the following subjects:

- Principal ideal domains; factorian domains; rings of fractions; Gauss theorem;
- Resultant and discriminant; Bezout theorem;
- Modules over rings, Jordan-Hölder and Krull-Schmidt theorems;
- Modules over the rings of principal ideal domain;
- integral extensions and algebraic integers;
- Noetherian rings; Hilbert's basis theorem and Hilbert's theorem on invariants; Hilbert Nullstellensatz.
- modules over the algebras and Schur's lemma.
- double centralizer theorem and classification of semisimple algebras.
- Representations of finite groups, Maschke's theorem, characters.
- Induced representations and Frobenius reciprocity;
- Representations of symmetric groups and Schur-Weyl duality.

[4] Standard basic algebra classes for I year bachelor students, fall 2019, 2 hours seminars per week, NRU Higher School of Economics

[5] Standard basic algebra classes for II year bachelor students, fall 2019, 2 hours seminars per week, NRU Higher School of Economics

[6] I am currently a scientific supervisor at NRU HSE of

- 4 bachelor students of the forth grade,
- 2 bachelor student of the third grade,
- 1 bachelor student of the second grade,
- 1 master student,