Conformal invariance in the critical Ising model: correlations and interfaces

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joint project with *C.Hongler* (Geneva → New York),
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& *S.Smirnov* (Geneva & St.Petersburg)

"Random Processes, Conformal Field Theory and Integrable Systems"

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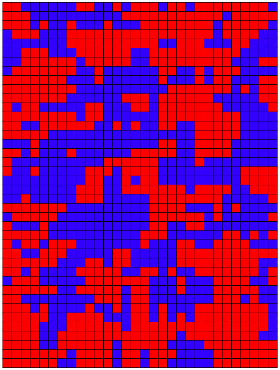
2D Ising model:
(square grid)



Spins $\sigma_i = +1$ or -1.

Hamiltonian:

$$H = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \, .$$

Partition function:

$$\mathbb{P}(conf.) \sim e^{-\beta H} \sim x^{\# \langle +- \rangle},$$

where

$$x = e^{-2\beta} \in [0, 1] \, .$$

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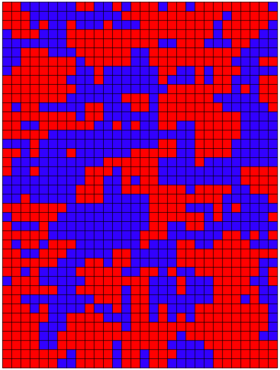

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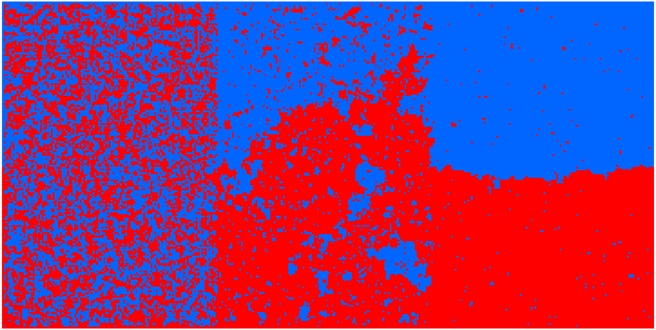
Other "lattices" (planar graphs): $H = -\sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j \,.$

$$\mathbb{P}(conf.) \sim \prod_{\langle ij \rangle : \sigma_i \neq \sigma_j} x_{ij}, \quad x_{ij} \in [0,1].$$

Phase transition, criticality:



$x > x_{\mathrm{crit}}$ $\qquad$ $x = x_{\mathrm{crit}}$ $\qquad$ $x < x_{\mathrm{crit}}$

(Dobrushin boundary values: two marked points a, b on the boundary; $+1$ on the arc (ab), -1 on the opposite arc (ba))

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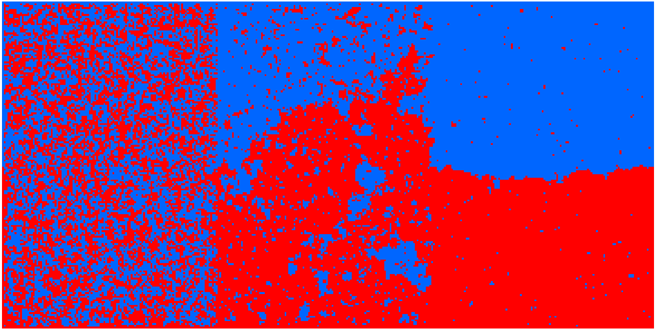

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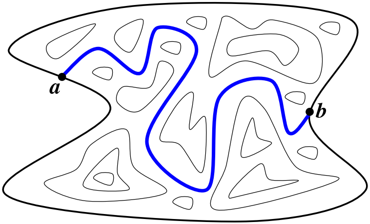
[Peierls '36; Kramers-Wannier '41]: $x_{\text{crit}} = \frac{1}{\sqrt{2}+1}$

Conformal invariance
(in the scaling limit):



Geometry:

single interface,
the whole loop ensemble

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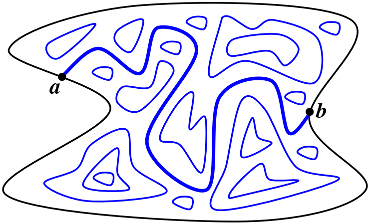

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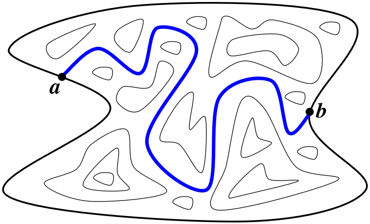

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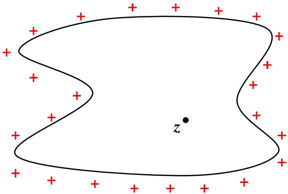

spin correlations, "boundary change operators", energy density, *fermionic observables*

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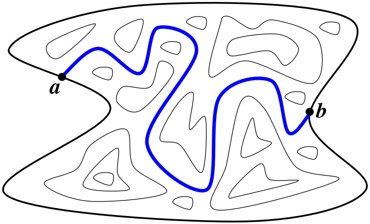

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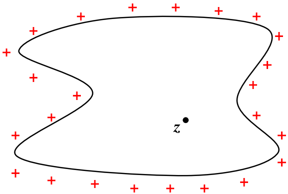

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$$\langle \sigma(z) \rangle_+^{\Omega} := \lim_{\delta \to 0} \delta^{-\frac{1}{8}} \mathbf{E}_+^{\Omega^\delta}[\sigma(z^\delta)]$$

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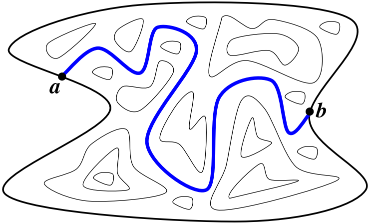

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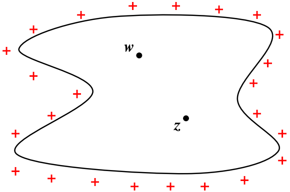

spin correlations, "boundary change operators", energy density, *fermionic observables*

$$\frac{\langle\sigma(z)\sigma(w)\rangle_+}{\langle\sigma(z)\rangle_+\langle\sigma(w)\rangle_+} := \lim_{\delta\to 0}\frac{\mathbf{E}_+[\sigma(z^\delta)\sigma(w^\delta)]}{\mathbf{E}_+[\sigma(z^\delta)]\mathbf{E}_+[\sigma(w^\delta)]}$$

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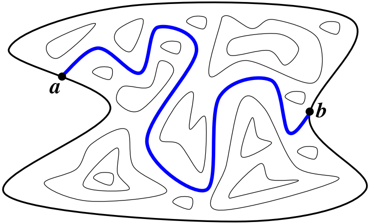

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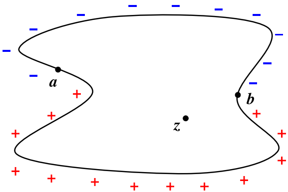

spin correlations, "boundary change operators", energy density, *fermionic observables*

$$\frac{\langle \sigma(z) \rangle_{ab}}{\langle \sigma(z) \rangle_{+}} := \lim_{\delta \to 0} \frac{\mathbf{E}_{ab}[\sigma(z^{\delta})]}{\mathbf{E}_{+}[\sigma(z^{\delta})]}$$

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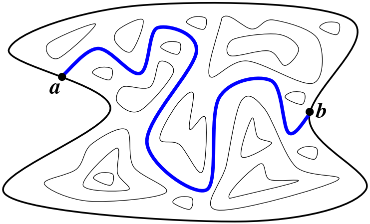

Theorem: (Smirnov-Ch., $\sim$08-10) *Let, as $\delta \to 0$, discrete domains $(\Omega^\delta; a^\delta, b^\delta)$ approximate a simply-connected domain $(\Omega; a, b)$. Then the corresponding (random) discrete interfaces γ^δ converge to the (random) conformally invariant curves* $\mathbf{SLE_3(\Omega; a, b)}$.

Remarks: (i) $\mathrm{SLE}_\varkappa, \varkappa \geqslant 0$ (Stochastic Loewner Evolution or Schramm-Loewner Evolution) is the one-parameter family of random conformally invariant curves introduced by O.Schramm. They are constructed dynamically in the half-plane $(\mathbb{C}_+; 0, \infty)$ via the classical Loewner equation with the random driving force $\sqrt{\varkappa}B_t$

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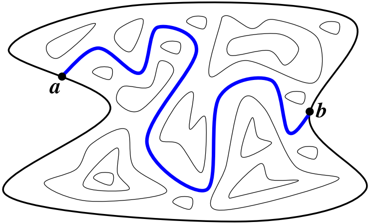

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Remarks: (ii) We firstly prove convergence to the conformally covariant limits of the so-called *basic fermionic observables* $F^\delta_{(\Omega^\delta; a^\delta, b^\delta)}(z^\delta)$ which are discrete **holomorphic** (in z^δ) **functions** having the (discrete) **martingale property** w.r.t. the growing interface (for any fixed z^δ). Then, we identify the limiting law with SLE_3 using the so-called *conformal martingale principle*.

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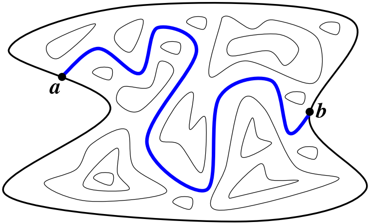

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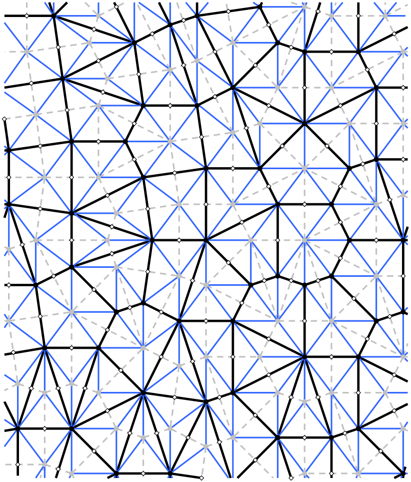
Remarks: (iii) The proof is *"lattice-independent"* and works not only for the square grid, but also for the critical Ising models on triangular and hexagonal lattices, and, more generally, for the particular Ising model defined on an arbitrary isoradial graph. Note that the conformally invariant limit is independent of the lattice.

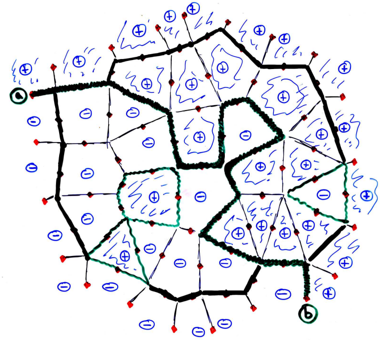
Isoradial graphs. Definition.



- *isoradial graph* Γ (black vertices, all faces can be inscribed into circles of equal radii δ (the “lattice” mesh);
- dual isoradial graph Γ^* (gray vertices);
- *rhombic lattice* ($\Lambda = \Gamma \cup \Gamma^*$, blue edges)
- and the set $\diamondsuit = \Lambda^*$ (white “diamonds”).

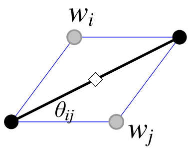
(♠): we assume that rhombi angles are uniformly bounded away from 0 and π.

Self-dual (critical) Ising model on isoradial graphs.



[C. Mercat '01; V. Riva, J. Cardy '06; C. Boutillier, B. de Tilière '09; ...]

$$Z = \sum_{\text{config.}} \prod_{w_i \neq w_j} \tan \frac{\theta_{ij}}{2}$$



Basic fermionic observable:

$$F^\delta(z) := \frac{Z_{config.:a \rightsquigarrow z} \cdot e^{-\frac{i}{2}\text{winding}(a \rightsquigarrow z)}}{Z_{config.:a \rightsquigarrow b} \cdot e^{-\frac{i}{2}\text{winding}(a \rightsquigarrow b)}}, \quad z \in \diamondsuit.$$

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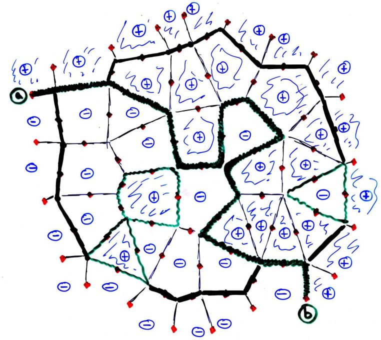

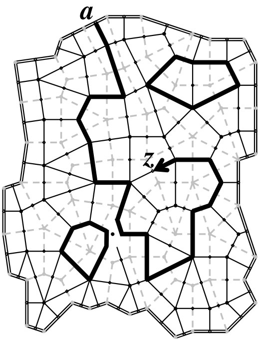

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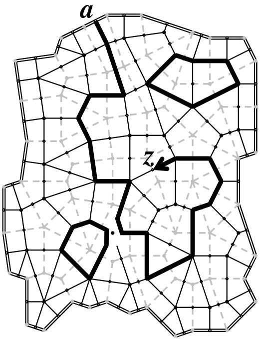
Basic fermionic observable and its discrete holomorphicity.

For critical weights, the function F^δ is discrete holomorphic, i.e., satisfies some discrete version of the Cauchy-Riemann identities.



Proof: Natural combinatorial bijection between the two sets of configurations involved into $F^\delta(z_1)$, $F^\delta(z_2)$ gives one <u>*real*</u> equation for any neighbors $z_{1,2}$.

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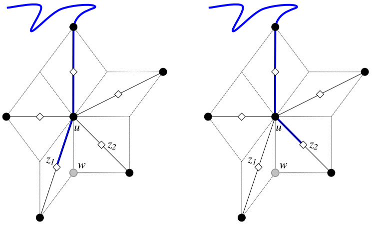

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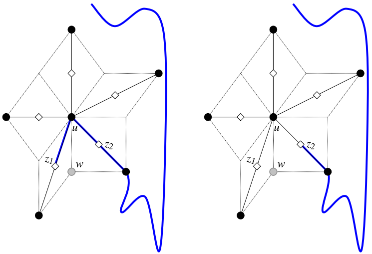

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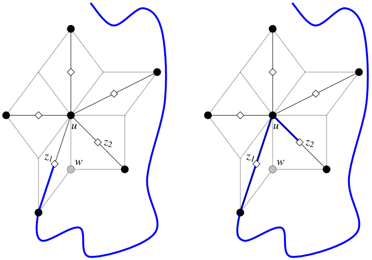

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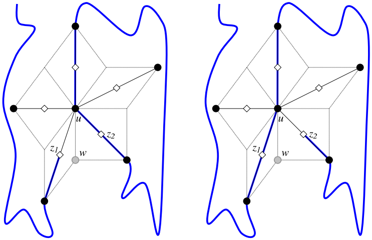

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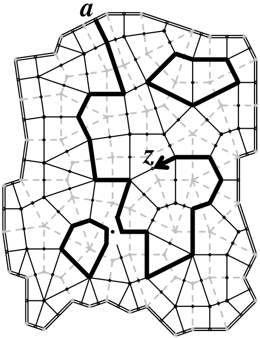

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Remarks: (i) there is a strong *physical motivation* for this definition (coming from the "order and disorder operators" technique),

but one can easily define the observable and derive holomorphicity using simple combinatorial arguments ("local rearrangements");

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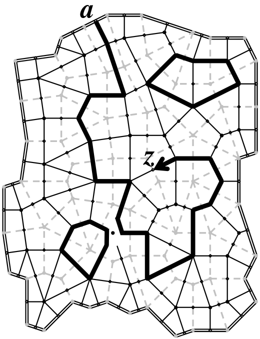

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Remarks: (i) there is a strong *physical motivation* for this definition (coming from the "order and disorder operators" technique);
(ii) *this observable was suggested by S.Smirnov (∼06)* as a crucial tool for the rigorous proof of the Ising model conformal invariance

(for arbitrary planar domains, and not only the Moebius invariance for the Ising model defined in a whole plane or a half-plane);

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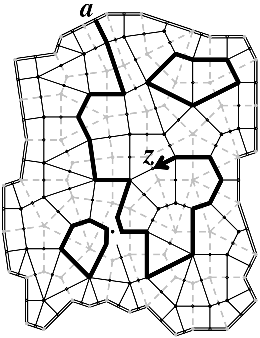

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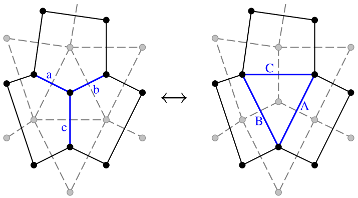
(ii) *this observable was suggested by S.Smirnov (∼06)* as a crucial tool for the rigorous proof of the Ising model conformal invariance;
(iii) several hard *technical problems arises when passing to the limit* (non-smooth boundaries, Riemann-type boundary conditions etc).

Isoradial graphs.
$Y - \Delta$ invariance.



$$\frac{AB + C}{ab} = \frac{BC + A}{bc}$$

$$= \frac{CA + B}{ca} = \frac{ABC + 1}{1}$$

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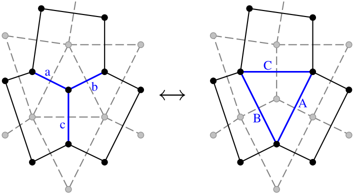

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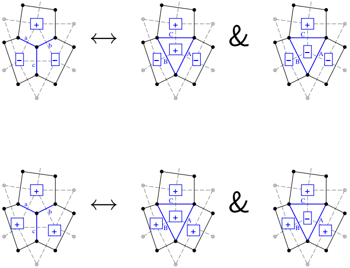

[R. Costa-Santos '06] Local weights satisfying $Y - \Delta$ relation naturally lead to the isoradial embedding of the graph.

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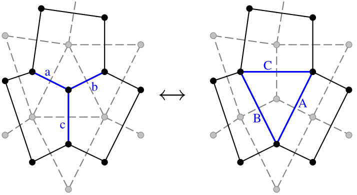

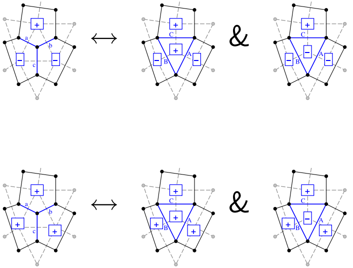

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[R. Costa-Santos '06] Local weights satisfying $Y - \Delta$ relation naturally lead to the isoradial embedding of the graph.

Remark: There exists a strong connection between (a) the critical Ising model, (b) the discrete complex analysis on isoradial graphs, and (c) the "consistency approach" to discrete integrable systems.

Conformal invariance (in the scaling limit):

Correlations:

• *Basic fermionic observables: done (Smirnov-Ch., ∼09).*

Theorem: As $\delta \to 0$, properly normalized discrete holomorphic observables $\delta^{-1/2}F^{\delta}$ converge to holomorphic functions $\Psi_{(\Omega;a,b)}$ such that

$$\Psi_{(\Omega;a,b)}(z) = (\phi'(z))^{1/2} \cdot \Psi_{(\phi\Omega;\phi a,\phi b)}(\phi z)$$

for any conformal mapping $\phi : \Omega \to \phi\Omega$.

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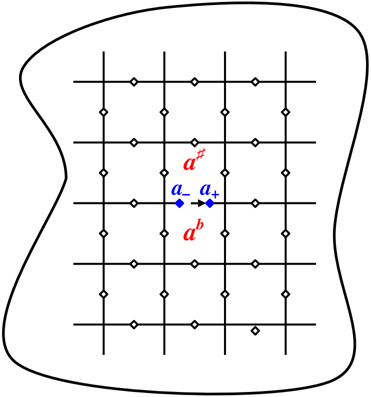
- *Basic fermionic observables: done (Smirnov-Ch., ∼09).*
- *Energy density field: done (Hongler-Smirnov, ∼10).*

Definition: For an edge a in Ω^δ, denote

$$\varepsilon_+^\delta(a) := \mathbf{E}_+[\sigma(a^\sharp)\sigma(a^\flat)]$$



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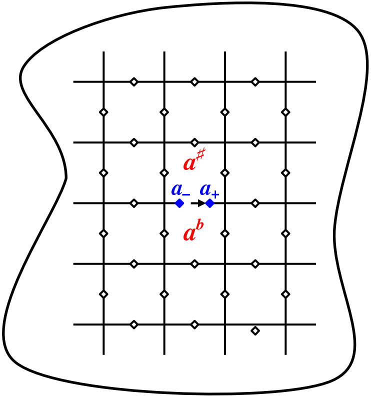
Theorem: As $\delta \to 0$, properly re-normalized discrete energy densities $\delta^{-1} \cdot (\varepsilon_{+}^{\delta}(a) - \sqrt{2}/2)$ converge to the continuum limit $\mathcal{E}_\Omega$ having the following covariance under conformal mappings:

$$\mathcal{E}_\Omega(a) = |\phi'(z)| \cdot \mathcal{E}_{\phi\Omega}(\phi a).$$



Conformal invariance (in the scaling limit):

Correlations:

- *Basic fermionic observables: done (Smirnov-Ch., ∼09).*
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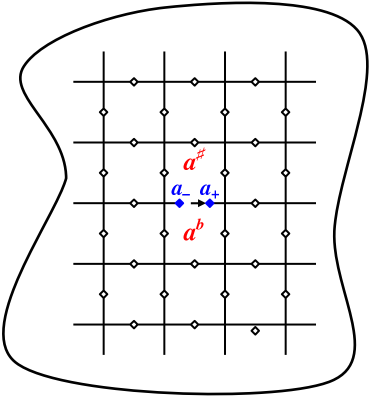
Moreover (*C.Hongler* ∼10), all correlations of the renormalized discrete energy densities $\delta^{-1} \cdot (\varepsilon_+^\delta(a_j) - \sqrt{2}/2)$ converge to the continuum limits, and this result extends to any number of boundary points b_k, where the boundary conditions change "+" to "−".



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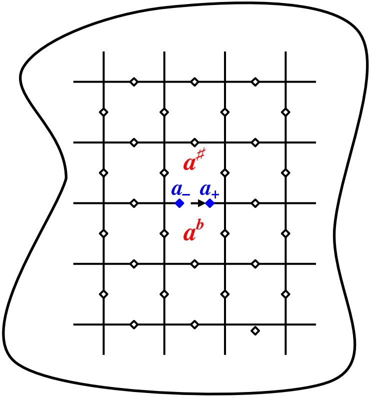
Main idea: Consider the similar observable with a "source point" a_+. Then $\mathbf{F(a_+)}$ counts configurations *without* a, while $-\mathbf{F(a_-)}$ counts configurations *with* a:

$$\varepsilon(a) = \frac{F(a_+) - (-F(a_-))}{F(a_+) + (-F(a_-))}\,.$$



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Theorem: As $\delta \to 0$, the ratio

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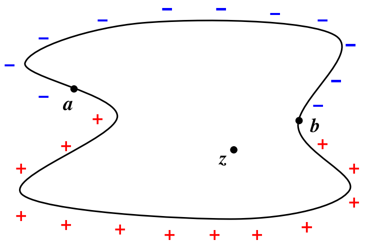
tends to the conformally invariant limit (namely, $\cos[\pi \mathrm{hm}(z, ab, \Omega)]$).



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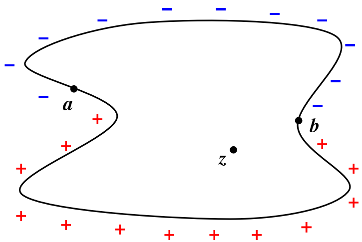

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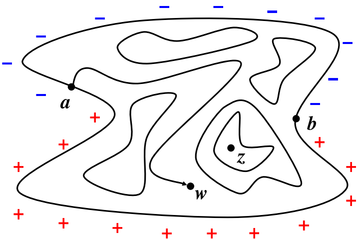
$\widetilde{F}^{\delta}(w) := Z_{config.:a \rightsquigarrow w}$

$\times e^{-\frac{i}{2}\text{winding}(a \rightsquigarrow w)}$

$\times (-1)^{\#[\text{loops around z}]}$

× sign ±1 depending on the sheet of $\widetilde{\Omega}^{\delta}$

$\widetilde{F}^{\delta}$ is a *spinor holomorphic observable* defined on a *double-cover* $\widetilde{\Omega}^{\delta}$ of Ω^{δ}.



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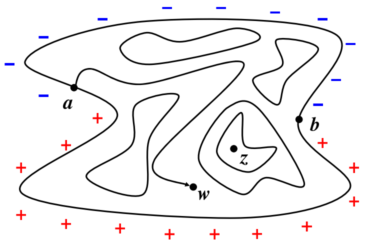

Then

$$\frac{\mathbf{E}_{ab}[\sigma(w^{\delta})]}{\mathbf{E}_{+}[\sigma(w^{\delta})]} = \frac{\widetilde{F}^{\delta}(b)F^{\delta}(a)}{\widetilde{F}^{\delta}(a)F^{\delta}(b)}.$$

Ratios of the spin correlations:

Theorem *(Izyurov-Ch., ∼11)*: Let $\Omega \subset \mathbb{C}$ be a bounded multiple connected domain with two marked points a, b on the outer boundary γ_0, and $\gamma_1, \ldots, \gamma_m$ be some of the inner components of $\partial\Omega$. If discrete domains Ω^δ approximate Ω as $\delta \to 0$, then

$$\frac{\mathbb{E}_{a^\delta b^\delta}[\,\sigma(\gamma_1^\delta)\sigma(\gamma_2^\delta)\ldots\sigma(\gamma_m^\delta)\,]}{\mathbb{E}_{+}[\,\sigma(\gamma_1^\delta)\sigma(\gamma_2^\delta)\ldots\sigma(\gamma_m^\delta)\,]} \;\to\; \vartheta_{ab}^{(\Omega)}(\gamma_1,\ldots,\gamma_m)\,,$$

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Remark:

If $\gamma_j = \{w_j\}$ *are just single points*, then it does not matter, whether γ_j^δ are *single faces* approximating w_j or *small boundary components* shrinking to $\{w_j\}$ as $\delta \to 0$: our proof works in both cases.

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Corollary: For $2n+2$ boundary points the following is fulfilled:

$$\frac{\mathbb{E}_{a_0^\delta \ldots a_{2n+1}^\delta}[\,\sigma(\gamma_1^\delta)\ldots\sigma(\gamma_m^\delta)\,]}{\mathbb{E}_+[\sigma(\gamma_1^\delta)\ldots\sigma(\gamma_m^\delta)\,]} \to \frac{\mathrm{Pf}\,[\,\zeta_{a_j a_k}^{-1}\vartheta_{a_j a_k}^{(\Omega)}(\gamma_1, \ldots, \gamma_m)\,]_{j<k}}{\mathrm{Pf}\,[\,\zeta_{a_j a_k}^{-1}\,]_{0\leqslant j<k\leqslant 2n+1}}\,,$$

where $\zeta_{ab}^{\Omega} = \zeta_{ab}^{\Omega}$ are conformal invariants of $(\Omega; a, b)$ independent of single-point inner components. In particular, $\zeta_{ab}^{\mathbb{C}_+\setminus\{w_1,..,w_m\}} = |b-a|$.

Ratios of the spin correlations:

Exact computations in the half-plane:

In order to find $\vartheta^{\mathbb{C}_+}_{\infty,0}(w_1,..w_m)$ one should solve the following *"interpolation problem"*:

Find a holomorphic spinor f defined on a double cover of $\mathbb{C}_+ \setminus \{w_1,..,w_m\}$ and branching around each of w_j such that

(i) $f(z) = \pm 1 + O(z^{-1})$ as $z \to \infty$;

(ii) $f(\zeta) \in \mathbb{R}$ for any $\zeta \in \mathbb{R}$;

(iii) f^2 has simple poles at all w_j and $\operatorname{res}_{z=w_j}(f(z))^2 \in i\mathbb{R}_+$.

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Remark. This problem can be solved *explicitly* for any $m \geqslant 1$. The answer includes some $m \times m$ determinants, but do not involve complicated analysis of the space of (system of) PDE's solutions which is usual for classical CFT methods.

Conformal invariance in the scaling limit. Summary.

Geometry:

- single interface: done;
- the whole loop ensemble: *[? in progress ?]*

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- some ratios of spin correlations: done (`arXiv:1105.5709`);
- magnetization, spin-spin correlations: *in progress*

  (taking the "source" point just nearby the branching, one can express the logarithmic derivative of the magnetization via spinor observables but some rather involved technical problems appear; ongoing project with C.Hongler and K.Izyurov).

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THANK YOU!