Algebraic and arithmetic area for m planar Brownian paths

LPTMS ORSAY CNRS/Université Paris-Sud

with Jean Desbois J. Stat. Mech. (2011) P05024 arXiv:1101.4135

about some recent results for the arithmetic area of m Brownian paths

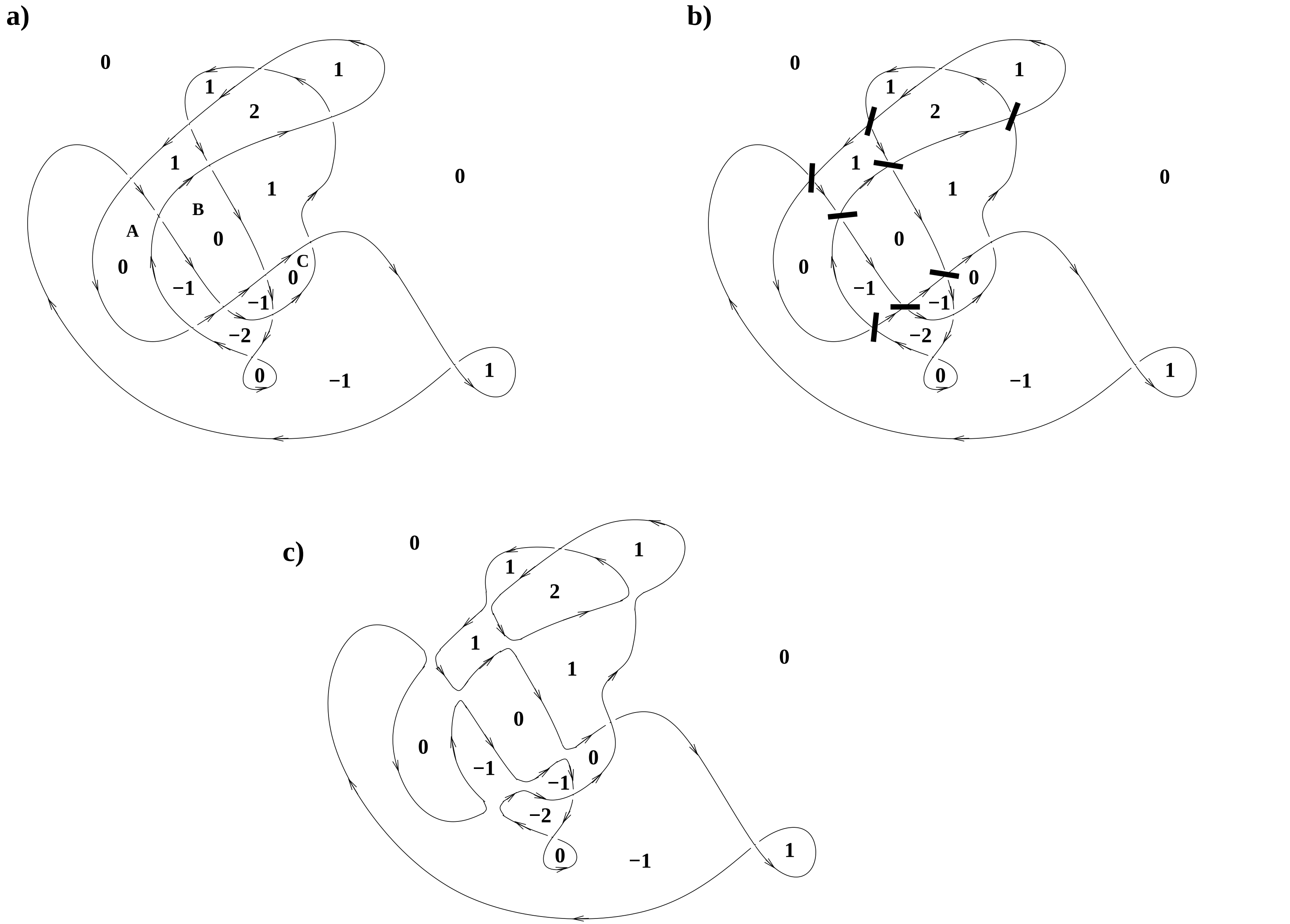
exact scaling $\log(m)$ when $m \to \infty \Rightarrow$ SLE ??

link with Satya Majumdar talk : convex envelopp of m Brownian paths

also algebraic area (easier, not discussed today)

Winding Sectors

points enclosed n times by the path (or $-n$ times)


a)
0
1
1
2
1
1
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A
B
0
C
0
0
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−1
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b)
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1
1
2
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0
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−1
−1
−2
0
−1
1
c)
0
1
1
2
1
1
0
0
0
−1
0
−1
−2
0
−1
1

Comtet, Desbois, S.O. (1990)

random variable S_n = arithmetic area of the n-winding sectors inside a Brownian path of lentgh t

average $\langle S_n \rangle$ on all closed paths of length t

scaling properties of Brownian paths $\rightarrow \langle S_n \rangle$ scales like t

$\langle S_n \rangle$ can be computed by path integral technics

$\Rightarrow$ arithmetic area $\langle S \rangle = \sum_{n=-\infty}^{\infty} \langle S_n \rangle = \langle S_0 \rangle + 2\sum_{n=1}^{\infty} \langle S_n \rangle$

here $\langle S_0 \rangle$ means inside

also $\langle S_n \rangle = \langle S_{-n} \rangle$

density probability for Brownian path starting from $\vec{r}_0$ at time $t = 0$ and reaching $\vec{r}$ at time t

$G(\vec{r}, t|\vec{r}_0, 0) = \frac{1}{2\pi t} e^{-\frac{(\vec{r}-\vec{r}_0)^2}{2t}} = \int_{\vec{r}(0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}} D\vec{r} e^{-\int_0^t \frac{\dot{\vec{r}}^2(\tau)}{2} d\tau}$

close path $\vec{r} = \vec{r}_0$

density probability $P(\vec{r}_0, n, t)$ to wind n times around the origin after time t:

$\theta = 2\pi n \to n = \frac{1}{2\pi} \int_0^t \dot{\theta}(\tau) d\tau$

$\Rightarrow$ constraint $\delta_{n, \frac{1}{2\pi} \int_0^t \dot{\theta}(\tau) d\tau} = \int_0^1 e^{i2\pi\alpha(n - \frac{1}{2\pi} \int_0^t \dot{\theta}(\tau) d\tau)} d\alpha$

in the path integral : $P(\vec{r}_0, n, t) = \int_0^1 d\alpha e^{i2\pi\alpha n} \int_{\vec{r}(0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}_0} D\vec{r} e^{-\int_0^t (\frac{\dot{\vec{r}}^2(\tau)}{2} + i\alpha\dot{\theta}(\tau)) d\tau}$

in the action $\alpha\dot{\theta} = \vec{A}.\dot{\vec{r}} \Rightarrow$ vector potential $\vec{A} = \alpha\vec{\partial}\theta$

$\Rightarrow$ quantum particle coupled to an Aharonov-Bohm flux $2\pi\alpha$ at the origin

singular flux line pierced by an infinite magnetic field $\vec{B} = 2\pi\alpha\delta(\vec{r})$

$\langle S_n \rangle =$ integrate over flux position $\Longleftrightarrow$ integrate over $\vec{r}_0$

$P(n,t) = \int d\vec{r}_o P(\vec{r}_0, n, t) = \int_0^1 d\alpha e^{i2\pi\alpha n} Z_t(\alpha)$

$P(n,t) =$ Fourier transform of $Z_t(\alpha)$

$Z_t(\alpha) =$ Aharonov-Bohm partition function at inverse temperature t

$\Rightarrow$ the result :

$n \neq 0 : \langle S_n \rangle = \frac{t}{2\pi n^2}$

$n = 0 : \langle S_0 \rangle = \infty$

normal : 0-winding sector = inside (finite) + outside (infinite) = infinite

$\Rightarrow \langle S_0 \rangle$ inside is not known note : for the algebraic area $A=\sum_{n=-\infty}^{\infty} nS_n$ same thing

density probability $P(\vec{r}_0, A, t)$ to enclose the algebraic area A after time t:

$A=\int_0^t \frac{\vec{r}(\tau)\times\dot{\vec{r}}(\tau)}{2}.\vec{k}d\tau$

$\Rightarrow$ constraint $\delta(A-\int_0^t \frac{\vec{r}(\tau)\times\dot{\vec{r}}(\tau)}{2}.\vec{k}d\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iB(A-\int_0^t \frac{\vec{r}(\tau)\times\dot{\vec{r}}(\tau)}{2}.\vec{k}d\tau)}dB$

in the action $\frac{B\vec{k}\times\vec{r}}{2}.\dot{\vec{r}}=\vec{A}.\dot{\vec{r}}$

quantum particle coupled to a magnetic field $\Rightarrow$ Landau partition function

Fourier transform $\rightarrow$ Levy's law

back to winding sectors :

1994 : W. Werner thesis ”Sur l’ensemble des points autour desquels le mouvement Brownien plan tourne beaucoup”

$$n \to \infty : n^2 S_n \to \langle n^2 S_n \rangle = \frac{t}{2\pi}$$

+ recent progress on winding sectors (SLE) $\to$ 0-winding sectors

2005 : Garban, Trujillo Ferreras ”The expected area of the filled planar Brownian loop is $\pi/5$”

total arithmetic area is known

$$\langle S \rangle = t\,\frac{\pi}{5} = \langle S_0 \rangle + 2\sum_{n=1}^{\infty} \langle S_n \rangle$$

$$\Rightarrow \langle S_0 \rangle = t\,\frac{\pi}{30}$$

Now what about m independant Brownian paths starting from and coming back to the same point $\vec{r}_0$?

possible interest in ecology : animals looking for food may be approximated by random walkers

question : if you multiply the animal population by 10, should you multiply the size of the natural reserve by 10 ?

so far for $m = 1$ Brownian path

$$< \vec{r}_0 | e^{-tH} | \vec{r}_0 > = \int_{\vec{r}(0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}_0} \mathcal{D}\vec{r}(\tau) e^{-\frac{1}{2}\int_0^t \dot{\vec{r}}^2(\tau)d\tau + i\alpha\int_0^t \dot{\theta}(\tau)d\tau} = \frac{1}{2\pi t} e^{-\frac{r_0^2}{t}} \sum_{k=-\infty}^{+\infty} I_{|k-\alpha|}\left(\frac{r_0^2}{t}\right)$$

integrate over initial position $\vec{r}_0 \Longleftrightarrow$ integrate over the flux line position

$\Rightarrow$ count the arithmetic areas S_n of the n-winding sectors

$$\text{set} \quad r_0^2/t = x \rightarrow \pi t \int_0^{\infty} dx \left(e^{-x} \sum_{k=-\infty}^{+\infty} I_{|k-\alpha|}(x) \right) = \sum_{n \neq 0} \langle S_n \rangle e^{i\alpha 2\pi n} + \langle S_0 \rangle = \infty$$

$$\alpha = 0 \rightarrow e^{-x} \sum_{k=-\infty}^{+\infty} I_{|k|}(x) = 1 \rightarrow \pi t \int_0^{\infty} dx = \sum_{n \neq 0} \langle S_n \rangle + \langle S_0 \rangle = \infty$$

$$\rightarrow \pi t \int_0^{\infty} dx \left(1 - e^{-x} \sum_{k=-\infty}^{+\infty} I_{|k-\alpha|}(x) \right) = \sum_{n \neq 0} \langle S_n \rangle \left(1 - e^{i\alpha 2\pi n} \right)$$

$$\int_0^{\infty} dx \left(\dots \right) = \alpha(1-\alpha) \Rightarrow \langle S_n \rangle = -\pi t \int_0^1 d\alpha \, \alpha(1-\alpha) \, e^{-i\alpha 2\pi n} = \frac{t}{2\pi n^2}$$

$$\text{and} \quad \sum_{n \neq 0} \langle S_n \rangle = \langle S - S_0 \rangle = \pi t \int_0^1 d\alpha \, \alpha(1-\alpha) = \frac{\pi t}{6}$$

for m Brownian paths:

$$\pi t \int_0^\infty dx \left(1 - \left(e^{-x} \sum_{k=-\infty}^{+\infty} I_{|k-\alpha|}(x)\right)^m \right) = \sum_{n\neq 0} \langle S_n(m) \rangle \left(1 - e^{i\alpha 2\pi n}\right)$$

$$\Rightarrow \sum_{n\neq 0} \langle S_n(m) \rangle = \langle S(m) - S_0(m) \rangle = \pi t \int_0^1 d\alpha \int_0^\infty dx \left(1 - \left(e^{-x} \sum_{k=-\infty}^{+\infty} I_{|k-\alpha|}(x)\right)^m \right)$$

evaluate when $m \to \infty$: one finds $\langle S(m) - S_0(m) \rangle \simeq \frac{\pi t}{2} \log(m)$

$\langle S(m) - S_0(m) \rangle \leq \langle S(m) \rangle \Rightarrow \frac{\pi t}{2} \log(m) \leq \langle S(m) \rangle$

use Comtet, Majumdar, Randon-Furling's result : $\langle S_{\text{convex}}(m) \rangle \simeq \frac{\pi t}{2} \log(m)$

$\langle S(m) \rangle \leq \langle S_{\text{convex}}(m) \rangle \Rightarrow \langle S(m) \rangle \leq \frac{\pi t}{2} \log(m)$

$$\frac{\pi t}{2} \log(m) \leq \langle S(m) \rangle \leq \frac{\pi t}{2} \log(m) \Rightarrow \langle S(m) \rangle \simeq \frac{\pi t}{2} \log(m)$$

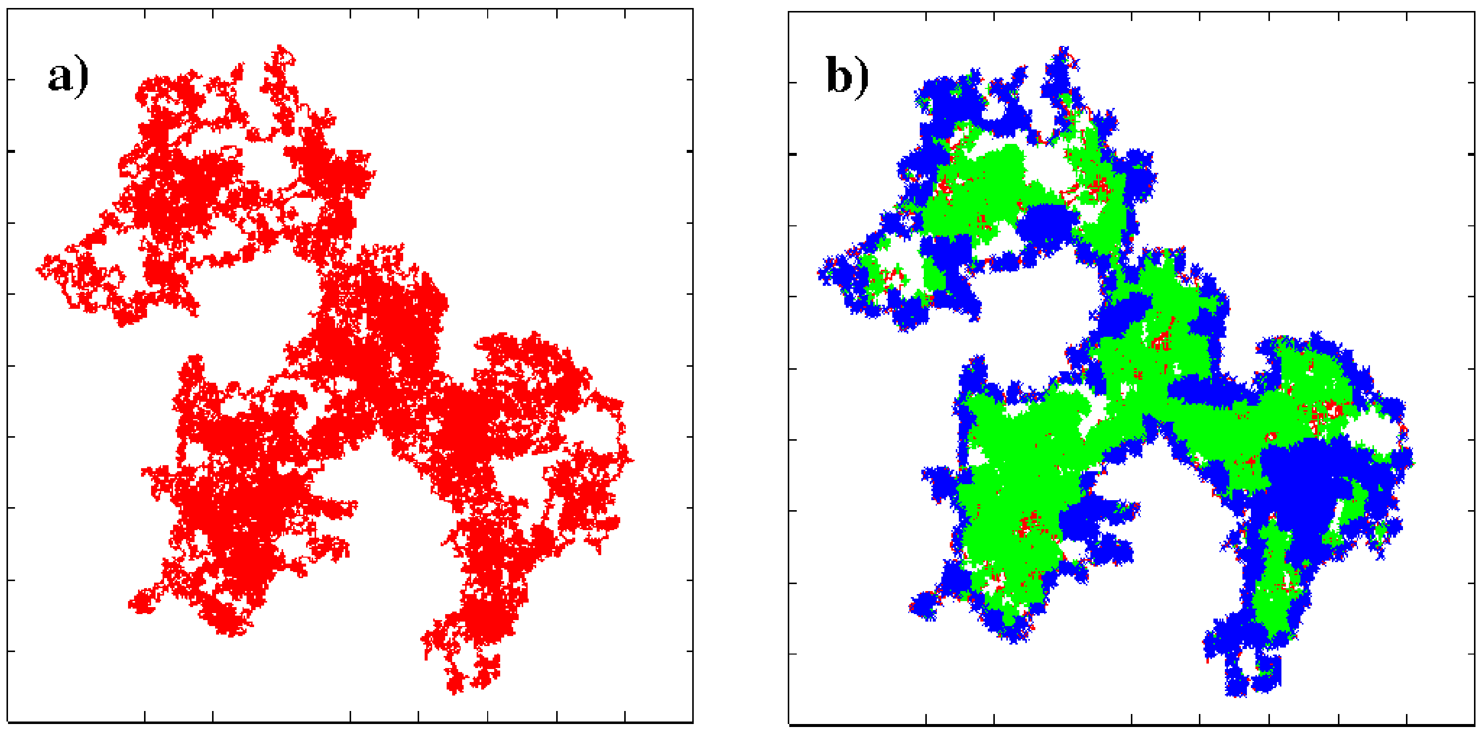
it means that in the $m \to \infty$ limit:

$\langle S_0(m) \rangle$ subleading

convex envelopp is filled

question: can this asymptotic result be obtained via SLE ?

question: $\langle S_0(m) \rangle$?



a)
b)