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Effective bounds for convergence, descriptive complexity, and natural examples of simple and hypersimple sets[☆]

Andrej Muchnik[*], Alexei Semenov[1]

Dorodnicyn Computing Centre of the Russian Academy of Sciences, Department of Cybernetics, Vavilov st. 40, 119991 Moscow GSP-1, Russia

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Abstract

Let μ be a universal lower enumerable semi-measure (defined by L. Levin). Any computable upper bound for μ can be effectively separated from zero with a constant (this is similar to a theorem of G. Marandzhyan).

Computable positive lower bounds for μ can be nontrivial and allow one to construct natural examples of hypersimple sets (introduced by E. Post).

Keywords: Plane entropy; Prefix entropy; Simple set; Hypersimple set; Convergent series; Effective bound

1. Introduction

During the first year of university education, students learn classic examples of convergent series

$$a_m^0 = \frac{1}{m^2}, \qquad a_m^1 = \frac{1}{m(\log m)^2}, \qquad a_m^2 = \frac{1}{m \log m (\log\log m)^2}, \ \ldots$$

and divergent series

$$b_m^0 = \frac{1}{m}, \qquad b_m^1 = \frac{1}{m \log m}, \qquad b_m^2 = \frac{1}{m \log m \log\log m}, \ \ldots.$$

It appears natural to draw a “borderline” between convergent and divergent positive series. This is hard to do if we stay within the limits of classic calculus.

Let us consider a class of convergent positive series that are superior limits of computable sequences of series. Then in this class there is a largest up to multiplicative constant series μ (the stated accuracy is not surprising because the property of convergence does not change on multiplying by a positive number). Series μ ultimately overtakes

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* Corresponding address: Institute of New Technologies, 10 Nizhnyaya Radishevskaya, 109004 Moscow, Russia. Fax: +7 495 9156963.
E-mail addresses: muchnik@lpcs.math.msu.su (A. Muchnik), alsemenov@mtu-net.ru (A. Semenov).
1 Fax: +7 495 9156963.

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series a^i_m, but it has a disadvantage: it is not computable. Using a method of G. Marandzhyan we can prove that any computable upper bound for μ is trivial: we can effectively separate from zero with a constant all its terms. However, computable positive lower bounds for μ can be nontrivial and allow us to construct interesting examples of hypersimple sets (the concept of the hypersimple set was introduced by E. Post in the famous article [5] of 1944 with the purpose of construction of tt-incomplete undecidable enumerable sets).

2. Convergence of series

Let us denote by $\log^{[i]} x$ the ith iteration of a binary logarithm. Classic examples of convergent series are $a^0_m = \frac{1}{m^2}$, $a^1_m = \frac{1}{m(\log m)^2}$, $a^2_m = \frac{1}{m \log m (\log^{[2]} m)^2}$ and so on. Here each series is essentially larger than the previous one; that is, $\forall i\ (a^{i+1}_m / a^i_m \to \infty$ as $m \to \infty)$. Classic examples of divergent series are $b^0_m = \frac{1}{m}$, $b^1_m = \frac{1}{m \log m}$, $b^2_m = \frac{1}{m \log m \log^{[2]} m}$ and so on. Here each series is essentially smaller than the previous one; that is, $\forall i\ (b^{i+1}_m / b^i_m \to 0$ as $m \to \infty)$. Series a^i_m and b^i_m are very close; their ratio is decreasing, but very slowly. Thus there is a question: does the largest convergent or smallest divergent positive series exist? The answer is negative. However, we are interested in computable positive series.

Hereinafter we consider only positive series. So without loss of generality it can be assumed that terms of series have values of the form 2^{-n} only. Indeed, we can change each positive number a to a number of the form 2^{-n} which is the nearest from below to a. Then a is decreasing not more than twice, and convergence or divergence of the series holds.

We can essentially decrease any computable divergent series and at the same time we can keep it divergent.

Proposition 1. *For any computable divergent series α_m there exists a computable divergent series β_m such that $\beta_m/\alpha_m \to 0$.*

Proof. Construct a monotonic sequence $\{m_i\}$ such that $\sum_{m=m_i}^{m_{i+1}-1} \alpha_m > 2^i$, $m_1 = 1$. Assume $\beta_m = \alpha_m 2^{-i}$ for $m_i \le m < m_{i+1}$. □

We can essentially increase any computable *effectively* convergent series and at the same time we can keep it effectively convergent.

Proposition 2. *For any computable effectively convergent series α_m there exists a computable effectively convergent series β_m such that $\beta_m/\alpha_m \to \infty$.*

Proof. Construct a monotonic sequence $\{m_i\}$ such that $\sum_{m=m_i}^{\infty} \alpha_m < 2^{-2i}$. Assume $\beta_m = \alpha_m 2^i$ for $m_i \le m < m_{i+1}$ and $\beta_m = \alpha_m$ for $m < m_1$. □

Surprisingly there is a computable convergent series such that it is impossible to increase it essentially and keep it convergent.

Theorem 1. *There exists a computable convergent series α_m such that there is no computable convergent series β_m such that $\beta_m/\alpha_m \to \infty$.*

The proof is given below.

Proposition 3. *There does not exist a largest up to multiplicative constant computable convergent series.*

Proof. Suppose α_m is a computable convergent series. Construct a sequence $\{m_i\}$ such that $\alpha_{m_i} < 2^{-2i}$. Assume $\beta_{m_i} = \alpha_{m_i} 2^i$ and $\beta_m = \alpha_m$ for other m. □

A remarkable discovery was made that, contrary to what was previously thought, in a natural extension of the class of computable series there exists a largest up to multiplicative constant convergent series. This extension is the class of computably approximable from below series. (At each moment of time the lower approximation for the series is equal to zero on some cofinite set. Note that we do not suppose that the approximation is *uniform*.)

Now we produce the corresponding exact proposition. For convenience of notation we will allow terms of series to be equal to zero. It is easy to replace them by very small positive terms. We will write the number of terms of the series in parentheses and not as an index.

Theorem 2 (*Levin, 1970*). *There exists a computably approximable from below series $\mu(x)$ such that its sum is ≤ 1 and for any computably approximable from below series $\nu(x)$ such that its sum is ≤ 1 there exists a constant C such that $\forall x\ \mu(x) > \nu(x)2^{-C}$.*

Proof. As in the case of enumerable sets there exists a universal computable function $U: n \mapsto \nu_n$ such that it enumerates all computably approximable from below series and any other function V with the same property is m-reducible to U.

Any computably approximable from below series ν can be effectively changed to a computably approximable from below series ν' such that $\sum \nu'(x) \leq 1$ and if $\sum \nu(x) \leq 1$, then $\forall x\ \nu'(x) = \nu(x)$. The program, which approximates from below series ν', runs the same way as the program, which approximates from below ν, while the sum of values of the current approximation is not more than 1; when this sum is more than 1, the program for ν' stops.

The required series μ can be assumed as $\sum_n 2^{-n}\nu'_n$. □

Obviously, series μ (which exists as we just proved) is unique up to a multiplicative constant. Let us fix an arbitrary such μ.

Levin proved that the series μ is not computable. Let us consider the question of its computable upper and lower bounds.

The series μ does not have any nontrivial partially computable upper bounds (this is similar to Marandzhyan's theorem for the plane entropy).

Theorem 3. *For any partially computable function γ there exists a constant C such that*

$$\forall x \in Dom(\gamma)\ \ \mu(x) \leq \gamma(x) \quad \Rightarrow \quad \forall x \in Dom(\gamma)\ \ \gamma(x) \geq 2^{-C}.$$

Proof. Consider the following computably approximable from below series $\delta = \lim \delta_n$. Approximation δ_1 is identically equal to zero. Approximation δ_{n+1} coincides with approximation δ_n everywhere except possibly for one argument x, which is obtained in the following way. We compute γ at all arguments simultaneously and are seeking for x such that $\gamma(x) < 2^{-2n}$. Assume $\delta_{n+1}(x) = \max\{\delta_n(x), 2^{-n}\}$ at the first x found (if we found any x). It is clear that $\sum_x \delta(x) \leq 1$. Hence we can find c (effectively from the number of γ) such that $\forall x\ \mu(x) > \delta(x)2^{-c}$. Combining this with $\forall x \in Dom(\gamma)\ \ \mu(x) \leq \gamma(x)$ we get that for $n \geq c$ the number x in the definition of δ_n will not be found and $\forall x\ \gamma(x) \geq 2^{-2c}$. □

We can suppose that for μ there is no "well" computable lower bound; that is, the ratio of μ and any computable lower bound for μ tends to infinity. The following theorem contradicts this supposition and at the same time proves Theorem 1.

Theorem 4. *There exists a computable series α such that α is a lower bound for μ and α is equal to μ on the infinite set.*

Proof. Construct an auxiliary computable series β as result of the computable process of filling the next table.

1					
2					
⋮					
i	$\beta(x_i^1) = 2^{-i}$ $\mu(x_i^1) > 2^{-i}$	$\beta(x_i^2) = 2^{-i}$ $\mu(x_i^2) > 2^{-i}$	…	$\beta(x_i^j) = 2^{-i}$	
⋮					

We return to each line infinitely often and at the same time we approximate from below the series μ. When we address to the ith line for the first time, we take the first undefined term of the series β (denote its number by x_i^1) and assume $\beta(x_i^1) = 2^{-i}$. When we address to the ith line the next time, we compare the current approximation $\mu(x_i^1)$ with 2^{-i}; if this approximation is $>2^{-i}$, then we indicate this fact in a lower part of the table's cell and again take the first undefined term of β (denote its number by x_i^2) and assume $\beta(x_i^2) = 2^{-i}$; and so on. For any number, there exists a unique x in the table which is equal to that number.

Since $\sum_x \mu(x) \le 1$, the length of the ith line is less than 2^i. Consider the set D of numbers x_i^j such that $\mu(x_i^j) \le 2^{-i}$. The sum of the series $\beta(x)$ over this set is equal to $\sum_i 2^{-i} = 1$. The sum of the series $\beta(x)$ over D's complement is less than $\sum_x \mu(x) \le 1$. Thus the series $\beta(x)$ is convergent and $\exists c\, \forall x\ \mu(x) > \beta(x)2^{-c}$.

On the other hand, $\mu(x) \le \beta(x)$ on the infinite set D. For any natural number C consider the set $M_C = \{x : \mu(x) \le \beta(x)2^{-C}\}$. For $C = 0$ this set is infinite; for $C = c$ this set is empty. When C increases, M_C decreases nonstrictly. Consider the largest d such that M_d is infinite. Since $\mu(x) > \beta(x)2^{-(d+1)} \Leftrightarrow \mu(x) \ge \beta(x)2^{-d}$, it follows that $\mu(x) \ge \beta(x)2^{-d}$ on the complement of the finite set M_{d+1} and $\mu(x) = \beta(x)2^{-d}$ on the infinite set $M_d \setminus M_{d+1}$.

The required series α is defined by the formula $\alpha(x) = \min\{\mu(x), \beta(x)2^{-d}\}$. □

Theorem 5. *If a computable series α estimates μ from below and equals μ on the infinite set, then the set $\{x : \mu(x) > \alpha(x)\}$ is hypersimple.*

(Recall that an enumerable set with an infinite complement is called *hypersimple* if in any computable infinite sequence of non-intersecting segments of natural numbers there exists a segment such that it is entirely embedded in this set.)

Proof. Suppose T_j is a computable sequence of non-intersecting segments of natural numbers.

Construct an auxiliary computable series β. For any m find $j_m > m$ such that $\sum_{x \in T_{j_m}} \alpha(x) < 2^{-2m}$. This is possible since the series $\mu(x)$ is convergent, $\alpha(x) \le \mu(x)$, and hence the computable series $\alpha(x)$ is convergent. Assume $\beta(x) = \alpha(x)2^m$ on the segments T_{j_m}; otherwise $\beta(x) = \alpha(x)$. Since

$$\sum_{x \in \bigcup_m T_{j_m}} \beta(x) < \sum_m 2^m \cdot 2^{-2m} = 1,$$

the series $\beta(x)$ is convergent. Therefore $\exists c\, \forall x\ \mu(x) > \beta(x)2^{-c}$. By definition of β we have that on the segment T_{j_c} there is no x such that $\alpha(x) = \mu(x)$. □

3. Descriptive complexity

Plane entropy of a natural number (introduced by Kolmogorov in [1]) is the minimal length of its code obtained using optimal encoding.

An encoding is called *prefix* if any code is not an extension to the right of another code. The condition of being prefix allows one to transmit a sequence of encoded messages without using an auxiliary symbol (for example white space). *Prefix entropy* of a natural number (introduced by Levin in [7]) is the minimal length of its code obtained using optimal prefix encoding.

However, definitions are possible which do not use any encoding (see [2]). Let us expound them.

Plane entropy is a minimal up to additive constant enumerable from above function KS (with values in $\mathbb{N}$) such that $|\{x : KS(x) < n\}| < 2^n$ for each n.

Prefix entropy is a minimal up to additive constant enumerable from above function KP (with values in $\mathbb{N}$) such that $\sum_x 2^{-KP(x)} \le 1$.

From our definitions it obviously follows that the functions KS and KP tend to infinity and $\forall x\ KS(x) \le KP(x) + O(1)$. It is much more difficult to prove that the difference $(KP - KS)$ tends to infinity. We gave this proof at the international workshop dedicated to Kolmogorov's centenary in spring of 2003 in Heidelberg; M. Li and P. Vitányi formulate this fact in their monograph [3] without proof and refer to the unpublished manuscript [6] of R. Solovay.

The functions KS and KP have rare, but unexpected falls. G. Marandzhyan proved in 1969 [4] that the function KS does not have any nontrivial partially computable lower bound (for the function KP, the argumentation is similar).

Consider computable one-to-one enumeration of all binary words, such that increasing of number implies non-decreasing of length of the word having that number. Denote by $\ell(x)$ the length of a binary word with a number x.

A computable upper bound of plane entropy is for example $\ell(x) + c$.

Using considerations of cardinality, we get

$$\forall n\, \exists x\ \ell(x) = n \,\&\, KS(x) \ge n.$$

On the other hand, by definition of KS we have $\exists c \forall x \; KS(x) < \ell(x) + c$. For any C consider the set $M_C = \{x : \ell(x) + C \le KS(x)\}$. For $C = 0$ this set is infinite; for $C = c$ this set is empty. When C increases, M_C decreases nonstrictly. Consider the largest d such that M_d is infinite. It is clear that $\ell(x) + d \ge KS(x)$ on the complement of the finite set M_{d+1} and $\ell(x) + d = KS(x)$ on the infinite set $M_d \setminus M_{d+1}$. Assume the function f is equal to KS on the finite set M_{d+1} and f is equal to $\ell(x) + d$ otherwise. It is obvious that f is a computable function which estimates KS from above and f is equal to KS on the infinite set.

Theorem 6. *If an everywhere defined computable function f estimates KS from above and equals KS on the infinite set, then the set $\{x : KS(x) < f(x)\}$ is simple.*

(Recall that an enumerable set with an infinite complement is called *simple* if there is no infinite enumerable subset in its complement. This concept was introduced by E. Post in 1944 [5] with the purpose of construction of a btt-incomplete undecidable enumerable set.)

Proof. Suppose there exists an infinite enumerable set R such that $\forall x \in R \, f(x) = KS(x)$. Let f' be the restriction of f to R. Then f' is the partially computable lower bound for KS. This contradicts the assertion $KS(x) \to \infty$ and Marandzhyan's theorem. □

Theorems 4 and 5 in the language of prefix entropy are given without proofs in the monograph [3] of M. Li and P. Vitányi with reference to an unpublished manuscript [6] of R. Solovay. Let us formulate them.

Theorem 7. *There exists an everywhere defined computable function f such that f estimates KP from above and equals KP on the infinite set.*

Theorem 8. *If an everywhere defined computable function f estimates KP from above and equals KP on the infinite set, then the set $\{x : KP(x) < f(x)\}$ is hypersimple.*

Note that in the formulation of Theorem 6 the set $\{x : KS(x) < f(x)\}$ must not be hypersimple. For that let us change function KS to the function $KS' = \min\{KS + 1, \ell + 1\}$. It is clear that the function KS' is enumerable from above, that $|\{x : KS'(x) < n\}| < 2^n$ and that for this function (as for function KS) the property of being minimal up to an additive constant holds. For x such that $KS(x) \ge \ell(x)$ (there are infinitely many of them) it will be the case that $KS'(x) \ge \ell(x) + 1$. On the other hand, for any x we have $KS'(x) \le \ell(x) + 1$. In the construction which is given before Theorem 6, the parameter d for function KS' will be equal to 1. Let us divide the natural scale into segments T_j which consist of the numbers of all the words of length j. In each of these segments there is a number x such that $KS'(x) = \ell(x) + 1$, which contradicts the property of hypersimplicity.

4. Conclusion

The constructive mathematics of Andrei Markov revising the classical one is fed and fertilized by its ideas and contexts. In its turn the constructivist consideration of classical concepts helps to answer relevant questions of the theory of algorithms itself. This supports our belief in the future development of ideas and approaches of A. Markov.

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