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Ultrasecond quantization of a classical version of superfluidity in nanotubes[1]

Victor P. Maslov

1. In order to distinguish the classical theory in its modern understanding from the quantum theory, it is necessary to modify (somewhat) the ideology habitual to physicists, for whom the classical theory is simply the whole body of physics as it existed in the 19th century before the appearance of quantum theory. Actually, the correct meaning is that the classical theory is the limit of the quantum one as $h \to 0$.

Thus, Feynman correctly understood that spin is a notion of classical mechanics. Indeed, it is obtained via a rigorous passage from quantum mechanics to classical mechanics [**1**]. In a similar same way, the polarization of light does not disappear when the frequency is increased, and is therefore a property of geometric rather than wave optics, contrary to the generally accepted belief, which arose because the polarization of light was discovered as the result of the appearance of wave optics.

Consider a "Lifshits hole", i.e. a one-dimensional Schrödinger equation with potential symmetric with respect to the origin of coordinates with two troughs. Its eigenfunctions are symmetric or antisymmetric with respect to the origin. As $h \to 0$ this symmetry remains, and since the square of the modulus of the eigenfunction corresponds to the probability of the particle to remain in the troughs, it follows that in the limit as $h \to 0$, i.e., in the "classical theory", for energies less than those required to pass over the barrier, the particle is simultaneously located in two troughs, although a classical particle cannot pass through the barrier. Nevertheless, this simple example shows how the ideology of the "classical theory" must be modified.

To understand this paradox, one must take into consideration the fact that the symmetry must be very precise, up to "atomic precision", and that stationary state means a state that arises in the limit for "infinitely long" time.

When we deal with nanotubes whose width is characterized by "atomic" or "quantum" dimensions, then new unexpected effects occur in the "classical" theory. Thus, already in 1958 [**2**], I discovered a strange effect of the standing longitudinal wave type in a slightly bent infinite narrow tube, for the case in which its radius is the same everywhere with atomic precision.

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It was was impossible at the time to implement this effect in practice, which would have allowed to obtain a unimode laser, despite A.M.Prokhorov's great interest in the effect.

2. Now let us discuss the notion known as "collective oscillations" in classical physics and as "quasiparticles" in quantum physics. In classical physics, it is described by the Vlasov equation for selfcompatible (or mean) fields, in quantum physics, by the Hartrey (or the Hartrey-Fock) equation.

(1) Variational equations depend on where (i.e., near what solutions of the original equation) we consider the variations. For example, in [**3, 4, 5**] we considered variations near a microcanonical distribution in an ergodic construction, while in [**7, 8, 9, 10**] this was done near a nanocanonical distribution concentrated on an invariant manifold of lesser dimension, i.e., not on a manifold of constant energy but, for example, on a Lagrangian manifold of dimension coinciding with that of the configuration space.

(2) Let us note the following crucial circumstance. The solution of the variational equation for the Vlasov equation *does not coincide* with the classical limit for variational equations for the mean field equations in quantum theory.

Consider the mean field equation in the form

$$
\begin{aligned}
ih\frac{\partial}{\partial t}\varphi^t(x) &= \Big(-\frac{h^2}{2m}\Delta + W_t(x)\Big)\varphi^t(x),\\
W_t(x) &= U(x) + \int V(x,y)|\varphi^t(y)|^2 dy,
\end{aligned}
\tag{1}
$$

with the initial condition $\varphi|_{t=0} = \varphi_0$, where φ_0 belongs to $W_2^\infty(\mathbf{R}^\nu)$ and satisfies $\int dx|\varphi_0(x)|^2 = 1$.

In order to obtain asymptotics of the complex germ type [**11**] one must write out the system consisting of the Hartrey equation and its dual, then consider the corresponding variational equation, and, finally, replace the variations $\delta\varphi$ and $\delta\varphi^*$ by the independent functions F and G. For the functions F and G, we obtain the following system of equations:

$$
\begin{aligned}
i\frac{\partial F^t(x)}{\partial t} &= \int dy\Big(\frac{\delta^2 H}{\delta\varphi^*(x)\delta\varphi(y)}F^t(y) + \frac{\delta^2 H}{\delta\varphi^*(x)\delta^*\varphi(y)}G^t(y)\Big);\\
-i\frac{\partial G^t(x)}{\partial t} &= \int dy\Big(\frac{\delta^2 H}{\delta\varphi(x)\delta\varphi(y)}F^t(y) + \frac{\delta^2 H}{\delta\varphi(x)\delta^*\varphi(y)}G^t(y)\Big).
\end{aligned}
\tag{2}
$$

The classical equations are obtained from the quantum ones, roughly speaking, by means of a substitution of the form $\varphi = \chi e^{\frac{i}{h}S}$ (the VKB

method), $\varphi^* = \chi^* e^{\frac{i}{h}S^*}, where\ S = S^*,\ \chi = \chi(x,t) \in C^\infty,\ S = S(x,t) \in C^\infty$.

To obtain the variational equations, it is natural to take the variation not only of the limit equation for χ and χ^*, but also for the functions S and S^*. This yields a new important term of the equation for collective oscillations.

Let us describe this fact for the simplest example, which was studied in N.N.Bogolyubov's famous paper concerning "weakly ideal Bose gas" [**12**].

Suppose $U = 0$ in equation (1) in a three-dimensional cubical box of edge L, the wave functions satisfying the periodicity condition (i.e., the problem being defined on the 3-torus with generators of lengths L, L, L). Then the function

$$\varphi(x) = L^{-3/2} e^{i/h(px - \Omega t)}, \tag{3}$$

where $p = 2\pi n/L$, n is an integer, satisfies the equation (1) for

$$\Omega = \frac{p^2}{2m} + L^{-3} \int dx V(x). \tag{4}$$

For $\lambda = 2\pi n/L$, n a nonzero integer, consider the functions $F^{(\lambda)}(x)$ and $G^{(\lambda)}(x)$ given by

$$\begin{aligned} F^{(\lambda)t}(x) &= L^{-3/2} \rho_\lambda e^{\frac{i}{h}|(p+\lambda)x + (\beta - \Omega)t|}, \\ G^{(\lambda)t}(x) &= L^{-3/2} \sigma_\lambda e^{\frac{i}{h}|(-p+\lambda)x + (\beta + \Omega)t|}; \end{aligned} \tag{5}$$

here

$$\begin{aligned} -\beta_\lambda \rho_\lambda &= \left(\frac{(p+\lambda)^2}{2m} - \frac{p^2}{2m} + \widetilde{V}_\lambda \right) \rho_\lambda + V_\lambda \sigma_\lambda, \\ \beta_\lambda \rho_\lambda &= \left(\frac{(p-\lambda)^2}{2m} - \frac{p^2}{2m} + \widetilde{V}_\lambda \right) \sigma_\lambda + V_\lambda \rho_\lambda, \\ |\sigma_\lambda|^2 - |\rho_\lambda|^2 &= 1, \qquad \widetilde{V}_\lambda = L^{-3} \int dx V(x) e^{\frac{i}{h}\lambda x}. \end{aligned} \tag{6}$$

From the system (6), we find

$$\beta_\lambda = -p\lambda + \sqrt{\left(\frac{\lambda^2}{2m} + \widetilde{V}_\lambda \right)^2 - \widetilde{V}_\lambda^2}. \tag{7}$$

In this example $u = e^{\frac{i}{h}s(x,t)}$, $u^* = e^{-\frac{s(x,t)}{h}}$, where $s(x,t) = px + \beta t$, while the variation of the action for the vector $\left(\delta u, \delta u^*\right)$ equals $\lambda x \pm \Omega t$.

Under a more accurate passage to the limit, we obtain

$$\widetilde{V}_\lambda \to V_0 = L^{-3} \int dx V(x)$$

.

Thus, in the classical limit, we have obtained the famous Bogolyubov relation (7). In the case under consideration $u(x) = 0$ and, as in the linear Schrödinger equation, the exact solution coincides with the quasiclassical one. In the paper [**10**], the case $u(x) \neq 0$ is investigated, and it turns out that the relation similar to (7) is the classical limit as $h \to 0$ of the variational equation in this general case. The curve showing the dependence of β_λ on λ is known as the *Landau curve* and determines the superfluid state. The value λ_{cr} for which superfluidity disappears is called the *Landau critical level*. Bogolyubov explains the superfluidity phenomenon in the following terms: "the 'degenerate condensate' can move without friction relatively to elementary perturbations with any sufficiently small velocity" [**10**, p. 210].

However, there is no Bose-Einstein condensate whatever in these mathematical considerations, it is just that the spectrum defined for $\lambda < \lambda_{\text{cr}}$ is a positive spectrum of quasiparticles. This means it is metastable (see [**13**]). The Bose-Einstein condensate is not involved here, it is only needed only to show that it would be wrong to believe that this argument works for a classical liquid, as one might think from the considerations above.

Indeed, for example, the molecules of a classical nondischarged liquid are, as a rule, Bose particles. For such a liquid, one can write out the N-particle equation, having in mind that each particle (molecule) is neutral and consists of an even number l of neutrons. Thus each ith particle is a point in $3(2k + l)$-dimensional space, where k is the number of electrons, $x_i \in R^{6k+3l}$, depends on the potential $u(x_i)$, $x_i \in R^{6k+3l}$ and we can consider the N-particle equation for $x_i, i = 1, \ldots, N$, with pairwise interaction $V(x_i - x_j)$.

3. However, there is a purely mathematical explanation of this paradox. The thing is that Bogolyubov found only one series of points in the spectrum of the many particle problem. Landau wrote "N.N.Bogolyubov recently succeeded, by means of a clever application of second quantization, in finding the general form of the energy spectrum of a Bose-Einstein gas with weak interaction between the particles" ([**14**, p. 43]). But this series is not unique, i.e., the entire energy spectrum was not obtained.

In 2001, the author proposed the method of ultra second quantization [**15**]; see also [**16**], [**17**], [**18**], [**19**], [**20**], The ultra second quantization of the Schrödinger equation, as well as its ordinary second quantization, is

a representation of the N-particle Schrödinger equation, and this means that basically the ultra second quantization of the equation is the same as the original N-particle equation: they coincide in $3N$-dimensional space. However, the replacement of the creation and annihilation operators by c-numbers, in contrast with the case of second quantization, does not yield the correct asymptotics, but it turns out that it coincides with the result of applying the Schroeder variational principle or the Bogolyubov variational method.

For the exotic Bardin potential, the correct asymptotic solution coincides with the one obtained by applying the ultra second quantization method described above. In the case of general potentials, in particular for pairwise interaction potentials, the answer is not the same. Specifically, the ultra second quantization method gives other asymptotic series of eigenvalues corresponding to the N-particle Schrödinger equation, and these eigenvalues, unlike the Bogolyubov ones (7), are not metastable.

It turns out that the main point is not related to the Bose-Einstein condensate, but has to do with the width of the capillary (the nanotube) through which the liquid flows. If we consider a liquid in a capillary or a nanotube of sufficiently small radius the velocity corresponding to metastable states is not small. Hence at smaller velocities the flow will be without friction.

The condition that the liquid does not flow through the boundary of the nanotube is a Dirichlet condition. It yields a standing wave, which can be regarded as a pair particle–antiparticle: a particle with momentum p orthogonal to the boundary of the tube, and an antiparticle with momentum $-p$.

We consider a short action pairwise potential $V(x_i - x_j)$. This means that as the number of particles tends to infinity, $N \to \infty$, interaction is possible for only a finite number of particles. Therefore, the potential depends on N in the following way:

$$V_N = V((x_i - x_j)N^{1/3}).$$

If $V(y)$ is finite with support Ω_V, then as $N \to \infty$ the support engulfs a finite number of particles, and this number does not depend on N.

As the result, it turns out that for velocities less than $\min(\lambda_{\mathrm{cr}}, \frac{h}{2mR})$, where λ_{cr} is the critical Landau velocity and R is the radius of the nanotube, superfluidity occurs.

Now let me present my own considerations, which are not related to the mathematical exposition. Viscosity is due to the collision of particles: the higher the temperature, the greater the number of collisions. In a

nanotube, there are few collisions, and only with the walls, and those are taken into account by the author's series. It is precisely this circumstance, and not the Bose-Einstein condensate, which leads to the weakening of viscosity and so to superfluidity. What I am saying is that the main factor in the superfluidity phenomenon, even for liquid helium 4, is not the condensate, but the presence of an extremely thin capillary [**21**], [**22**]. It seems to me that a neutral gas like argon could be used for a crucial experiment.

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Policy iteration and max-plus finite element method

David McCaffrey

1. Introduction

We consider the finite horizon differential game

$$v(x,T) = \inf_{a(.)} \sup_{b(.)} \int_0^T \left\{ \frac{1}{2}x(s)^2 + \frac{1}{2}a(s)^2 - \frac{\gamma^2}{2}b(s)^2 \right\} ds + \phi(x(T)) \tag{1.1}$$

over trajectories $(x(.), a(.), b(.))$ satisfying $\dot{x}(s) = f(x(s)) + g(x(s))a(s) + h(x(s))b(s)$, $x(0) = x$, where $x(s) \in X \subseteq \mathbb{R}^n$, $a(s) \in U \subseteq \mathbb{R}^m$, $b(s) \in W \subseteq \mathbb{R}^r$.This problem arises, for example, as the differential game formulation of a well-known class of non-linear affine H_∞ control problems - see [**5, 6, 7, 4**] for details. In particular it is known that the value function $v(x,t)$ for the finite horizon problem is a (possibly non-smooth) solution to the Hamilton-Jacobi-Isaacs equation

$$H(x, \partial v/\partial x) = \partial v/\partial t \tag{1.2}$$

with initial condition $v(x,0) = \phi(x)$ for $(x,t) \in X \times (0,T]$, where the Hamiltonian is defined as

$$H(x,p) = \min_a \max_b \left\{ p\left(f(x) + g(x)a + h(x)b\right) + \frac{1}{2}x^2 + \frac{1}{2}a^2 - \frac{\gamma^2}{2}b^2 \right\}.$$

Note this Hamiltonian is non-convex in p.

Suppose we choose some feedback function $\hat{a}(x)$ and, on any solution trajectory $(x(.), a(.), b(.))$, define the control input $a(s) = \hat{a}(x(s))$ for all s. We can then define $f_{\hat{a}}(x,b) = f(x) + g(x)\hat{a}(x) + h(x)b$ and $l_{\hat{a}}(x,b) = \frac{1}{2}x^2 + \frac{1}{2}\hat{a}(x)^2 - \frac{\gamma^2}{2}b^2$, and consider the finite horizon optimal control problem

$$v_{\hat{a}}(x,T) = \sup_{b(.)} \int_0^T l_{\hat{a}}(x(s), b(s))ds + \phi(x(T)) \tag{1.3}$$

over trajectories $(x(.), b(.))$ satisfying $\dot{x}(s) = f_{\hat{a}}(x(s), b(s)), \quad x(0) = x$. In this case, the value function $v_{\hat{a}}(x,t)$ satisfies the Hamilton-Jacobi equation

$$H_{\hat{a}}(x, \partial v_{\hat{a}}/\partial x) = \partial v_{\hat{a}}/\partial t \tag{1.4}$$

with initial condition $v_{\hat{a}}(x,0) = \phi(x)$ for $(x,t) \in X \times (0,T]$, where the Hamiltonian is defined as $H_{\hat{a}}(x,p) = \max_b \{p f_{\hat{a}}(x,b) + l_{\hat{a}}(x,b)\}$. Note that this Hamiltonian is convex in p for all x.

A max-plus analogue of the finite element method (FEM) is set out in [**1**] for the numerical computation of the value function $v_{\hat{a}}$ solving this convex optimal control problem (1.3). In this note, we set out a policy iteration algorithm for the solution of the non-convex differential game (1.1). This involves the use of the max-plus FEM to solve (1.4) for a given fixed control feedback $a(x)$ in the value determination step of the algorithm, and then a QP to improve the control feedback in the policy improvement step.We show here that the algorithm converges. It can also be shown that the approximation error on the converged solution is of order $\sqrt{\Delta t} + \Delta x (\Delta t)^{-1}$, the same order as that obtained in [**1**] for the errors associated with the max-plus FEM. We do not give details of this result here, due to limited space.

2. The Max-Plus Finite Element Method

In the following, let S^t denote the evolution semi-group of the PDE (1.2). This associates to any function ϕ, the function $v^t = v(.,t)$ where v is the value function of the differential game (1.1). Similarly, let $S^t_{\hat{a}}$ denote the evolution semi-group of the PDE (1.4) for some fixed feedback function $\hat{a}(.)$. This associates to any function ϕ, the function $v^t_{\hat{a}} = v_{\hat{a}}(.,t)$ where $v_{\hat{a}}$ is the value function of the otimal control problem (1.3). Maslov [**3**] observed that the semi-group $S^t_{\hat{a}}$ is max-plus linear. We now briefly review the max-plus finite element method (FEM) set out in [**1**] for the numerical computation of $v_{\hat{a}}$

Let $\mathbb{R}_{\max}$ denote the idempotent semi-ring obtained from $\mathbb{R}$, with its usual order $\leq$, by defining idempotent addition as $a \oplus b := \max(a,b)$ and multiplication as $ab := a + b$. Then let $\bar{\mathbb{R}}_{\max} := \mathbb{R}_{\max} \cup \{+\infty\}$, with the convention that $-\infty$ is absorbing for the mutiplication.

For X a set, we consider the set $\bar{\mathbb{R}}^X_{\max}$ of $\bar{\mathbb{R}}_{\max}$ valued functions on X. This is a semimodule over $\bar{\mathbb{R}}_{\max}$ with respect to componentwise addition $(u,v) \longmapsto u \oplus v$, defined by $(u \oplus v)(x) = u(x) \oplus v(x)$, and componentwise scalar multiplication $(\lambda, u) \longmapsto u\lambda$, defined by $(u\lambda)(x) = u(x)\lambda$, where $u, v \in \bar{\mathbb{R}}^X_{\max}$, $\lambda \in \bar{\mathbb{R}}_{\max}$ and $x \in X$. Note that the natural order on $\bar{\mathbb{R}}^X_{\max}$ arising from the idempotent addition, i.e. the order defined by $u \leq v \iff$

$u \oplus v = v$, corresponds to the componentwise partial order $u \leq v \iff u(x) \leq v(x)$ for all $x \in X$.

Now let X and Y be sets and consider an operator $A : \bar{\mathbb{R}}_{\max}^{Y} \to \bar{\mathbb{R}}_{\max}^{X}$ from $\bar{\mathbb{R}}_{\max}$ valued functions on Y to $\bar{\mathbb{R}}_{\max}$ valued functions on X. Such an operator is called linear if , for all $u_1,\ u_2 \in \bar{\mathbb{R}}_{\max}^{Y}$ and $\lambda_1,\ \lambda_2 \in \bar{\mathbb{R}}_{\max}$, $A(u_1\lambda_1 \oplus u_2\lambda_2) = A(u_1)\lambda_1 \oplus A(u_2)\lambda_2$. Given some $\bar{\mathbb{R}}_{\max}$ valued function $a \in \bar{\mathbb{R}}_{\max}^{X\times Y}$ on $X \times Y$, we are then interested in the linear operator $A : \bar{\mathbb{R}}_{\max}^{Y} \to \bar{\mathbb{R}}_{\max}^{X}$ with kernel a which maps any function $u \in \bar{\mathbb{R}}_{\max}^{Y}$ to the function $Au \in \bar{\mathbb{R}}_{\max}^{X}$defined, in terms of the normal arithmetic operations on $\mathbb{R}$, by

$$Au(x) = \sup_{y\in Y} \{a(x, y) + u(y)\} \tag{2.1}$$

Then, as shown in the references cited in [**1**], this kernel operator A is residuated, i.e. for any $v \in \bar{\mathbb{R}}_{\max}^{X}$, the set $\{u \in \bar{\mathbb{R}}_{\max}^{Y} : Au \leq v\}$ has a maximal element. The residual map $A^{\#} : \bar{\mathbb{R}}_{\max}^{X} \to \bar{\mathbb{R}}_{\max}^{Y}$ then takes any $v \in \bar{\mathbb{R}}_{\max}^{X}$ to this maximal element in $\bar{\mathbb{R}}_{\max}^{Y}$ defined, again in terms of the normal arithmetic operations on $\mathbb{R}$, as the function

$$(A^{\#}v)(y) = \inf_{x\in X} \{-a(x, y) + v(x)\} \tag{2.2}$$

The next notion to be introduced, for a kernel operator $B : \bar{\mathbb{R}}_{\max}^{Y} \to \bar{\mathbb{R}}_{\max}^{X}$, is that of projection on the image $\mathrm{im}B$ of B. The projector is denoted $P_{\mathrm{im}B}$ and is a map $\bar{\mathbb{R}}_{\max}^{X} \to \bar{\mathbb{R}}_{\max}^{X}$ defined for all $v \in \bar{\mathbb{R}}_{\max}^{X}$ by $P_{\mathrm{im}B}(v) = \max\{w \in \mathrm{im}B : w \leq v\}$. Again as shown in the references cited in [**1**], this projector on the subsemimodule $\mathrm{im}B$ can be expressed as a composition $P_{\mathrm{im}B} = B \circ B^{\#}$ of B and its residual $B^{\#}$. If $b(x, y)$ denotes the kernel of B, then this formula can be expressed in the normal arithmetic of $\mathbb{R}$, as

$$B \circ B^{\#}(v)(x) = \sup_{y\in Y} \left(b(x, y) + \inf_{\xi\in X} (-b(\xi, y) + v(\xi)) \right) \tag{2.3}$$

Given a kernel operator $C : \bar{\mathbb{R}}_{\max}^{X} \to \bar{\mathbb{R}}_{\max}^{Z}$ with kernel $c(z, x)$, we can consider the transposed operator $C^* : \bar{\mathbb{R}}_{\max}^{Z} \to \bar{\mathbb{R}}_{\max}^{X}$ with kernel $c^*(x, z) = c(z, x)$. We can then define a dual projector on the $\bar{\mathbb{R}}_{\min}$-subsemimodule $-\mathrm{im}C^*$ in terms of $P^{-\mathrm{im}C^*}(v) = \min\{w \in -\mathrm{im}C^* : w \geq v\}$ for all $v \in \bar{\mathbb{R}}_{\max}^{X}$. Then, as above, this projector can be expressed as a composition $P^{-\mathrm{im}C^*} = C^{\#} \circ C$ which, in the normal arithmetic of $\mathbb{R}$, has the form

$$C^{\#} \circ C(v)(x) = \inf_{z\in Z} \left(-c(z, x) + \sup_{\xi\in X} (c(z, \xi) + v(\xi)) \right) \tag{2.4}$$

Now we can define the max-plus FEM for approximating the value function $v_{\hat{a}}^{t} = v_{\hat{a}}(., t)$ for the optimal control problem (1.3). Let $Y =$

$\{1,\dots,I\}$, $X=\mathbb{R}^n$ and $Z=\{1,\dots,J\}$. Consider a family $\{w_1,\dots,w_I\}$ of finite element functions $w_i : X \to \mathbb{R}_{\max}$, and a family $\{z_1,\dots,z_J\}$ of test functions $z_j : X \to \mathbb{R}_{\max}$. The vectors $\lambda=(\lambda_i)_{i=1,\dots,I} \in \bar{\mathbb{R}}_{\max}^I$ and $\mu=(\mu_j)_{j=1,\dots,J} \in \bar{\mathbb{R}}_{\max}^J$can be considered as $\bar{\mathbb{R}}_{\max}$ valued functions on Y and Z respectively. So, as above in equation (2.1), we can define max-plus kernel operators $W : \bar{\mathbb{R}}_{\max}^Y \to \bar{\mathbb{R}}_{\max}^X$ and $Z^* : \bar{\mathbb{R}}_{\max}^Z \to \bar{\mathbb{R}}_{\max}^X$ with kernels $W=\mathrm{col}(w_i)_{1\le i\le I}$ and $Z^*=\mathrm{col}(z_j)_{1\le j\le J}$. The action of W, which plays the role of operator B above, is as follows

$$W\lambda(x)=\sup_{i\in Y}\{w_i(x)+\lambda_i\}$$

while Z^* gives rise to the transposed operator $Z : \bar{\mathbb{R}}_{\max}^X \to \bar{\mathbb{R}}_{\max}^Z$ which plays the role of operator C above, and acts as follows

$$(Zv)_j=\sup_{x\in X}\{z_j(x)+v(x)\}=\langle z_j|v\rangle$$

where $\langle .|.\rangle$ denotes the max-plus scalar product. Then from equations (2.3) and (2.4), we can give the specific form of the corresponding two projectors

$$P_{\mathrm{im}W}(v)(x) = \sup_{i\in Y}\left(w_i(x)+\inf_{\xi\in X}\left(-w_i(\xi)+v(\xi)\right)\right) \tag{2.5}$$

$$P^{-\mathrm{im}Z^*}(v)(x) = \inf_{j\in Z}\left(-z_j(x)+\sup_{\xi\in X}\left(z_j(\xi)+v(\xi)\right)\right) \tag{2.6}$$

To start the algorithm off, we approximate the initial data $v_{\hat{a}}^0=\phi$ with the maximal element $\le v_{\hat{a}}^0$ in the space $\mathrm{im}W$ spanned by the finite element functions. The approximation of $v_{\hat{a}}^0$ is denoted with a subscript h and takes the form

$$v_{\hat{a}h}^0(x)=(W\lambda^0)(x)=\sup_{i\in Y}\left(w_i(x)+\lambda_i^0\right)$$

where the coefficients λ_i^0 are determined from the residuation of W given in formula (2.2) as

$$\lambda_i^0=\inf_{x\in X}\left(-w_i(x)+\phi(x)\right). \tag{2.7}$$

As an induction assumption, suppose that at time step $q\Delta t$ we have a vector of coefficients $\lambda_i^{q\Delta t}$ giving an approximation

$$v_{\hat{a}h}^{q\Delta t}(x)=\sup_{i\in Y}\left(w_i(x)+\lambda_i^{q\Delta t}\right)$$

of $v_{\hat{a}}^{q\Delta t}$ by the maximal element $\le v_{\hat{a}}^{q\Delta t}$ in the space $\mathrm{im}W$. Then the approximation $v_{\hat{a}h}^{(q+1)\Delta t}$ of $v_{\hat{a}}^{(q+1)\Delta t}$ at the next time step can be calculated

as

$$v_{\hat{a}h}^{(q+1)\Delta t}(.) = P_{\mathrm{im}W} \circ P^{-\mathrm{im}Z^*} \circ S_{\hat{a}}^{\Delta t} \circ v_{\hat{a}h}^{q\Delta t}(.)$$

The coefficients of this approximation are given, from equations (2.5) and (2.6), by

$$\lambda_i^{(q+1)\Delta t} = \inf_{\xi \in X} \left(-w_i(\xi) + \inf_{j \in Z} \left(-z_j(\xi) + \sup_{\eta \in X} \left(z_j(\eta) + S_{\hat{a}}^{\Delta t} \circ v_{\hat{a}h}^{q\Delta t}(\eta) \right) \right) \right)$$

It is shown in [**1**] that $v_{\hat{a}h}^{(q+1)\Delta t}$ is the maximal element in the space $\mathrm{im}W$ spanned by the finite element functions which satisfies

$$\left\langle z_j | v_{\hat{a}h}^{(q+1)\Delta t} \right\rangle \leq \left\langle z_j | S_{\hat{a}}^{\Delta t} v_{\hat{a}h}^{q\Delta t} \right\rangle$$

for each test function z_j. So $v_{\hat{a}h}^{(q+1)\Delta t}$ is the maximal solution to a max-plus variational formulation of the semi-group equation.

If (see Section 3.3 of [**1**]) we further approximate the semi-group action $S_{\hat{a}}^{\Delta t} v_{\hat{a}h}^{q\Delta t}$ by

$$\left(\tilde{S}_{\hat{a}}^{\Delta t} v_{\hat{a}h}^{q\Delta t} \right)(x) = \sup_{i \in Y} \left(w_i(x) + \lambda_i^{q\Delta t} + \Delta t H_{\hat{a}}(x, \partial w_i / \partial x) \right)$$

then $\lambda_i^{(q+1)\Delta t}$ can be written explicitly as

$$\begin{aligned} (2.8) \lambda_i^{(q+1)\Delta t} \quad = \quad & \inf_{\xi \in X} \left(-w_i(\xi) + \inf_{j \in Z} \left(-z_j(\xi) + \sup_{\eta \in X} \left(z_j(\eta) \right. \right. \right. \\ & \left. \left. \left. + \sup_{k \in Y} \left(w_k(\eta) + \lambda_k^{q\Delta t} + \Delta t H_{\hat{a}} \left(\eta, \partial w_k / \partial x |_\eta \right) \right) \right) \right) \right) \end{aligned}$$

Finally, choose two sets $(\hat{x}_i)_{i \in Y}$ and $(\hat{x}_j)_{j \in Z}$ of discretisation points, and take the finite element functions to be $w_i(x) = -\frac{c}{2} \left\| x - \hat{x}_i \right\|_2^2$, for some fixed Hessian c, and test functions to be $z_j(x) = -a \left\| x - \hat{x}_j \right\|_1$, for some fixed constant a. Then it is shown in Theorem 22 of [**1**] that the error $\left\| v_{\hat{a}h}^T - v_{\hat{a}}^T \right\|_\infty = O(\Delta t + \Delta x (\Delta t)^{-1})$, where Δx is the maximal radius of the cells of the two Voronoi tessellations centred on the points $(\hat{x}_i)_{i \in Y}$ and $(\hat{x}_j)_{j \in Z}$ respectively.

3. Policy Iteration with Max-Plus FEM in the Value Determination Step

Now let p denote the cycle index within the policy iteration algorithm, and let $q \in \{0, \ldots, N-1\}$ denote the time step index, so that the full time horizon T is divided into N equal steps of length Δt, i.e. $T = N\Delta t$, with the qth step running from $q\Delta t$ to $(q+1)\Delta t$. We restrict consideration of time-dependent feedback control policies $a(x,t)$ to those in the form of

sequences of N constant-in-time policy components $\left(a^0(x), \ldots, a^{N-1}(x)\right)$, and we then further restrict our choice of the individual policy components to functions $a^q(.)$ chosen from the set $A = \{a(.) : X \to U\}$ of functions which are locally constant with respect to x on cells of the Voronoi tessellation V_Y centred on the origins $(\hat{x}_i)_{i\in Y}$ of the finite element functions w_i.

So suppose, as an induction hypothesis, that on iteration p, we have a set of constants $\{a_p^{qi}\}$ for $q \in \{0, \ldots, N-1\}$ and $i \in Y$. These give rise to a fixed policy a_p which, for a given q, takes the form $a_p^q(x) = a_p^{q\mu(x)}$, where $\mu(x) \in Y$ is the index of the cell of the Voronoi tessellation V_Y containing x. Note, the process can be initiated, for $p = 0$, by choosing some fixed value a^i (say zero) such that $a_0^q(x) = a^i$ for all $q \in \{0, \ldots, N-1\}$ and for all $x \in$ cell i of V_Y, where cell i is the one centred on the origin $\hat{x}_i$ of finite element w_i.

3.1. Value Determination Step. The max-plus FEM outlined above can be applied to approximate the value function $v_{a_p}^t$ solving the optimal control problem (1.3) with fixed strategy a_p. The coefficients of the expansion of this approximation, with respect to the finite elements w_i, are obtained as follows. For $q = 0$ and $i \in Y$, the coefficients $\lambda_{pi}^0 = \lambda_i^0$ defined in (2.7) above. Then, using (2.8), for $q \in \{0, \ldots, N-1\}$ and $i \in Y$ we get

$$\begin{aligned}\lambda_{pi}^{(q+1)\Delta t} &= \inf_{\xi\in X}\Bigg(-w_i(\xi) + \inf_{j\in Z}\Bigg(-z_j(\xi) + \sup_{\eta\in X}\left(z_j(\eta)\right.\\ &\qquad \left.+ \sup_{k\in Y}\left(w_k(\eta) + \lambda_{pk}^{q\Delta t} + \Delta t H_{a_p^q}\left(\eta, \partial w_k/\partial x|_\eta\right)\right)\right)\Bigg)\Bigg)\end{aligned}$$

Note that in the Hamiltonian $H_{a_q^p}$ we apply the policy $a_p^q(\eta) = a_p^{q\mu(\eta)}$ where $\mu(\eta) \in Y$ is the index of the cell of the Voronoi tessellation V_Y containing η. The above can be re-arranged to give

$$\begin{aligned}\lambda_{pi}^{(q+1)\Delta t} &= \inf_{j\in Z}\Bigg(-\langle w_i|z_j\rangle + \sup_{k\in Y}\Bigg(\lambda_{pk}^{q\Delta t} + \sup_{\eta\in X}\left(z_j(\eta)\right.\\ &\qquad \left.+w_k(\eta) + \Delta t H_{a_p^q}\left(\eta, \partial w_k/\partial x|_\eta\right)\right)\Bigg)\Bigg)\end{aligned}$$

For a given policy a, let

$$T_{jka} = \sup_{\eta\in X}\left(z_j(\eta) + w_k(\eta) + \Delta t H_a\left(\eta, \partial w_k/\partial x|_\eta\right)\right)$$

In the normal max-plus FEM, the T_{jka} terms can be calculated offline. This would be difficult in the application of max-plus FEM to policy iteration, since we don't know the policies a in advance. The relevant a for each p iteration is known at the start of that iteration and so, in principle, the next set of T_{jka} terms for a given a could be calculated at the start of that iteration. However, this would be slow. An alternative is to approximate the T_{jka} online by

$$\tilde{T}_{jka} = \langle z_j | w_k \rangle + \Delta t H_a \left(\eta_{jk}^{opt}, \partial w_k / \partial x |_{\eta_{jk}^{opt}} \right)$$

where $\eta_{jk}^{opt} = \arg\sup \langle z_j | w_k \rangle = \arg\sup \left(z_j(\eta) + w_k(\eta) \right)$. Note, this approximation $\tilde{T}$ is presented in [**1**], where it is shown in Theorem 22 that the resulting error estimate on the max-plus FEM deteriorates to $\left\| v_{ah}^T - v_a^T \right\|_\infty = O(\sqrt{\Delta t} + \Delta x (\Delta t)^{-1})$. So, finally, the coefficients of the expansion of the approximation to the value function $v_{a_p}^t$ for fixed strategy a_p are given by

$$\lambda_{pi}^{(q+1)\Delta t} = \inf_{j \in Z} \left(- \langle w_i | z_j \rangle + \sup_{k \in Y} \left(\lambda_{pk}^{q\Delta t} + \tilde{T}_{jka_p^q} \right) \right) \tag{3.1}$$

3.2. Policy Improvement Step. For each i and q, there exists $\bar{j}(iq) \in Z$ which achieves the inf in (3.1), so that

$$\lambda_{pi}^{(q+1)\Delta t} = - \left\langle w_i | z_{\bar{j}} \right\rangle + \sup_{k \in Y} \left(\lambda_{pk}^{q\Delta t} + \tilde{T}_{\bar{j}ka_p^q} \right) \tag{3.2}$$

For each k, the Hamiltonian within $\tilde{T}_{\bar{j}ka_q^p}$ is evaluated at $\eta_{\bar{j}k}^{opt}$, and so the strategy a_p^q applied in the Hamiltonian term takes the value $a_p^{q\mu(\bar{j}k)}$, where $\mu(\bar{j}k) \in Y$ is the index of the cell of the Voronoi tessellation V_Y containing $\eta_{\bar{j}k}^{opt}$.

The policy improvement can be formulated for test functions given by $z_j(x) = -a \left\| x - \hat{x}_j \right\|_1$ for some constant a. Here, due to lack of space, we consider only the special case where the constant term $a \to \infty$ in the test functions $z_j(x)$, so that they are therefore defined as

$$z_j = \begin{cases} 0 \text{ at } x = \hat{x}_j \\ -\infty \text{ otherwise} \end{cases} \tag{3.3}$$

Then we have $\eta_{\bar{j}k}^{opt} = \hat{x}_{\bar{j}}$ for all $k \in Y$ and $\mu(\bar{j}k) = \mu(\bar{j}) \in Y$ is the index of the cell of the Voronoi tessellation V_Y containing $\hat{x}_{\bar{j}}$. It follows that $a_p^q(\hat{x}_{\bar{j}}) = a_p^{q\mu(\bar{j})}$ is the policy value applied in the Hamiltonian term in $\tilde{T}_{\bar{j}ka_p^q}$ for all k. So every term $\tilde{T}_{\bar{j}ka_p^q}$ uses the same policy value $a_p^{q\mu(\bar{j})}$ for all $k \in Y$ within the $\sup_{k \in Y}$ operation in (3.2).

Now let $\bar{k}(iq) = \arg\sup_{k\in Y}$ in (3.2), so that

$$\lambda_{pi}^{(q+1)\Delta t} = -\left\langle w_i | z_{\bar{j}} \right\rangle + \lambda_{p\bar{k}}^{q\Delta t} + \tilde{T}_{\bar{j}\bar{k}a_p^q}$$

Then we can improve the policy a_p^q in cell $\mu(\bar{j})$ of V_Y by taking

$$\min_{a\in U} \tilde{T}_{\bar{j}\bar{k}a} \tag{3.4}$$

subject to

$$\lambda_{p\bar{k}}^{q\Delta t} + \tilde{T}_{\bar{j}\bar{k}a} \geq \lambda_{pk}^{q\Delta t} + \tilde{T}_{\bar{j}ka} \tag{3.5}$$

for all $k \in Y$. This optimisation is feasible since the current policy value $a_p^{q\mu(\bar{j})}$ satisfies

$$\lambda_{p\bar{k}}^{q\Delta t} + \tilde{T}_{\bar{j}\bar{k}a_p^{q\mu(\bar{j})}} \geq \lambda_{pk}^{q\Delta t} + \tilde{T}_{\bar{j}ka_p^{q\mu(\bar{j})}}$$

for all $k \in Y$.

Let $\bar{a} = \arg\min_{a\in U} \tilde{T}_{\bar{j}\bar{k}a}$ subject to the constraints (3.5). In cell with index $\mu(\bar{j})$ of the Voronoi tessellation V_Y, take new policy

$$a_{p+1}^q(x) = a_{p+1}^{q\mu(\bar{j})} := \bar{a}$$

for all $x \in$ cell with index $\mu(\bar{j})$. Note that for each q, there may be some remaining cells of V_Y whose indices $\neq \mu(\bar{j}(iq))$ for any $i \in Y$. In these cells we leave the policy at time step q unchanged, i.e. if μ^* is the index of such a cell, then for all $x \in$ cell with index μ^*

$$a_{p+1}^q(x) = a_p^{q\mu^*}$$

Then the resulting new policy $a_{p+1} = \{a_{p+1}^{qi}\}$ is an improvement on the old one $a_p = \{a_p^{qi}\}$ in the sense that the corresponding $v_{a_p h}^{q\Delta t}$ and $v_{a_{(p+1)}h}^{q\Delta t}$, i.e. the approximations to the value functions which solve the optimal control problem (1.3) with fixed policies a_{p+1} and a_p respectively, satisfy

$$v_{a_{(p+1)}h}^{q\Delta t} \leq v_{a_p h}^{q\Delta t} \tag{3.6}$$

for all $q \in \{0, \ldots, N\}$.

To see this, note first that the policy improvement is unique. If, for a given q, there are two i giving rise to the same $\bar{j}(iq)$, then these both result in the same policy improvement $a_{p+1}^q(x) = \bar{a}$ in cell $\mu(\bar{j})$ since the term $\tilde{T}_{\bar{j}ka_p^q}$ in (3.2) does not depend on i.

Next, suppose with a view to induction on q, that $\lambda^{q\Delta t}_{(p+1)i} \leq \lambda^{q\Delta t}_{pi}$ for all i. Then

$$\begin{aligned}
\lambda^{(q+1)\Delta t}_{pi} &= -\langle w_i|z_{\bar{j}}\rangle + \lambda^{q\Delta t}_{p\bar{k}} + \tilde{T}_{\bar{j}\bar{k}a^q_p} \\
&= -\langle w_i|z_{\bar{j}}\rangle + \lambda^{q\Delta t}_{p\bar{k}} + \tilde{T}_{\bar{j}\bar{k}a^{q\mu(\bar{j})}_p} \\
&\geq -\langle w_i|z_{\bar{j}}\rangle + \lambda^{q\Delta t}_{p\bar{k}} + \tilde{T}_{\bar{j}\bar{k}\bar{a}} \\
&= -\langle w_i|z_{\bar{j}}\rangle + \sup_{k\in Y}\left(\lambda^{q\Delta t}_{pk} + \tilde{T}_{\bar{j}k\bar{a}}\right) \\
&\geq -\langle w_i|z_{\bar{j}}\rangle + \sup_{k\in Y}\left(\lambda^{q\Delta t}_{(p+1)k} + \tilde{T}_{\bar{j}k\bar{a}}\right) \\
&= -\langle w_i|z_{\bar{j}}\rangle + \sup_{k\in Y}\left(\lambda^{q\Delta t}_{(p+1)k} + \tilde{T}_{\bar{j}ka^q_{p+1}}\right)
\end{aligned}$$

since every term $\tilde{T}_{\bar{j}ka^q_{p+1}}$ uses the same policy value in the same cell $\mu(\bar{j})$. So

$$\lambda^{(q+1)\Delta t}_{pi} \geq \inf_j\left(-\langle w_i|z_j\rangle + \sup_{k\in Y}\left(\lambda^{q\Delta t}_{(p+1)k} + \tilde{T}_{jka^q_{p+1}}\right)\right) = \lambda^{(q+1)\Delta t}_{(p+1)i}$$

Since this holds for all i, then it follows that for any $x \in X$

$$\sup_i\left(\lambda^{(q+1)\Delta t}_{(p+1)i} + w_i(x)\right) \leq \sup_i\left(\lambda^{(q+1)\Delta t}_{pi} + w_i(x)\right)$$

i.e. $v^{(q+1)\Delta t}_{a_{(p+1)}h}(x) \leq v^{(q+1)\Delta t}_{a_ph}(x)$. So by induction, after noting that from (2.7), the coefficients at the initial time step $q = 0$ satisfy $\lambda^0_{(p+1)i} = \lambda^0_{pi} = \lambda^0_i$, it follows that $\lambda^{q\Delta t}_{(p+1)i} \leq \lambda^{q\Delta t}_{pi}$ and $v^{q\Delta t}_{a_{(p+1)}h}(x) \leq v^{q\Delta t}_{a_ph}(x)$ for all q.

Finally by taking $b = 0$ in (1.3), and by restricting our choice of initial data to functions $\phi \geq 0$, we can see that $v^t_{a_p} \geq 0$ for all p. Since $\left\|v^t_{a_ph} - v^t_{a_p}\right\|_\infty = O(\sqrt{\Delta t} + \Delta x(\Delta t)^{-1})$, it follows that

$$v^t_{a_ph} \geq -K(\sqrt{\Delta t} + \Delta x(\Delta t)^{-1}) \tag{3.7}$$

for some $K > 0$.

Hence, the above policy iteration algorithm converges to a time-discretised finite element approximation $v^t_h(.) = \left\{v^{\Delta t}_h(.), \ldots, v^{N\Delta t}_h(.)\right\}$ to the value function v^t of the differential game (1.1).

3.3. QP Optimisation. The Hamiltonian appearing in (1.4) has the form

$$\begin{aligned} H_a(x,p) &= \max_b \{pf_a(x,b) + l_a(x,b)\} \\ &= p(f+ga) + \frac{1}{2}x^2 + \frac{1}{2}a^2 - \frac{1}{2\gamma^2}phh^Tp \end{aligned}$$

and for z_j given by (3.3), $\tilde{T}_{\bar{j}\bar{k}a}$ has the specific form

$$\tilde{T}_{\bar{j}\bar{k}a} = w_{\bar{k}}(\hat{x}_{\bar{j}}) + \Delta t H_a\left(\hat{x}_{\bar{j}}, \partial w_{\bar{k}}/\partial x|_{\hat{x}_{\bar{j}}}\right)$$

The policy improvement optimisation set out in (3.4) and (3.5) can then be formulated as

$$\min_{a\in U} \Delta t H_a\left(\hat{x}_{\bar{j}}, \partial w_{\bar{k}}/\partial x|_{\hat{x}_{\bar{j}}}\right)$$

subject to

$$\begin{aligned} \Delta t H_a\left(\hat{x}_{\bar{j}}, \partial w_{\bar{k}}/\partial x|_{\hat{x}_{\bar{j}}}\right) \geq \lambda_{pk}^{q\Delta t} - \lambda_{p\bar{k}}^{q\Delta t} + w_k(\hat{x}_{\bar{j}}) - \\ -w_{\bar{k}}(\hat{x}_{\bar{j}}) + \Delta t H_a\left(\hat{x}_{\bar{j}}, \partial w_k/\partial x|_{\hat{x}_{\bar{j}}}\right) \end{aligned}$$

for all $k \in Y$. This can be simplified down to the following QP

$$\min_{a\in U}\left(\frac{\partial w_{\bar{k}}}{\partial x}ga + \frac{1}{2}a^2\right)$$

subject to

$$\begin{aligned} \left(\frac{\partial w_{\bar{k}}}{\partial x} - \frac{\partial w_k}{\partial x}\right)ga \quad \geq \quad & \frac{1}{\Delta t}\left(w_k - w_{\bar{k}}\right) + \frac{1}{\Delta t}\left(\lambda_{pk}^{q\Delta t} - \lambda_{p\bar{k}}^{q\Delta t}\right) \\ & + \frac{1}{2\gamma^2}\left(\frac{\partial w_k}{\partial x} - \frac{\partial w_{\bar{k}}}{\partial x}\right)^T hh^T\left(\frac{\partial w_k}{\partial x} - \frac{\partial w_{\bar{k}}}{\partial x}\right) \\ & + \left(\frac{\partial w_k}{\partial x} - \frac{\partial w_{\bar{k}}}{\partial x}\right)f \end{aligned}$$

for all $k \in Y$, and evaluated at $\hat{x}_{\bar{j}}$.

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Using max-plus convolution to obtain fundamental solutions for differential equations with quadratic nonlinearities[1]

William M. McEneaney

1. Introduction

We first consider time-invariant differential Riccati equations (DREs) of the form

$$\dot{P}_t = F(P_t) \doteq A'P_t + P_tA + C + P_t\Sigma P_t \tag{1.1}$$

where C is symmetric and $\Sigma = \sigma\sigma'$ is symmetric, nonnegative definite with at least one positive eigenvalue. Throughout, we assume that all of the matrices are $n \times n$. We suppose one has initial condition, $P_0 = p_0$ where p_0 is also symmetric. The Daivson-Maki approach uses the Bernoulli substitution to create a linear system of two matrices, each of the same size as P_t, thus leading to a fundamental solution. We obtain a completely different form of fundamental solution, with a particularly clear control-theoretic motivation. The new approach will be constructed through a finite-dimensional semigroup defined by this fundamental solution. The forward propagation of the fundamental solution is naturally defined by this operation through the semigroup property.

We will consider linear/quadratic control problems parameterized by $z \in I\!R^n$, and the value functions associated with these control problems are propagated forward by a max-plus linear semigroup, which we denote as S_τ. The space of semiconvex functions is a max-plus vector space (moduloid) [**10**], [**4**], [**2**], [**3**], [**7**]. Working in the semiconvex-dual space, S_τ has a

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semiconvex-dual operator, $\mathcal{B}_\tau$ which takes the form of a max-plus integral operator with kernel, $B_\tau(x,y)$, taking the form of a quadratic function. The matrix, β_τ, defining this quadratic kernel function will be the fundamental solution of the DRE. We will define a multiplication operation ($\circledast$-multiplication) with the semigroup property, specifically $\beta_{t+\tau} = \beta_t \circledast \beta_\tau$, where the $\circledast$ operation involves inverse, multiplication and addition $n \times n$-matrix operations (in the standard algebra). We will also define an exponentiation operation ($\circledast$-exponentiation) such that $\beta_t = \beta_1^{\circledast t}$. The solution of (1.1) will be obtained by $P_t = D_\psi^{-1} \beta_t D_\psi p_0$ where the D_ψ and D_ψ^{-1} operators are descended from the semiconvex dual and its inverse. It is important to note that the fundamental solution approach has the benefit that one only solves once for β_t, even if one wishes to solve the DRE for a variety of initial conditions.

This approach may be extended to a class of quasilinear, first-order PDEs, yielding a fundamental solution for a class of such PDEs. More specifically, we consider PDEs

$$0 = -P_t + A'P - BP_\lambda + \tfrac{1}{2} P\Sigma P \tag{1.2}$$

on the domain $[0,T] \times \mathcal{L}$ where $\mathcal{L} \doteq [0,L]$. For simplicity, we consider the scalar case, and so $A, B, \Sigma \in I\!R$, $\Sigma > 0$, where we specifically require $B \neq 0$ (otherwise this reduces to an ODE problem).

We again create a linear/quadratic control problem, where in this case, the state takes values in $L_2(\mathcal{L})$. The above PDE is essentially the "Riccati" equation for this virtual control problem. We again apply semiconvex duality, and max-plus vector space concepts. This leads to an extension of the $\circledast$ operator to this infinite-dimensional context, and finally, a fundamental solution for this class of PDEs.

2. The linear-quadratic control problem and semigroups

The proofs of the results in the sections on the DRE may be found in [**9**].

As indicated above, the fundamental solution to the DRE will be obtained through an associated optimal control problem. Recall that we are considering the DRE given by (1.1). Since we will be employing semiconvex duality (see below and [**4, 10**]), we will choose some (duality-parametrizing) symmetric matrix, Q, such that $F(Q) > 0$, where we note that, for any square matrix D, we will use the notation $D > 0$ to indicate that matrix D is positive definite throughout. We will need to consider the specific solution of DRE (1.1) with initial condition

$$\widetilde{P}_0 = Q. \tag{2.1}$$

We assume:
There exists a solution of DRE (1.1), $\widetilde{P}_t$, with initial condition (2.1), satisfying $\widetilde{P}_t > Q$ (i.e., $\widetilde{P}_t - Q$ positive-definite) for $t \in (0, \overline{T})$ with $\overline{T} > 0$, and we note specifically, that we may have $\overline{T} = +\infty$. (A.e)

We will be obtaining the fundamental solution β_t for solutions with initial conditions, $P_0 = p_0 > Q$. Note that we do not assume stability of the DRE, and finite-time blow-up is possible. We will let $\widetilde{T} = \widetilde{T}(p_0) = \sup\{t \geq 0 \mid P_t \text{ exists, and } P_t > Q\}$, and we let $\widehat{T} = \widehat{T}(p_0) \doteq \overline{T} \wedge \widetilde{T}$ where $\wedge$ indicates the minimum operation.

REMARK 2.1. Note that with $\Sigma \geq 0$ and at least one positive eigenvalue, we may take $Q = -kI$ for arbitrarily large k, so that one can ensure $F(Q) > 0$ (as well as for any $p_0 > Q$).

We will be using a control value function to motivate and develop the fundamental solution. Consider the Hamilton-Jacobi-Bellman partial differential equation (HJB PDE) problems on $[0, \overline{T}) \times I\!R^n$, indexed by $z \in I\!R^n$, given by

$$V^z_t = H(x, \nabla V^z) = (Ax)'\nabla V^z + \tfrac{1}{2}x'Cx + (\nabla V^z)'\Sigma\nabla V^z \tag{2.2}$$

$$V^z(0, x) = \psi(z, x) = \tfrac{1}{2}(x - z)'Q(x - z). \tag{2.3}$$

THEOREM 2.2. *For any $z \in I\!R^n$, there exists a solution to (2.2),(2.3) in $C^\infty([0, \overline{T}) \times I\!R^n) \cap C([0, \overline{T}) \times I\!R^n)$, and this is given by*

$$V^z = \tfrac{1}{2}(x - \Lambda_t z)'\widetilde{P}_t(x - \Lambda_t z) + z'R_t z \tag{2.4}$$

where $\widetilde{P}$ satisfies (1.1),(2.1), and Λ, r satisfy $\Lambda_0 = I$, $R_0 = 0$,

$$\dot{\Lambda} = \left[\widetilde{P}^{-1}C - A\right]\Lambda \quad \text{and} \quad \dot{R} = \Lambda' C \Lambda. \tag{2.5}$$

For $\phi : I\!R^n \to I\!R$ given by $\phi(x) = \frac{1}{2}(x - z)'p_0(x - z)$ (and actually for a much larger set of functions), we define the max-plus linear semigroup, S_τ, by

$$S_\tau[\phi](x) = V^z(\tau, x) = \tfrac{1}{2}(x - \Lambda_\tau z)'\widetilde{P}_\tau(x - \Lambda_\tau z) + z'R_\tau z. \tag{2.6}$$

We let $\oplus, \otimes$ denote the max-plus addition and multiplication operations. We say that ϕ is uniformly semiconvex with (symmetric matrix) constant K if $\phi(x) + \frac{1}{2}x'Kx$ is convex on $I\!R^n$, and we denote this space as $\mathcal{S}^K(I\!R^n)$. Recall that $\mathcal{S}^K$ is a max-plus vector space.

We will use the quadratic ψ given in (2.3) to define our semiconvex duality. The main duality result (c.f., [**10**], [**4**], where proofs may be found) is

THEOREM 2.3. *Let $\phi \in \mathcal{S}^K(\mathbb{R}^n)$ where $-K > Q$. Then, for all $x, z \in \mathbb{R}^n$,*

$$\phi(x) = \max_{z\in\mathbb{R}^n} [\psi(x,z) + a(z)] \doteq \int^{\oplus}_{\mathbb{R}^n} \psi(x,z) \otimes a(z)\, dz \doteq \psi(x,\cdot) \odot a(\cdot) \doteq \mathcal{D}_\psi^{-1}[a] \tag{2.7}$$

$$\begin{aligned} a(z) &= -\int^{\oplus}_{\mathbb{R}^n} \psi(x,z) \otimes [-\phi(x)]\, dx \\ &= -\{\psi(\cdot,z) \odot [-\phi(\cdot)]\} \doteq \mathcal{D}_\psi[\phi]. \end{aligned} \tag{2.8}$$

Using Theorem 2.3 and some technical arguments, for all $t \in (0, \overline{T})$ and all $x, z \in \mathbb{R}^n$

$$\begin{aligned} S_t[\psi(\cdot,z)](x) &= \int^{\oplus}_{\mathbb{R}^n} \psi(x,y) \otimes B_t(y,z)\, dy \\ &= \psi(x,\cdot) \odot B_t(\cdot,z), \end{aligned} \tag{2.9}$$

where for all $y \in \mathbb{R}^n$

$$\begin{aligned} B_t(y,z) &= -\int^{\oplus}_{\mathbb{R}^n} \psi(x,y) \otimes \left\{-S_t[\psi(\cdot,z)](x)\right\} dx \\ &= \left\{\psi(\cdot,y) \odot [S_t[\psi(\cdot,z)](\cdot)]^-\right\}^-. \end{aligned} \tag{2.10}$$

We define the time-indexed max-plus linear operators $\mathcal{B}_t$ by

$$\mathcal{B}_t[a](z) \doteq B_t(\cdot,z) \odot a(\cdot) = \int^{\oplus}_{\mathbb{R}^n} B_t(y,z) \otimes a(y)\, dy, \tag{2.11}$$

and one easily sees that these satisfy the semigroup property. (We may use a space of uniformly semiconcave functions as the domain.) We say that B_t is the kernel of max-plus integral operator $\mathcal{B}_t$.

THEOREM 2.4. *Let $\phi(x) \doteq \frac{1}{2}x' p_0 x$ and $a(z) = \mathcal{D}_\psi[\phi]$. Then, for $t \in (0, \widehat{T})$, $x \in \mathbb{R}^n$,*

$$S_t[\phi](x) = \psi(x,\cdot) \odot \mathcal{B}_t[a](\cdot) = \mathcal{D}_\psi^{-1}\mathcal{B}_t[a](x) = \mathcal{D}_\psi^{-1}\mathcal{B}_t\mathcal{D}_\psi[\phi](x). \tag{2.12}$$

Now, note that by (2.6) and (2.10),

$$B_t(x,y) = -\max_{x\in\mathbb{R}^n}\left\{\tfrac{1}{2}(x-y)'Q(x-y) - \left[\tfrac{1}{2}(x-\Lambda_t z)'\widetilde{P}_t(x-\Lambda z) + \tfrac{1}{2}z'R_t z\right]\right\} \tag{2.13}$$

where $t < \overline{T}$ guarantees strict concavity of the argument of the maximum.

LEMMA 2.5. *Let η and α be $2n \times 2n$ matrices with block structure given by*

$$\eta = \begin{bmatrix} \eta^{1,1} & \eta^{1,2} \\ \eta^{1,2\prime} & \eta^{2,2} \end{bmatrix} \quad \text{and} \quad \alpha = \begin{bmatrix} \alpha^{1,1} & \alpha^{1,2} \\ \alpha^{1,2\prime} & \alpha^{2,2} \end{bmatrix}, \tag{2.14}$$

Let

$$F(x,z) \doteq \max_{z \in \mathbb{R}^n} \left\{ \tfrac{1}{2} \begin{pmatrix} x \\ z \end{pmatrix}' \eta \begin{pmatrix} x \\ z \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} z \\ y \end{pmatrix}' \alpha \begin{pmatrix} z \\ y \end{pmatrix} \right\}.$$

Then,

$$F(x,y) = \tfrac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}' \gamma \begin{pmatrix} x \\ y \end{pmatrix}$$

where γ has identical block structure to η and α, and is given by $\gamma = \eta \circledast \alpha$ where the $\circledast$ operation is defined as

$$\gamma^{1,1} = \eta^{1,1} - \eta^{1,2} S^{-1} \eta^{1,2\prime}, \qquad \gamma^{1,2} = -\eta^{1,2} S^{-1} \alpha^{1,2},$$
$$\gamma^{2,1} = \gamma^{1,2\prime}, \qquad \gamma^{2,2} = \alpha^{2,2} - \alpha^{1,2\prime} S^{-1} \alpha^{1,2},$$

and $S \doteq \eta^{2,2} + \alpha^{1,1}$.

Combining (2.13) and Lemma 2.5, one obtains the following.

THEOREM 2.6.

$$B_t(x,y) = \tfrac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}' \beta_t \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.15}$$

where β_t has the same block structure as η above.

3. The DRE fundamental solution semigroup

Now we will use the semigroup nature of the S_t operators to obtain the semigroup nature of the $\mathcal{B}_t$ operators, and consequently the propagation of the B_t and β_t. The propagation of $\beta_t = (\beta_1)^{\circledast t}$ will be the dynamics of the fundamental solution of the DRE.

LEMMA 3.1. *Let $a(z) = \frac{1}{2}(z - \overline{z})' q_a (z - \overline{z}) + r_a$ with $q_a < -Q$, and $\phi = \mathcal{D}_\psi^{-1} a$. Then,*

$$\phi(x) = \tfrac{1}{2}(x - \overline{z})' \left[Q U^{-1} q_a \right] (x - \overline{z}) + r_a \tag{3.1}$$

where $U = Q + q_a$. Alternatively, let $\phi(x) = \frac{1}{2}(x - \overline{x})' q_p (x - \overline{x}) + r_p$ with $q_p > Q$, and let $a = \mathcal{D}_\psi \phi$. Then, with $\Delta \doteq Q - q_p$

$$a(z) = \tfrac{1}{2}(z - \overline{x})' \left[Q \Delta^{-1} q_p \right] (z - \overline{x}) + r_p. \tag{3.2}$$

Based on this lemma, it is natural to make the following definitions, which inherit notation from $\mathcal{D}_\psi$ and $\mathcal{D}_\psi^{-1}$. For symmetric $q_p > Q$, define $D_\psi[q_p] \doteq Q(Q-q_p)^{-1}q_p$, and for symmetric $q_a < -Q$, define $D_\psi^{-1}[q_a] = Q(Q+q_a)^{-1}q_a$. One may show (see [**9**]):

THEOREM 3.2. *For all $t_1, t_2 \geq 0$ such that $t_1 + t_2 < \overline{T}$,*

$$B_{t_1+t_2}(\zeta, x) = \int_{I\!R^n}^{\oplus} B_{t_1}(\zeta, z) \otimes B_{t_2}(z, x)\, dz \qquad \forall\, x, \zeta \in I\!R^n.$$

THEOREM 3.3. *The forward propagation of semigroup β_t is given by*

$$\beta_{t_1+t_2} = \beta_{t_1} \circledast \beta_{t_2} \tag{3.3}$$

where the $\circledast$ operation is given in Lemma 2.5.

To summarize, suppose one wishes to obtain the solution of (1.1) at time t with initial condition $P_0 = p_0$. Then, one performs the following steps:

- Obtain q_0 from p_0 via $q_0 = D_\psi p_0 = Q(Q-p_0)^{-1}p_0$.
- Obtain q_t from β_t and q_0 via $q_t = \beta_t^{1,1} - \beta_t^{1,2}\left(\beta_t^{2,2} + q_0\right)^{-1}\beta_t^{1,2\prime} \doteq \beta_t \circledast' q_0$.
- Obtain P_t from from q_t via $P_t = D_\psi^{-1} q_t = Q(Q+q_t)^{-1}q_t$.

4. Exponentiation and a Semiring

Recall that for a standard-algebra linear system, one views the fundamental solution as $e^{At} = (e^A)^t$. We would like some similar exponential-type representation here. Naturally, we define $\circledast$-exponentiation for positive integer powers through $\beta^{\circledast 2} = \beta \circledast \beta$, $\beta^{\circledast 3} = [\beta^{\circledast 2}] \circledast \beta$, et cetera. Using Theorem 3.3, this immediately yields $\beta_{nt} = \beta_t^{\circledast n}$. However, this only works for integer powers. We will extend this to positive real powers so that we may simply write $\beta_t = (\beta_1)^{\circledast t}$ for any $t > 0$.

Let $\mathcal{Q}$ denote the set of rationals. Given any $t \in (0,\infty)$, let $e_t \doteq \{s \in (0,\infty) \,|\, \exists p \in \mathcal{Q} \text{ such that } s = pt\}$. As is well-known, the collection of such e_t forms an uncountable set of equivalence classes covering $(0,\infty)$. Suppose $s \in e_t$. Then, there exists $p = m/n$ with $m, n \in \mathcal{N}$ such that $s = pt$. Let $\tau = t/n$. Then, $t = n\tau$ and $s = m\tau$. Consequently, by Theorem 3.3, $\beta_s = \beta_\tau^{\circledast m}$ and $\beta_t = \beta_\tau^{\circledast n}$. With this in mind, we make the following extension of $\circledast$-exponentiation.

DEFINITION 4.1. Let $s = pt$ with $p = m/n$, $m, n \in \mathcal{N}$. We define $\beta_t^{\circledast p} \doteq \beta_\tau^{\circledast m}$ where $\tau = t/n$.

We need to demonstrate that the definition is independent of the choice of $m, n \in \mathcal{N}$. That is, suppose $p = m_0/n_0 = m_1/n_1$. Let $\tau_0 = t/n_0$ and $\tau_1 = t/n_1$. We must show $\beta_{\tau_0}^{\circledast m_0} = \beta_{\tau_1}^{\circledast m_1}$. We will use the following, trivially-verified result.

LEMMA 4.2. $\left[\beta_t^{\circledast n}\right]^{\circledast m} = \beta_t^{\circledast (nm)}$.

With this lemma, the above independence is easily proven. Lastly, one extends the $\circledast$-exponentiation definition to exponents which may not be rational using continuity.

There are underlying semirings with the $\oplus, \circledast$ operations, and this seems to be quite interesting. These semirings are related to the convolution semiring of [**5**]. We only touch on the matter here. Let $a, b \in [0, +\infty) \cup \{+\infty\} \doteq \mathcal{W}^+$. Then define $a \circledast b \doteq ab/(a+b)$ which defines the $\circledast$ operation on $\mathcal{W}^+$. Also, define $\oplus$ on $\mathcal{W}^+$ by $a \oplus b = \max\{a, b\}$.

THEOREM 4.3. *$\langle \mathcal{W}^+, \oplus, \circledast \rangle$ is a commutative idempotent semiring.*

5. First-Order Quasilinear PDE

The same approach, which was used above in the case of the DRE, can be applied to a first-order, quasilinear PDE with a quadratic nonlinearity. This PDE will take the form of a Riccati equation, and we will refer to it as the fully-first-order Riccati PDE (the FFOR PDE). The FFOR PDE will be

$$0 = -P_t + AP - BP_\lambda + \tfrac{1}{2} P\Sigma P \tag{5.1}$$

where the domain will be $[0, T] \times \mathcal{L}$ where $\mathcal{L} \doteq [0, L]$. For simplicity, we consider the scalar case, and so $A, B, \Sigma \in I\!R$, $\Sigma > 0$, where we specifically require $B \neq 0$ (otherwise this reduces to an ODE problem). We let

$$\mathcal{E}_P^B = \begin{cases} (0, T] \times \{0\} & \text{if } B > 0 \\ (0, T] \times \{L\} & \text{if } B < 0. \end{cases}$$

The initial and boundary conditions will be

$$P(0, \lambda) = p_0(\lambda) \qquad \forall\, \lambda \in \mathcal{L} \tag{5.2}$$

$$P(t, \hat{\lambda}) = 0 \qquad \forall\, (t, \hat{\lambda}) \in \mathcal{E}_P^B. \tag{5.3}$$

We will obtain a fundamental solution for (5.1)–(5.3) using technology analogous to that used for the DRE. In order to do so, we must devise a virtual control problem for which the time-reversed version of (5.1) is the associated Riccati equation.

We begin by defining the dynamics of the virtual control problem. The state will take values in $\mathcal{X} = L_2(\mathcal{L}; I\!R)$. The control will take values in $\mathcal{W} = L_2(\mathcal{L}; I\!R)$. In particular, we consider the control space $\mathcal{W}^s = L_2([-T,0], \mathcal{W})$. The domain for the dynamics will be $[-T,0] \times \mathcal{L}$, and for $t \in [-T,0]$, we will have state, $\xi(t,\cdot) \in \mathcal{X}$. Let

$$\mathcal{E}_X^B = \begin{cases} (-T,0] \times \{L\} & \text{if } B > 0 \\ (-T,0] \times \{0\} & \text{if } B < 0. \end{cases}$$

The virtual control problem dynamics is given by first-order PDE, initial condition and boundary condition

$$\xi_t(t,\lambda) = A\xi(t,\lambda) + B\xi_\lambda(t,\lambda) + \sigma w(t,\lambda), \tag{5.4}$$

$$\xi(-T,\cdot) = x_0(\cdot) \in \mathcal{X} \tag{5.5}$$

$$\xi(t,\hat{\lambda}) = 0 \qquad \forall\, (t,\hat{\lambda}) \in \mathcal{E}_X^B. \tag{5.6}$$

Let the inner product and norm on $\mathcal{X}, \mathcal{W}$ be denoted by $\langle x, y\rangle$ and $\|x\|$, respectively. Let $C, Q > 0$. The payoff and value are given by

$$J^z(-T,x,w) = \int_{-T}^{0} \tfrac{1}{2}\langle \xi(t,\cdot), C\xi(t,\cdot)\rangle - \tfrac{1}{2}\|w(t,\cdot)\|^2\, dt + \psi(\xi(0,\cdot), z) \tag{5.7}$$

$$\widehat{W}^z(-T,x) = \sup_{w \in \mathcal{W}^s} J^z(-T,x,w) \tag{5.8}$$

where $\psi(x,z) \doteq \frac{1}{2}\langle \xi(0,\cdot) - z, Q(\xi(0,\cdot) - z)\rangle$.

The first step is to obtain the verification result.

THEOREM 5.1. *Suppose $V^z \in C([0,T] \times \mathcal{X}) \cap C^1((0,T] \times \mathcal{X})$ satisfies*

$$\begin{aligned} 0 = -V_t^z + \langle (\nabla_x V^z, Ax\rangle - \langle (\nabla_x v)_\lambda, Bx\rangle + \\ + \tfrac{1}{2}\|\sigma' \nabla_x V^z\|^2 + \tfrac{1}{2}\langle x, Cx\rangle, \end{aligned} \tag{5.9}$$

$$V^z(0,x) = \psi(x,z), \tag{5.10}$$

$$\nabla_x V^z(t,\hat{\lambda}, x(\hat{\lambda})) = 0 \qquad \forall\, (t,\hat{\lambda}) \in \mathcal{E}_P^B,\ x \in H^1(\mathcal{L}). \tag{5.11}$$

Then, $V^z(T,x) \geq J^z(-T,x,w)$ for all $w \in \mathcal{W}$, for all $x \in H^1(\mathcal{L})$. Further, if there exists a solution, ξ^* to (5.4)–(5.6) with $w^*(t,\lambda,\xi^*(t,\lambda))$, then letting $\tilde{w}^*(t,\lambda) = w^*(t,\lambda,\xi^*(t,\lambda))$, one has $V^z(T,x) = J^z(-T,x,\tilde{w}^*)$, and consequently $V^z(T,x) = \widehat{W}^z(-T,x)$.

The next step is to note that the solution for this problem has a simple form.

THEOREM 5.2. *For any $z \in \mathcal{X}$, there is a solution to (5.9)–(5.11) in $V^z \in C([0,T] \times \mathcal{X}) \cap C^1((0,T] \times \mathcal{X})$ of the form*

(5.12)
$$V^z(t,x) = \tfrac{1}{2}\langle (x - Z(t,\cdot)), \widetilde{P}(t,\cdot)(x - Z(t,\cdot))\rangle + \tfrac{1}{2}\langle Z(t,\cdot), R(t)Z(t,\cdot)\rangle,$$

where $\widetilde{P}$ satisfies (5.1)–(5.3) with

$$p_0(\lambda) = Q \qquad \forall\, \lambda \in \mathcal{L}, \tag{5.13}$$

$Z \in C^1([0,T] \times \mathcal{L})$ satisfies

$$0 = \widetilde{P}Z_t + [A\widetilde{P} + C]Z + B\widetilde{P}Z_\lambda, \tag{5.14}$$

$$Z(0,\lambda) = z(\lambda) \qquad \forall\, \lambda \in \mathcal{L}, \tag{5.15}$$

$$Z(t,\hat{\lambda}) = 0 \qquad \forall\, (t,\hat{\lambda}) \in \mathcal{E}_X^B, \tag{5.16}$$

and $R \in C^1([0,T])$ satisfies

$$R_t = C, \qquad R(0) = 0. \tag{5.17}$$

Note that in the case where $C = 0$, the Z PDE takes the simpler form

$$0 = Z_t + AZ + BZ_\lambda. \tag{5.18}$$

Note also that $Z(t,\cdot)$ is given by a linear operator acting on z, denoted as

$$Z(t,\cdot) = \mathcal{M}(t)[z](\cdot) \doteq \int_{\mathcal{L}} M(t;\cdot,\eta)z(\eta)\, d\eta. \tag{5.19}$$

Using this in (5.12), one has

(5.20)
$$V^z(t,x) = \tfrac{1}{2}\langle (x - \mathcal{M}(t)z), \widetilde{P}(t,\cdot)(x - \mathcal{M}(t)z)\rangle + \tfrac{1}{2}\langle \mathcal{M}(t)z, R(t)\mathcal{M}(t)z\rangle.$$

We may think of $V^z(t,\cdot)$ as given by the max-plus linear semigroup $V^z(t,x) = S_t[\psi(\cdot,z)](x)$. Introducing the semiconvex dual, one may propagate instead in the dual space. The dual-space semigroup operator is naturally found in the form of a max-plus integral operator with some kernel, which we denote by $B(t;x,z)$ for $t \in [0,T]$ and $x, z \in \mathcal{X}$. One obtains $B(t;x,z)$ from

$$B(t;y,z) = -\max_{x \in \mathcal{X}} \Big\{ \tfrac{1}{2}\langle (y-x), Q(y-x)\rangle - \Big[\tfrac{1}{2}\langle (x - \mathcal{M}(t)[z], \widetilde{P}(x - \mathcal{M}(t)[z]\rangle + \tfrac{1}{2}\langle \mathcal{M}(t)z, R(t)\mathcal{M}(t)z\rangle \Big] \Big\}.$$

Further details will appear in the full paper.

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Polynomial quantization on para-hermitian symmetric spaces from the viewpoint of overgroups: an example[1]

Vladimir F. Molchanov

Quantization in the spirit of Berezin on para-Hermitian symmetric spaces G/H was constructed by the author in [2]. One of the variants of quantization is the so-called polynomial quantization (here for the initial algebra of operators, one has to take a representation of the universal enveloping algebra). A construction of polynomial quantization on para-Hermitian symmetric spaces G/H was presented in [4]. For rank one, explicit formulas were given in [3]. In this paper we consider a new approach to the polynomial quantization using the notion of an "overgroup". This

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approach gives the Berezin covariant and contravariant symbols and the Berezin transform in a highly natural and transparent way. In the paper we restrict ourselves to a simple but crucial example: $G = \mathrm{SL}(2,\mathbb{R})$ with the diagonal subgroup H and $\widetilde{G} = G \times G$.

1. Groups, subgroups, a cone, sections

The group $G = \mathrm{SL}(2,\mathbb{R})$ consists of real matrices

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

Its subgroups H, Z, N of G consist of matrices

$$h = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad z_\xi = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}, \quad n_\eta = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix},$$

respectively. The Gauss and "anti-Gauss" decompositions of G are defined by $G = \overline{NHZ}$ and $G = \overline{ZHN}$. The group G acts on Z and N by fractional linear transformations:

$$\xi \mapsto \xi \cdot g = \frac{\alpha\xi + \gamma}{\beta\xi + \delta}, \quad \eta \mapsto \eta \circ g = \frac{\delta\eta + \beta}{\gamma\eta + \alpha}.$$

These actions are obtained when we decompose $z_\xi g$ "by Gauss" and $n_\eta g$ "by anti-Gauss". We can reduce the second action to the first one: $\eta \circ g = \eta \cdot \widehat{g}$ where

$$\widehat{g} = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}.$$

We assume that the groups act from the right, in accordance with this we will write vectors in the row form.

Let us take the following bilinear form in the space $\mathbb{R}^4$:

$$[x, y] = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4.$$

Realize $\mathbb{R}^4$ as the space $\mathrm{Mat}(2,\mathbb{R})$ of real 2×2 matrices:

$$x = \frac{1}{2} \begin{pmatrix} x_1 - x_4 & -x_2 + x_3 \\ x_2 + x_3 & x_1 + x_4 \end{pmatrix}.$$

Denote the matrix corresponding to the vector xJ, $J = \mathrm{diag}\{1,-1,-1,-1\}$, by $x^\natural$. Then the form $[x, y]$ can be written in terms of matrices: $[x, y] = -2\,\mathrm{tr}\,(x^\natural y)$.

As an overgroup for G, we take the direct product $\widetilde{G} = G \times G$. It acts on $\mathrm{Mat}(2,\mathbb{R})$ as follows: to a pair $(g_1, g_2) \in \widetilde{G}$ we assign the transformation

$$x \mapsto g_1^{-1} x g_2. \tag{1.1}$$

This action preserves $\det x = -(1/4)[x,x]$. Therefore $\widetilde{G}$ covers the group $\mathrm{SO}_0(2,2)$ with multiplicity 2, and the kernel of the homomorphism consists of two pairs: (e,e) and $(-e,-e)$, e being the unit matrix in G.

Let $\mathcal{C}$ be the cone in $\mathbb{R}^4$ defined by $[x,x]=0, x\neq 0$ (or $\det x = 0,\ x\neq 0$). Let us take the following two points in $\mathcal{C}$:

$$s^- = (1,0,0,-1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad s^+ = (1,0,0,1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and two parabolic sections $\Gamma^- = \{[x,s^+]=-2\}$ and $\Gamma^+ = \{[x,s^-]=-2\}$ containing s^- and s^+ respectively.

Consider in $\widetilde{G}$ two unipotent subgroups Q^- and Q^+ consisting of pairs (z_ξ, n_η) and (n_η, z_ξ) respectively. They act simply transitively on sections Γ^- and Γ^+ respectively and transfer points s^- and s^+ to the points

$$\begin{aligned} u &= u(\xi,\eta) = (1-\xi\eta,\ -\xi-\eta,\ -\xi+\eta,\ -1-\xi\eta), \\ v &= v(\xi,\eta) = (1-\xi\eta,\ \xi+\eta,\ \xi-\eta,\ 1+\xi\eta), \end{aligned}$$

respectively. Let $u = u(\xi_1,\eta_1)$ and $v = v(\xi_2,\eta_2)$, then

$$[u,v] = -2N(\xi_1,\eta_2)N(\xi_2,\eta_1), \tag{1.2}$$

where

$$N(\xi,\eta) = 1-\xi\eta.$$

In terms of matrices, the vectors u and v are written as follows:

$$u = \begin{pmatrix} 1 & \eta \\ -\xi & -\xi\eta \end{pmatrix} = \begin{pmatrix} 1 \\ -\xi \end{pmatrix} (1 \quad \eta), \ \ v = \begin{pmatrix} -\xi\eta & -\eta \\ \xi & 1 \end{pmatrix} = \begin{pmatrix} -\eta \\ 1 \end{pmatrix} (\xi \quad 1),$$

The relation between u and v is given by: $u = vJ$ or $u = v^\natural$.

Let $\mathcal{X}$ be the section of the cone $\mathcal{C}$ by the plane $x_1 = 1$ (or $\mathrm{tr}\ x = 1$). It is a hyperboloid of one sheet: $-x_2^2 + x_3^2 + x_4^2 = 1$, in $\mathbb{R}^3$.

Using maps of points along generating lines in $\mathcal{C}$, we obtain actions of $\widetilde{G}$ on sections. Let $(g_1,g_2) \in \widetilde{G}$. For $\mathcal{X}$ we have:

$$x \mapsto \widetilde{x} = \frac{g_1^{-1} x g_2}{\mathrm{tr}(g_1^{-1} x g_2)}. \tag{1.3}$$

For Γ^- and Γ^+, these actions are given by fractional linear transformations of ξ and η:

$$u(\xi,\eta) \mapsto u(\xi\cdot g_1, \eta\circ g_2), \quad v(\xi,\eta) \mapsto v(\xi\cdot g_2, \eta\circ g_1).$$

Each of the sections Γ^- and Γ^+ is mapped on $\mathcal{X}$ along the generating lines (almost everywhere):

$$u \mapsto x = \frac{u}{u_1} = \frac{u(\xi,\eta)}{N(\xi,\eta)}, \quad v \mapsto y = \frac{v}{v_1} = \frac{v(\xi,\eta)}{N(\xi,\eta)}.$$

These maps give the following two systems of coordinates ξ, η on $\mathcal{X}$:

$$x = \Big(1, -\frac{\xi+\eta}{N(\xi,\eta)}, -\frac{\xi-\eta}{N(\xi,\eta)}, -\frac{1+\xi\eta}{N(\xi,\eta)}\Big),$$
$$y = \Big(1, \frac{\xi+\eta}{N(\xi,\eta)}, \frac{\xi-\eta}{N(\xi,\eta)}, \frac{1+\xi\eta}{N(\xi,\eta)}\Big).$$

Let us call these coordinates the *horospherical coordinates* corresponding to Γ^- and Γ^+ respectively. The relation between these two systems is given by: $x = yJ$ or $x = y^\natural$.

Let us take the following measures on the sections $\mathcal{X}$, Γ^- and Γ^+:

$$dx = |x_4|^{-1} dx_2\, dx_3, \quad du = d\xi\, d\eta, \quad dv = d\xi\, d\eta.$$

Under the maps mentioned above, the measures are related as follows:

$$dx = dy = 2N(\xi,\eta)^{-2} d\xi\, d\eta.$$

2. Representations of $G = \mathrm{SL}(2,\mathbb{R})$

The representations $T_{\sigma,\varepsilon}$, $\sigma \in \mathbb{C}$, $\varepsilon = 0, 1$, of G act on functions $\varphi(\xi)$ on $\mathbb{R}$ by:

$$(T_{\sigma,\varepsilon}(g)\varphi)(\xi) = \varphi(\xi \cdot g)(\beta\xi + \delta)^{2\sigma,\varepsilon},$$

where we use the notation:

$$t^{\lambda,\nu} = |t|^\lambda \mathrm{sgn}^\nu t, \quad \lambda \in \mathbb{C}, \quad \nu = 0, 1, \quad t \in \mathbb{R} \setminus \{0\}.$$

Together with these representations, we consider the representations

$$\widehat{T}_{\sigma,\varepsilon}(g) = T_{\sigma,\varepsilon}(\widehat{g}),$$

so that

$$(\widehat{T}_{\sigma,\varepsilon}(g)\psi)(\eta) = \psi(\eta \circ g)(\gamma\eta + \alpha)^{2\sigma,\varepsilon}$$

(notice that $\widehat{T}_{\sigma,\varepsilon}$ and $T_{\sigma,\varepsilon}$ are equivalent). The operator $A_{\sigma,\varepsilon}$ defined by

$$(A_{\sigma,\varepsilon}\varphi)(\eta) = \int_{-\infty}^{\infty} (1-\xi\eta)^{-2\sigma-2,\varepsilon} \varphi(\xi)\, d\xi,$$

intertwines $T_{\sigma,\varepsilon}$ with $\widehat{T}_{-\sigma-1,\varepsilon}$ and also $\widehat{T}_{\sigma,\varepsilon}$ with $T_{-\sigma-1,\varepsilon}$. The product $A_{-\sigma-1,\varepsilon}A_{\sigma,\varepsilon}$ is a scalar operator:

$$A_{-\sigma-1,\varepsilon}A_{\sigma,\varepsilon} = \omega(\sigma,\varepsilon) \cdot \mathrm{id},$$

where

$$\omega(\sigma,\varepsilon) = \frac{2\pi}{2\sigma+1} \tan\frac{2\sigma-\varepsilon}{2}\pi.$$

Notice that

$$\omega(-\sigma-1,\varepsilon)=\omega(\sigma,\varepsilon). \tag{2.1}$$

3. Representations of $\widetilde{G}=G\times G$ associated with a cone

For $\lambda\in\mathbb{C}$, $\nu=0,1$, let $\mathcal{D}_{\lambda,\nu}(\mathcal{C})$ denote the space of functions $f\in C^\infty(\mathcal{C})$ satisfying the condition:

$$f(tx)=t^{\lambda,\nu}f(x),\quad x\in\mathcal{C},\quad t\in\mathbb{R}\setminus\{0\}.$$

Let $R_{\lambda,\nu}$ be the representation of $\widetilde{G}$ on $\mathcal{D}_{\lambda,\nu}(\mathcal{C})$ by translations:

$$(R_{\lambda,\nu}(g_1,g_2)f)(x)=f(g_1^{-1}xg_2).$$

In fact, it is the representation of the group $\mathrm{SO}_0(2,2)$ associated with a cone [1]. This representation can be realized on functions on sections of the cone $\mathcal{C}$, see § 1. In the realization on $\mathcal{X}$, the representation $R_{\lambda,\nu}$ is given by (see (1.3)):

$$(R_{\lambda,\nu}(g_1,g_2)f)(x)=f(\widetilde{x})\left\{\mathrm{tr}(g_1^{-1}xg_2)\right\}^{\lambda,\nu},\ x\in\mathcal{X}.$$

On Γ^- and on Γ^+ we have respectively:

$$(R_{\lambda,\nu}(g_1,g_2)f)(\xi,\eta)=f(\xi\cdot g_1,\eta\circ g_2)\Big\{(\beta_1\xi+\delta_1)(\gamma_2\eta+\alpha_2)\Big\}^{\lambda,\nu}, \tag{3.1}$$

$$(R_{\lambda,\nu}(g_1,g_2)f)(\xi,\eta)=f(\xi\cdot g_2,\eta\circ g_1)\Big\{(\beta_2\xi+\delta_2)(\gamma_1\eta+\alpha_1)\Big\}^{\lambda,\nu}. \tag{3.2}$$

Formulas (3.1) and (3.2) show that $R_{\lambda,\nu}(g_1,g_2)$ is the tensor product $T_{\sigma,\varepsilon}(g_1)\otimes\widehat{T}_{\sigma,\varepsilon}(g_2)$ and $T_{\sigma,\varepsilon}(g_2)\otimes\widehat{T}_{\sigma,\varepsilon}(g_1)$ respectively with $\sigma=\lambda/2$.

Define the operator $B_{\lambda,\,\nu}$ in the $\mathcal{X}$-realization by

$$(B_{\lambda,\,\nu}f)(x)=\int_{\mathcal{X}}[x,y]^{-\lambda-2,\nu}f(y)\,dy,\ x\in\mathcal{X}. \tag{3.3}$$

It intertwines $R_{\lambda,\nu}$ with $R_{-\lambda-2,\nu}$. It acts from Γ^- to Γ^+ by

$$(B_{\lambda,\,\nu}f)(u)=2\int_{\Gamma^+}[u,v]^{-\lambda-2,\nu}f(v)\,dv,\quad u\in\Gamma^-,$$

and similarly from Γ^- to Γ^+. By (1.2) it can be written as

$$(B_{\lambda,\,\nu}f)(\xi_1,\eta_1)=$$
$$(-1)^\nu 2^{-\lambda-1}\int_{\Gamma^+}\Big[N(\xi_1,\eta_2)N(\xi_2,\eta_1)\Big]^{-\lambda-2,\nu}f(\xi_2,\eta_2)\,d\xi_2 d\eta_2.$$

It shows that

$$B_{\lambda,\,\nu}=(-1)^\nu 2^{-\lambda-1}A_{\sigma,\nu}\otimes A_{\sigma,\nu},\quad \sigma=\lambda/2.$$

Therefore

$$B_{\lambda,\nu}B_{-\lambda-2,\nu} = \big[\omega(\lambda/2,\nu)\big]^2 \cdot \mathrm{id}. \tag{3.4}$$

Let us go back to the $\mathcal{X}$-realization and use the *both* horospherical coordinate systems. Then

$$(B_{\lambda,\nu}f)(x) = (-1)^\nu 2^{-\lambda-2}\int_{\mathcal{X}}\Big[\frac{N(\xi_1,\eta_2)N(\xi_2,\eta_1)}{N(\xi_1,\eta_1)N(\xi_2,\eta_2)}\Big]^{-\lambda-2,\nu} f(y)\,dy, \tag{3.5}$$

where x and y have coordinates ξ_1,η_1 and ξ_2,η_2 in the horospherical coordinate systems corresponding to Γ^- and Γ^+, respectively.

4. The Berezin symbols and the Berezin transform

The group $\widetilde{G}$ contains three subgroups isomorphic to G. The first one is the diagonal consisting of (g,g), $g\in G$. It preserves $\mathcal{X}$ under the action (1.1), hence $\mathcal{X}=G/H$. The measure dx is invariant. The representation $R_{\lambda,\nu}$ is the representation by translations:

$$R_{\lambda,\nu}(g,g)f(x) = f(g^{-1}xg).$$

Other two subgroups G_1 and G_2 consist of pairs (g,e) and (e,g), $g\in G$, respectively. By (3.2) we have on Γ^+:

$$(R_{\lambda,\nu}(e,g)f)(\xi,\eta) = f(\xi\cdot g,\eta)(\beta\xi+\delta)^{\lambda,\nu}.$$

Therefore, in the horospherical coordinates on $\mathcal{X}$ corresponding to Γ^+, we have that

$$(R_{\lambda,\nu}(e,g)f)(\xi,\eta) = \Big[\frac{1}{N(\xi,\eta)}\Big]^{\lambda,\nu} f(\xi\cdot g,\eta)N(\xi\cdot g,\eta)^{\lambda,\nu}(\beta\xi+\delta)^{\lambda,\nu}. \tag{4.1}$$

This equation can be rewritten as follows. Denote

$$\Phi_{\lambda,\nu}(\xi,\eta) = N(\xi,\eta)^{\lambda,\nu}.$$

It is the kernel of the intertwining operator for G (see Sect. 1) and it is an analogue of the Berezin supercomplete system. Then (4.1) is

$$(R_{\lambda,\nu}(e,g)f)(\xi,\eta) = \frac{1}{\Phi_{\lambda,\nu}(\xi,\eta)}(T_{\lambda/2,\nu}(g)\otimes 1)\big[f(\xi,\eta)\Phi_{\lambda,\nu}(\xi,\eta)\big].$$

Similarly, in the horospherical coordinates on $\mathcal{X}$ corresponding to Γ^-, we obtain that

$$(R_{\lambda,\nu}(e,g)f)(\xi,\eta) = \frac{1}{\Phi_{\lambda,\nu}(\xi,\eta)}(1\otimes \widehat{T}_{\lambda/2,\nu}(g))\big[f(\xi,\eta)\Phi_{\lambda,\nu}(\xi,\eta)\big].$$

(and similar formulas for (g,e)). Let us go from the group G to its universal enveloping algebra $\mathrm{Env}(\mathfrak{g})$ and retain symbols for representations. Then

the dependence of representations on ν disappears and we omit ν for them. Now take for f the function f_0 equal to 1 identically, then for $X \in \mathrm{Env}(\mathfrak{g})$ we obtain that

$$(R_\lambda(0,X)f_0)(\xi,\eta) = \frac{1}{\Phi_{\lambda,\nu}(\xi,\eta)}(T_{\lambda/2}(X)\otimes 1)\Phi_{\lambda,\nu}(\xi,\eta),$$

$$(R_{-\lambda-2}(0,X)f_0)(\xi,\eta) = \frac{1}{\Phi_{-\lambda-2,\nu}(\xi,\eta)}(1\otimes \widehat{T}_{-\lambda/2-1}(X))\Phi_{-\lambda-2,\nu}(\xi,\eta).$$

The right hand sides of these formulas are just the *covariant* and *contravariant* symbols of the operator $T_{\lambda/2}(X)$ in the polynomial quantization.

We can normalize the operator $B_{-\lambda-2,\nu}$ so that the normalized operator $Q_{\lambda,\nu}$ will satisfy the condition

$$Q_{\lambda,\nu}Q_{-\lambda-2,\nu} = \mathrm{id}.$$

Namely,

$$(Q_{\lambda,\,\nu}f)(x) = c(\lambda,\nu)\int_{\mathcal{X}}\Big[\frac{(1-\xi_1\eta_2)(1-\xi_2\eta_1)}{(1-\xi_1\eta_1)(1-\xi_2\eta_2)}\Big]^{\lambda,\nu} f(y)\,dy, \tag{4.2}$$

where x and y have coordinates ξ_1,η_1 and ξ_2,η_2 as in (3.5) and

$$c(\lambda,\nu)^{-1} = 2\omega(\lambda/2,\nu), \tag{4.3}$$

see (2.1), (3.4), (3.5). The kernel in (4.2) (with the factor $c(\lambda,\nu)$) is nothing but the Berezin kernel. Therefore, the operator $Q_{\lambda,\nu}$ is the Berezin transform. It transfers contravariant symbols to covariant ones. Note that if we want to write the Berezin transform using only *one* coordinate system, then we will have to change the operator (3.3), namely, we will have to write $[x,yJ]$ instead of $[x,y]$.

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The structure of max-plus hyperplanes[1]

V. Nitica and I. Singer

A max-plus hyperplane (briefly, a hyperplane) is the set of all points $x = (x_1, ..., x_n)$ in $\mathbb{R}^n_{\max}$ satisfying an equation of the form

$$a_1 x_1 \oplus ... \oplus a_n x_n \oplus a_{n+1} = b_1 x_1 \oplus ... \oplus b_n x_n \oplus b_{n+1},$$

that is,

$$\max(a_1 + x_1, ..., a_n + x_n, a_{n+1}) = \max(b_1 + x_1, ..., b_n + x_n, b_{n+1}),$$

with $a_i, b_i \in \mathbb{R}_{\max}$ $(i = 1, ...n + 1)$, where each side contains at least one term, and where $a_i \neq b_i$ for at least one index i. We show that the complements of (max-plus) semispaces at finite points $z \in \mathbb{R}^n$ are "building blocks" for the hyperplanes in $\mathbb{R}^n_{\max}$ (recall that a semispace at z is a maximal –with respect to inclusion– max-plus convex subset of $\mathbb{R}^n_{\max} \backslash \{z\}$). Namely, observing that, up to a permutation of indices, we may write the equation of any hyperplane H in one of the following two forms:

$$\begin{aligned} &a_1 x_1 \oplus ... \oplus a_p x_p \oplus a_{p+1} x_{p+1} \oplus ... \oplus a_q x_q \\ &= a_1 x_1 \oplus ... \oplus a_p x_p \oplus a_{q+1} x_{q+1} \oplus ... \oplus a_m x_m \oplus a_{n+1}, \end{aligned}$$

where $0 \leq p \leq q \leq m \leq n$ and all a_i $(i = 1, ..., m, n + 1)$ are finite, or,

$$\begin{aligned} &a_1 x_1 \oplus ... \oplus a_p x_p \oplus a_{p+1} x_{p+1} \oplus ... \oplus a_q x_q \oplus a_{n+1} \\ &= a_1 x_1 \oplus ... \oplus a_p x_p \oplus a_{q+1} x_{q+1} \oplus ... \oplus a_m x_m \oplus a_{n+1}, \end{aligned}$$

where $0 \leq p \leq q \leq m \leq n$, and all a_i $(i = 1, ..., m)$ are finite (and a_{n+1} is either finite or $-\infty$), we give a formula that expresses a nondegenerate strictly affine hyperplane (i.e., with $m = n$ and $a_{n+1} > -\infty$) as a union of complements of semispaces at a point $z \in \mathbb{R}^n$, called the "center" of H, with the boundary of a union of complements of other semispaces at z. Using this formula, we obtain characterizations of nondegenerate strictly affine hyperplanes with empty interior. We give a description of the boundary of a nondegenerate strictly affine hyperplane with the aid of complements of semispaces at its center, and we characterize the cases in which the boundary bd H of a nondegenerate strictly affine hyperplane H is also a hyperplane. Next, we give the relations between nondegenerate strictly affine hyperplanes H, their centers z, and their coefficients a_i. In the converse direction we show that any union of complements of semispaces at a point $z \in \mathbb{R}^n$ with the boundary of any union of complements

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of some other semispaces at that point z, is a nondegenerate strictly affine hyperplane. We obtain a formula for the total number of strictly affine hyperplanes. We give complete lists of all strictly affine hyperplanes for the cases $n = 1$ and $n = 2$. We show that each linear hyperplane H in $\mathbb{R}^n_{\max}$ (i. e., with $a_{n+1} = -\infty$) can be decomposed as the union of four parts, where each part is easy to describe in terms of complements of semispaces, some of them in a lower dimensional space.

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Image processing based on a partial differential equation satisfying the pseudo-linear superposition principle[1]

E. Pap and M. Štrboja

We consider a general form of PDE-based methods for image restoration, and give a short overview of the underlying models. In these models, the original image is transformed through a process that can be represented by a second-order partial differential equation. Typically, this role is played by some nonlinear generalization of the heat equation, and it is possible to analyse the solutions from the viewpoint of pseudo-linear (idempotent) analysis. Our main result is that the generalization of the heat equation proposed by Perona and Malik satisfies the pseudo-linear superposition principle.

1. Introduction

The approach based on partial differential equations is well-known in image processing ([**A, C, T**]). In this approach, a restored image can be seen as a version of the initial image at a special scale. Image u is an instance of an evolution process, denoted by $u(t, \cdot)$. The original image is taken at time $t = 0$, $u(0, \cdot) = u_0(\cdot)$. The original image is then transformed, and this process can be described by the equation $\frac{\partial u}{\partial t}(t, x) + F\left(x, u(t, x), \nabla u(t, x), \nabla^2 u(t, x)\right) = 0$, where $x \in \Omega$. Some possibilities for F to restore an image are considered in [**A**].

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Pseudo-linear superposition principle means the following. Instead of the field of real numbers, think of a semiring defined on a real interval $[a, b] \subset [-\infty, \infty]$. This is a structure equipped with pseudo-addition $\oplus$, which is typically idempotent ($x \oplus x = x$), and with pseudo-multiplication $\odot$. The pseudo-linear superposition principle says that some nonlinear equations (ODE, PDE, difference equations, etc.) turn out to be linear over such structures, meaning that if u_1 and u_2 are two solutions of the considered equation, then $a_1 \odot u_1 \oplus a_2 \odot u_2$ is also a solution for any constants a_1 and a_2 from $[a, b]$.

By pseudo-analysis we mean analysis over such semirings, in the framework of [**L, M, N, O, P, Q, R**]. One of its key ideas is the pseudo-linear superposition principle stated above. This (pseudo-) linear intuition leads to the concepts of $\oplus$-measure, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc.

Similar ideas were developed independently by Maslov and his collaborators in the framework of idempotent analysis and idempotent mathematics, with some applications [**G, H, J, K**]. In particular, idempotent analysis is fundamental for the theory of weak solutions to Hamilton-Jacobi equations with non-smooth Hamiltonians, see [**G, H, K**] and also [**Q, R**] (which use the language of pseudo-analysis). In some cases, this theory enables one to obtain exact solutions in the similar form as for the linear equations. Some further developments relate more general pseudo-operations with applications to nonlinear partial differential equations, see [**S**]. Recently, these applications have become important in the field of image processing [**Q, R**].

Our report is organized as follows. In Sect. 2 we consider a general form of PDE for image restoration. The starting PDE in image restoration is the heat equation. Because of its oversmoothing property (edges get smeared), it is necessary to introduce some nonlinearity. We consider then the following model ([**A, T**])

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(c\left(|\nabla u|^2\right)\nabla u\right),$$

where we choose the function c such that the equation remains to be of the parabolic type. We take $c(s) \approx 1/\sqrt{s}$ as $s \to \infty$, because we want to preserve the discontinuities [**A**]. Because of this behavior, it is not possible to apply general results from parabolic equations theory. An appropriate framework to study this equation is nonlinear semigroup theory ([**A, B, D**]). In Sect. 3 we show that Perona and Malik equation satisfies the pseudo-linear superposition principle.

2. PDE-based method in image processing

PDE-methods for restoration can be written in the following general form:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(t,x) + F\left(x, u(t,x), \nabla u(t,x), \nabla^2 u(t,x)\right) = 0 \text{ in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial N}(t,x) = 0 \text{ on } (0,T)\times\partial\Omega \text{ (Neumann boundary condition)}, \\ u(0,x) = u_0(x) \text{ (initial condition)}, \end{array} \right\}$$

where $u(t,x)$ is the restored version of the initial degraded image $u_0(x)$. The idea is to construct a family of functions $\{u(t,x)\}_{t>0}$ representing successive versions of $u_0(x)$. As t increases, the image $u(t,x)$ becomes more and more simplified. We would like to attain two goals. The first is that $u(t,x)$ should represent a smooth version of $u_0(x)$, where the noise has been removed. The second, $u(t,x)$ should be able to preserve some features such as edges, corners, which may be viewed as singularities.

The heat equation is the basic PDE for image restoration:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = 0, \ t \geq 0, \ x \in \mathbb{R}^2, \\ u(0,x) = u_0(x). \end{array} \right.$$

The heat equation has been successfully applied in image processing but it has some drawback. It is too smoothing and because of that edges can be lost or severely blurred. In [**A**] authors consider models that are generalizations of the heat equation. Suppose that the domain image is a bounded open set Ω of $\mathbb{R}^2$. The following equation was initially proposed by Perona and Malik [**T**]:

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} = \operatorname{div}\left(c\left(|\nabla u|^2\right)\nabla u\right) & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial N} = 0 \text{ on } (0,T)\times\partial\Omega, & \\ u(0,x) = u_0(x) \text{ in } \Omega & \end{array} \right. \tag{2.1}$$

where $c : [0,\infty) \to (0,\infty)$. If we choose $c \equiv 1$, then it is reduced to the heat equation. If we assume that $c(s)$ is a decreasing function satisfying $c(0) = 1$ and $\lim\limits_{s\to\infty} c(s) = 0$, then inside the regions where the magnitude of the gradient of u is weak, equation (2.1) acts like the heat equation and the edges are preserved.

For each point x where $|\nabla u| \neq 0$ we can define the vectors $\mathbf{N} = \frac{\nabla u}{|\nabla u|}$ and $\mathbf{T}$ with $\mathbf{T}\cdot\mathbf{N} = 0$, $|\mathbf{T}| = 1$. For the first and second partial derivatives of u we use the usual notation u_{x_1}, u_{x_2}, $u_{x_1x_1,\ldots}$ We denote by $u_{\mathbf{NN}}$ and $u_{\mathbf{TT}}$

the second derivatives of u in the **N**-direction and **T**-direction, respectively:

$$u_{\mathbf{NN}} = \mathbf{N}^t \nabla^2 u \mathbf{N} = \frac{1}{|\nabla u|^2}(u_x^2 u_{xx} + u_y^2 u_{yy} + 2u_x u_y u_{xy}),$$
$$u_{\mathbf{TT}} = \mathbf{T}^t \nabla^2 u \mathbf{T} = \frac{1}{|\nabla u|^2}\left(u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy}\right).$$

The first equation in (2.1) can be written as

$$\frac{\partial u}{\partial t}(t,x) = c\left(|\nabla u(t,x)|^2\right) u_{\mathbf{TT}} + b\left(|\nabla u(t,x)|^2\right) u_{\mathbf{NN}}, \tag{2.2}$$

where $b(s) = c(s) + 2sc'(s)$. Therefore, (2.2) is a sum of a diffusion in the **T**-direction and a diffusion in the **N**-direction. The functions c and b act as weighting coefficients. Since **N** is normal to the edges, it would be preferable to smooth more in the tangential direction **T** than in the normal direction. Because of that we impose

$$\lim_{s\to\infty}\frac{b(s)}{c(s)} = 0 \quad \text{or} \quad \lim_{s\to\infty}\frac{sc'(s)}{c(s)} = -\frac{1}{2} \tag{2.3}$$

If $c(s) > 0$ with power growth, then (2.3) implies that $c(s) \approx 1/\sqrt{s}$ as $s \to \infty$. The equation (2.1) is parabolic if $b(s) > 0$.

The assumptions imposed on $c(s)$ are

$$\begin{cases} c : [0,\infty) \to (0,\infty) \text{ decreasing}, \\ c(0) = 1,\ c(s) \approx \frac{1}{\sqrt{s}} \text{ as } s \to \infty, \\ b(s) = c(s) + 2sc'(s) > 0. \end{cases} \tag{2.4}$$

Consider $c(s) = \frac{1}{\sqrt{1+s}}$, an often used function satisfying (2.4). Because of the behavior $c(s) \approx 1/\sqrt{s}$ as $s \to \infty$, it is not possible to apply general results from parabolic equations theory. An appropriate framework to study this equation is nonlinear semigroup theory (see [**A, B, D**]).

3. Pseudo-linear superposition principle for Perona and Malik equation

Let $[a,b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty,+\infty]$. We consider here the total order $\leq$ on $[a,b]$. The operation $\oplus$ (pseudo-addition) is a commutative, non-decreasing, associative function $\oplus : [a,b]\times[a,b] \to [a,b]$ with a zero (neutral) element denoted by $\mathbf{0}$. Denote $[a,b]_+ = \{x : x \in [a,b], x \geq \mathbf{0}\}$. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a,b]\times[a,b] \to [a,b]$ which is commutative, positively non-decreasing, i.e., $x \leq y$ implies $x \odot z \leq y \odot z, z \in [a,b]_+$,associative and for

which there exist a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b]$, $1 \odot x = x$. We assume $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

The structure $([a,b], \oplus, \odot)$ is called a *semiring* (see [**I, O**]). In this paper we shall consider only the min-plus (or tropical) semiring. It is defined on the interval $(-\infty, +\infty]$ and has the following continuous operations: $x \oplus y = \min\{x,y\}$, $x \odot y = x + y$. Note that the pseudo-addition is idempotent, while the pseudo-multiplication is not. We have $\mathbf{0} = -\infty$ and $\mathbf{1} = 0$.

We show that the pseudo-linear superposition principle holds for Perona and Malik equation.

THEOREM 3.1. *If $u_1 = u_1(t,x)$ and $u_2 = u_2(t,x)$ are solutions of the equation*

$$\frac{\partial u}{\partial t} - div\left(c\left(|\nabla u|^2\right)\nabla u\right) = 0, \tag{3.1}$$

then $(\lambda_1 \odot u_1) \oplus (\lambda_2 \odot u_2)$ is also a solution of (3.1) on the set

$$D = \{(t,x) \,|\, t \in (0,T), x \in \mathbb{R}^2, u_1(t,x) \neq u_2(t,x)\},$$

with respect to the operations $\oplus = \min$ and $\odot = +$.

The obtained results will serve for further investigation of the weak solutions of the equation (3.1) in the sense of Maslov [**G, H**] and Gondran [**E, F**], as well as some important applications.

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Tropical analysis on plurisubharmonic singularities

Alexander Rashkovskii

1. Plurisubharmonic singularities

Recall that an upper semicontinuous, real-valued function on an open set in $\mathbb{C}^n$ is called *plurisubharmonic* (psh) if its restriction to every complex line is a subharmonic function. A basic example is $\log|f|$ for an analytic function f. Moreover, by Bremermann's theorem [**1**], every psh function u can be written as

$$u(z) = \limsup_{y\to z} \limsup_{m\to\infty} \frac{1}{m} \log|f_m(y)|.$$

Let $\mathcal{O}_0$ denote the ring of germs of analytic functions f at $0 \in \mathbb{C}^n$, and let $\mathfrak{m}_0 = \{f \in \mathcal{O}_0 : f(0) = 0\}$ be its maximal ideal. The log-transformation $f \mapsto \log|f|$ maps $\mathcal{O}_0$ into the collection of germs of psh functions at 0.

We will say that a psh germ u has singularity at 0 if $u(0) = -\infty$. For functions $u = \log|f|$ this means $f \in \mathfrak{m}_0$; asymptotic behaviour of arbitrary psh functions can be much more complicated. By $PSHG_0$ we denote the collections of all psh germs singular at 0.

The operations on $\mathcal{O}_0$ induce a natural tropical structure on $PSHG_0$ with the addition $u \oplus v := \max\{u, v\}$ (which is based on Maslov's dequantization: $f + g \mapsto \frac{1}{N}\log|f^N + g^N| \to \log|f| \oplus \log|g|$ as $N \to \infty$) and multiplication $u \otimes v := u + v$ (simply by $fg \mapsto \log|fg| = \log|f| \otimes \log|g|$). Thus $PSHG_0$ becomes a tropical semiring, closed under (usual) multiplication by positive constants.

A partial order on $PSHG_0$ is given as follows: $u \preceq v$ if $u(z) \le v(z) + O(1)$ as $z \to 0$, which leads to the equivalence relation $u \sim v$ if $u(z) = v(z) + O(1)$. The equivalence class $\mathrm{cl}(u)$ of u is called the *plurisubharmonic singularity* of the germ u. The collection of psh singularities $PSHS_0 = PSHG_0/\sim$ has the same tropical structure $\{\oplus, \otimes\}$ and the partial order: $\mathrm{cl}(u) \le \mathrm{cl}(v)$ if $u \preceq v$. It is endowed with the following topology: $\mathrm{cl}(u_j) \to \mathrm{cl}(u)$ if there exists a neighbourhood ω of 0 and psh functions $v_j \in \mathrm{cl}(u_j)$, $v \in \mathrm{cl}(u)$ in ω such that $v_j \to v$ in $L^1(\omega)$.

By abusing the notation, we will right occasionally u for $\mathrm{cl}(u)$.

2. Characteristics of singularities

The main characteristic of an analytic germ $f \in \mathfrak{m}_0$ is its multiplicity (vanishing order) m_f: if $f = \sum P_j$ is the Taylor expansion of f in homogeneous polynomials, $P_j(tz) = t^j P(z)$, then $m_f = \min\{j : P_j \not\equiv 0\}$.

The basic characteristic of singularity of $u \in PSHG_0$ is its *Lelong number*

$$\nu(u) = \lim_{t\to-\infty} \frac{1}{t} M(u,t) = \liminf_{z\to 0} \frac{u(z)}{\log|z|} = dd^c u \wedge (dd^c \log|z|)^{n-1}(0);$$

here $M(u,t)$ is the mean value of u over the sphere $\{|z| = e^t\}$, $d = \partial + \bar\partial$, $d^c = (\partial - \bar\partial)/2\pi i$. If $f \in \mathfrak{m}_0$, then $\nu(\log|f|) = m_f$. This characteristic of singularity gives an important information on the asymptotic behaviour of u at 0: $u(z) \le \nu(u)\log|z| + O(1)$.

Since $\nu(v) = \nu(u)$ for all $v \in \mathrm{cl}(u)$, Lelong number can be considered as a functional on $PSHS_0$ with values in the tropical semiring $\mathbb{R}_+(\min, +)$ of non-negative real numbers with the operations $x \bar\oplus y = \min\{x, y\}$ and $x \otimes y = x + y$. As such, it is

(i) positive homogeneous: $\nu(cu) = c\,\nu(u)$ for all $c > 0$,
(ii) additive: $\nu(u \oplus v) = \nu(u) \bar{\oplus} \nu(v)$,
(iii) multiplicative: $\nu(u \otimes v) = \nu(u) \otimes \nu(v)$, and
(iv) upper semicontinuous: $\nu(u) \geq \limsup \nu(u_j)$ if $u_j \to u$.

Lelong numbers are independent of the choice of coordinates. Let us now fix a coordinate system (centered at 0). The *directional Lelong number* of u in the direction $a \in \mathbb{R}^n_+$ (introduced by C. Kiselman [**5**]) is

$$\nu(u, a) = \lim_{t \to -\infty} \frac{1}{t} M(u, ta) = \liminf_{z \to 0} \frac{u(z)}{\phi_a(z)}, \tag{2.1}$$

where $M(u, ta)$ is the mean value of u over the distinguished boundary of the polydisk $\{|z_k| < \exp(ta_k)\}$ and $\phi_a(z) = \oplus_k\, a_k^{-1} \log |z_k|$. It has the same properties (i)–(iv), and the collection $\{\nu(u, a)\}_a$ gives a refined information on the singularity u. In particular, $\nu(u) = \nu(u, (1, \dots, 1))$.

A general notion of Lelong number with respect to a plurisubharmonic weight was introduced by J.-P. Demailly [**2**]. Let $\varphi \in PSHG_0$ be continuous and locally bounded outside 0. Then the mixed Monge–Ampère current $dd^c u \wedge (dd^c \varphi)^{n-1}$ is well defined for any psh function u and is equivalent to a positive Borel measure. Its mass at 0, $\nu(u, \varphi) = dd^c u \wedge (dd^c \varphi)^{n-1}(\{0\})$, is called *the generalized Lelong number*, or *the Lelong–Demailly number*, of u with respect to the weight φ. Since it is constant on $\mathrm{cl}(u)$, we have a different kind of functional on $PSHS_0$. It still has the above properties (i), (iii), and (iv), however in general is only subadditive: $\nu(u \oplus v, \varphi) \leq \nu(u, \varphi) \bar{\oplus} \nu(v, \varphi)$.

Note that $\nu(u, a) = a_1 \dots a_n\, \nu(u, \phi_a)$.

3. Additive functionals

Another generalization of the notion of Lelong number was introduced in [**12**]. Let $\varphi \in PSHG_0$ be locally bounded and *maximal* outside 0 (that is, satisfies $(dd^c \varphi)^n = 0$ on a punctured neighbourhood of 0); the collection of all such germs (weights) will be denoted by MW_0. The *type of* $u \in PSHS_0$ *relative to* $\varphi \in MW_0$,

$$\sigma(u, \varphi) = \liminf_{z \to 0} \frac{u(z)}{\varphi(z)},$$

gives the bound $u \leq \sigma(u, \varphi)\varphi$.

This functional is positive homogeneous, additive, supermultiplicative, and upper semicontinuous. Actually, relative types give a general form for all "reasonable" additive functionals on $PSHS_0$:

THEOREM 3.1. ([**12**]) *Let a functional* $\sigma : PSHS_0 \to [0,\infty]$ *be such that*

1) $\sigma(cu) = c\,\sigma(u)$ *for all* $c > 0$;
2) $\sigma(\oplus u_k) = \bar{\oplus}\sigma(u_k)$, $k = 1, 2$;
3) *if* $u_j \to u$, *then* $\limsup \sigma(u_j) \le \sigma(u)$;
4) $\sigma(\log|z|) > 0$;
5) $\sigma(u) < \infty$ *if* $u \not\equiv -\infty$.

Then there exists a weight $\varphi \in MW_0$ *such that* $\sigma(u) = \sigma(u,\varphi)$ *for every* $u \in PSHS_0$. *The representation is essentially unique: if two maximal weights* φ *and* ψ *represent* σ, *then* $\mathrm{cl}(\varphi) = \mathrm{cl}(\psi)$.

4. Relative types and valuations

Recall that a *valuation* on the analytic ring $\mathcal{O}_0$ is a nonconstant function $\mu : \mathcal{O}_0 \to [0, +\infty]$ such that

$$\mu(f_1 f_2) = \mu(f_1) + \mu(f_2), \quad \mu(f_1 + f_2) \ge \min\{\mu(f_1), \mu(f_2)\}, \quad \mu(1) = 0;$$

a valuation μ is *centered* if $\mu(f) > 0$ for every $f \in \mathfrak{m}_0$, and *normalized* if $\min\{\mu(f) : f \in \mathfrak{m}_0\} = 1$. Every weight $\varphi \in MW_0$ generates a functional σ_φ on $\mathcal{O}_0$, $\sigma_\varphi(f) = \sigma(\log|f|, \varphi)$, with the properties

$$\sigma_\varphi(f_1 f_2) \ge \sigma_\varphi(f_1) + \sigma_\varphi(f_2),$$
$$\sigma_\varphi(f_1 + f_2) \ge \min\{\sigma_\varphi(f_1), \sigma_\varphi(f_2)\}, \quad \sigma_\varphi(1) = 0.$$

It is a valuation, provided $\sigma(u,\varphi)$ is tropically multiplicative; σ_φ is centered iff $\sigma(\log|z|, \varphi) > 0$, and normalized iff $\sigma(\log|z|, \varphi) = 1$.

One can thus consider linear (both additive and multiplicative) functionals on $PSHS_0$ as tropicalizations of certain valuations on $\mathcal{O}_0$.

For example, the (usual) Lelong number is the tropicalization of the multiplicity valuation m_f. The types relative to the directional weights ϕ_a are multiplicative functionals on $PSHS_0$, and σ_{ϕ_a} are monomial valuations on $\mathcal{O}_0$; they are normalized if $\min_k a_k = 1$. It was shown in [**4**] that an important class of valuations (quasi-monomial valuations, or Abhyankar valuations of rank 1) can be realized as σ_φ with certain weights $\varphi \in MW_0$; when $n = 2$, all other centered valuations are limits of increasing sequences of the quasi-monomial ones [**3**].

5. Local indicators as Maslov's dequantizations

Consideration of psh germs is the first step of Maslov's dequantization of analytic functions $\mathcal{O}_0 \ni f \mapsto \log|f| \in PSHG_0$. One can perform the next step – namely, passage to the logarithmic scale in the arguments z.

For a fixed coordinate system at 0, let $\nu(u,a)$ be the directional Lelong numbers of $u \in PSHS_0$ in the directions $a \in \mathbb{R}^n_+$ (2.1). Then the function $\psi_u(t) = -\nu(u,-t)$, $t \in \mathbb{R}^n_-$, is convex and increasing in each t_k, so $\psi_u(\log|z_1|,\dots,|z_n|)$ can be extended (in a unique way) to a function $\Psi_u(z)$ plurisubharmonic in the unit polydisk $\mathbb{D}^n = \{z \in \mathbb{C}^n : |z_k| < 1, 1 \le k \le n\}$. This function is called the *local indicator* of u at 0 [**7**]. It is easy to see that it has the homogeneity property

(5.1)
$$\Psi_u(z_1,\dots,z_n) = \Psi_u(|z_1|,\dots,|z_n|) = c^{-1}\Psi_u(|z_1|^c,\dots,|z_n|^c) \quad \forall c > 0.$$

It was shown in [**9**] that $\Psi_u(z)$ can be represented as the (unique) weak limit of the functions $m^{-1}u(z_1^m,\dots,z_n^m)$ as $m \to \infty$, so the indicator can be viewed as the tangent (in the logarithmic coordinates) for the function u at 0. This means that for $u = \log|f|$, $f \in \mathfrak{m}_0$, the sublinear function $\psi_u(t)$ on $\mathbb{R}^n_-$ is just a Maslov's dequantization of f.

The indicator is a psh characteristic of asymptotic behaviour near 0. Namely, if u is psh in the unit polydisk $\mathbb{D}^n$, then $u(z) \le \Psi_u(z) + \sup_{\mathbb{D}} u$. When u has isolated singularity at 0, this implies the following relation between the residual Monge-Ampère masses: $(dd^c u)^n(0) \ge (dd^c\Psi_u)^n(0)$.

Since Ψ_u is much simpler than the original function u, one can compute explicitly the value of its residual mass. The first equation in (5.1) suggests us to pass from plurisubharmonic functions to convex ones and from the complex Monge-Ampère operator to the real one, while the second equation allows us to calculate the real Monge-Ampère measure in terms of volumes of gradient images. Denote

$$\Theta_{u,x} = \{b \in \mathbb{R}^n_+ : \sup_{\sum a_k = 1} [\nu(u,a) - \langle b, a\rangle] \ge 0\}.$$

The convex image $\psi_u(t)$, $t \in \mathbb{R}^n_-$, of the indicator is just the support function to the convex set $\Gamma_u = \mathbb{R}^n_+ \setminus \Theta_{u,x}$: $\psi_u(t) = \sup\{\langle t, a\rangle : a \in \Gamma_u\}$. This gives

THEOREM 5.1. ([**9**]) *The residual Monge-Ampère mass of $u \in PSHG_0$ with isolated singularity at 0 has the lower bound*

$$(dd^c u)^n(0) \ge (dd^c\Psi_u)^n(0)) = n!\,\mathrm{Vol}(\Theta_{u,x}).$$

If $F = (f_1,\dots,f_n)$ is a holomorphic mapping with isolated zero at 0, then its multiplicity at 0 equals $(dd^c\log|F|)^n(0)$ and the set $\Gamma_{\log|F|}$ is the convex hull of the union of the Newton polyhedra $\mathrm{conv}\{\alpha + \mathbb{R}^n_+ : D^{(\alpha)}f_j(0) \ne 0\}$ of f_j at 0, $1 \le j \le n$. In this case, Theorem 2 gives us Kushnirenko's theorem on multiplicity of holomorphic mappings [**6**].

The results on local indicators have global counterparts concerning psh functions of logarithmic growth in $\mathbb{C}^n$ (i.e., $u(z) \le A\log(1+|z|)+B$ everywhere in $\mathbb{C}^n$, a basic example being logarithm of modulus of a polynomial), see [**10**] and [**11**]. Similar notions concerning Maslov's dequantization in $\mathbb{C}^n$ and generalized Newton polytops were also introduced and studied in [**8**].

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Minimal elements and cellular closures over the max-plus semiring[1]

Sergeĭ Sergeev

This report is based on the publications [**2**] (part 1) and [**7**] (part 2). In part 1, I outline some simple consequences of the observation that extremals are minimal elements with respect to the certain preorder relation. Part 2 is occupied with some extensions of algebraic closure operation, which arise from the cellular decomposition considered in [**3**]. The common feature of these works is a bit of interplay between max-algebra [**1**] and tropical convexity [**3**], [**5**]. All results are obtained in the setting of $\mathbb{R}^n_{\max,\mathrm{m}}$, the n-dimensional free semimodule over the semiring $\mathbb{R}_{\max,\mathrm{m}} = (\mathbb{R}_+, \oplus = \max, \odot = *)$.

1. Extremals as minima

An element u of a (sub)semimodule $K \subseteq \mathbb{R}^n_{\max,\mathrm{m}}$ is an *extremal*, if $u = x \oplus y$, $x, y \in K$ implies that $u = x$ or $u = y$. The preorder relation $\leq_j$ is defined by

$$u \leq_j v \Leftrightarrow u_j \neq 0,\ v_j \neq 0,\ u/u_j \leq v/v_j. \tag{1.1}$$

The role of $\leq_j$ is explained in the following.

Proposition 1.1. [**5, 2**] *The following are equivalent:*

(1) *y is a (max-)linear combination of $x^1, \ldots, x^m \in \mathbb{R}^n_{\max,m}$;*
(2) *for any $j \in supp(y)$, there exists some x^l from $x^1, \ldots, x^m$ such that $x^l \leq_j y$.*

Proposition 1.2. [**2**] *Let a semimodule K be generated by a subset S of $\mathbb{R}^n_{\max,m}$. The following are equivalent.*

(1) *y is an extremal of K;*
(2) *for some j, this y is a minimal element of S (and, equivalently, of K) with respect to $\leq_j$.*

Proposition 1.2 enables to treat idempotent extremals as minima. The problem of finding partial maxima (and minima) in n-dimensional real space was investigated by F. Preparata et al. [**6**]. The following estimate is derived from their results.

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THEOREM 1.3. [**5**](for $n = 3$),[**2**] *Let K be a semimodule in $\mathbb{R}^n_{\max,m}$ generated by k elements. The problem of finding all extremals of K requires not more than $O(k \log_2 k)$ operations, if $n = 3$, and not more than $O(k(\log_2 k)^{n-3}$ operations, if $n > 3$ (with n fixed).*

Propositions 1.1 and 1.2 imply a number of statements for generators of idempotent semimodules in $\mathbb{R}^n_{\max,\mathrm{m}}$, see [**2**] for details. Here I mention two of them.

THEOREM 1.4. [**8, 2**] *Let K be a semimodule in $\mathbb{R}^n_{\max,m}$ generated by S, and let E be the set of extremals of K such that $||u|| = 1$[2] for all $u \in E$. Then $S = E \cup F$, where F is redundant in the sense that $S - \{u\}$ generates K for any $u \in F$.*

As a corollary, the weak basis of a semimodule is essentially unique whenever it exists. The following is a tropical version of Minkowski's theorem.

THEOREM 1.5. [**4, 2**] *A closedsemimodule in $\mathbb{R}^n_{\max,m}$ is generated by its extremals.*

2. Cellular closures

Algebraic closure of a square matrix A is the series $I \oplus A \oplus A^2 \oplus \ldots$, where I is the identity matrix. This series converges iff $\lambda(A) \leq 1$, where $\lambda(A)$ is the maximal cycle mean of A. This $\lambda(A)$ is also the maximal eigenvalue of the problem $Ax = \lambda x$. The corresponding eigenspace will be denoted by $\mathrm{eig}(A)$.

The following theorem and its corollary are the "ground stone" of this section.

THEOREM 2.1. [**7**] *Let A and B be two square matrices such that $\lambda(A) \leq 1$ and $\lambda(B) \leq 1$. Then $A^* = B^*$ iff the spaces generated by columns of A^* and B^* coincide.*

A square matrix A is *definite*, if $\lambda(A) = 1$ and all the diagonal entries equal 1.

COROLLARY 2.2. *Let A and B be two definite matrices. Then $A^* = B^*$ iff $eig(A) = eig(B)$.*

I consider now the concepts of [**3**]. Let A be an $n \times m$ matrix over $\mathbb{R}_{\max,\mathrm{m}}$ and y an n-component vector. Denote the collection $S = \{S_j : j \in$

[2]the choice of norm does not matter

$\operatorname{supp}(y)\}$, where $S_j = \{i\colon y \geq_j A_{\cdot i}\}$, by $\operatorname{type}(y \mid A)$ and call it the *combinatorial type* of y with respect to A. Combinatorial types can be formally defined as arbitrary collections of not more than n possibly empty subsets of $\{1, \ldots, m\}$. Denote the set of indices i, whose S_i are present in the type, by $\operatorname{supp}(S)$. If $S = \operatorname{type}(y \mid A)$ for some y, then $\operatorname{supp}(S) = \operatorname{supp}(y)$. The types are partially ordered by the rule $S \subseteq S'$ if $\operatorname{supp}(S') \subseteq \operatorname{supp}(S)$ and $S_i \subseteq S'_i$ for all $i \in \operatorname{supp}(S)$. The set

$$X^S = \{z\colon S \subseteq \operatorname{type}(z \mid A)\}$$

is the *region* of S. If $A_{ik} \neq 0$ for all $i \in S_k$, then S is *compatible* and we introduce the matrix A^S by

$$A^S_{\cdot i} = \begin{cases} \bigoplus_{k \in S_i} A_{\cdot k}/A_{ik}, & \text{if } i \in \operatorname{supp}(S) \text{ and } S_i \neq \emptyset; \\ e_i, & \text{if } i \in \operatorname{supp}(S) \text{ and } S_i = \emptyset; \\ \mathbf{0}, & \text{if } i \notin \operatorname{supp}(S). \end{cases}$$

If the region X^S is not empty, then $X^S = \operatorname{eig}(A^S)$. Hence any region is (essentially) the eigenspace of a definite matrix, and we use Corollary 2.2.

THEOREM 2.3. *If S and T are (compatible) types such that X^S and X^T are not empty and $X^S = X^T$, then $(A^S)^* = (A^T)^*$.*

Theorem 2.3 enables to define various *cellular closures* of A to be $(A^S)^*$. This operation is correctly defined for every region, being independent of the type.

Consider now the case when A is a square $n \times n$ matrix with a permutation σ whose weight $\odot_{i=1}^n A_{i\sigma(i)}$ is nonzero. A permutation with maximal weight is called *maximal*. We define D^σ to be the matrix such that $D^\sigma_{ij} = A_{ij}$ if $j = \sigma(i)$ and $D_{ij} = 0$ otherwise. If σ is maximal, then $(D^\sigma)^{-1}A$ is definite and is called the *definite form* of A [**7**]. Different maximal permutations lead to different definite forms. But we have that eigenspaces of all definite forms coincide (see [**7**]), and by Corollary 2.2 closures of all definite forms are equal.

Thus, for any square matrix A with nonzero permutations, we can define its *definite closure* to be $(D^\sigma)^{-1}A)^*$, where σ is a maximal permutation. Definite closure is a cellular closure, since $\operatorname{eig}((D^\sigma)^{-1}A)$ is the same as X^S with $S = (\{\sigma(1)\}, \ldots, \{\sigma(n)\})$, where σ is any maximal permutation.

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Semiclassical quantization of field theories[1]

Oleg Yu. Shvedov

§1. It is well-known that equations of quantum field theory (QFT) are ill-defined [**1**]. One usually investigates the perturbative QFT instead of "exact" QFT: all quantities are presented as formal series in a small perturbation parameter; the QFT divergences are eleminated within a perturbation framework only.

Semiclassical approximation [**2**] may be also viewed as an expansion in a small parameter. Since the well-defined results are obtained within the perturbation theory, it seems to be more reasonable to talk about semiclassical quantization rather than semiclassical approximation.

Consider the field theory model with the Lagrangian $\mathcal{L}$ depending on the small parameter h ("Planck constant") as follows [**3**] (the scalar case is considered for the simplictiy):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{h}V(\sqrt{h}\varphi). \tag{1}$$

with $V(\Phi)$ being a scalar potential.

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In the formal quantum theory, field $\hat{\varphi}(\mathbf{x})$ and momentum $\hat{\pi}(\mathbf{x})$ are viewed as operators satisfying the canonical commutation relations. Semiclassical states depend on the small parameter h as:

$$\Psi(t) \simeq e^{\frac{i}{h}S(t)} e^{\frac{i}{\sqrt{h}}\int d\mathbf{x}[\Pi(\mathbf{x},t)\hat{\varphi}(\mathbf{x})-\Phi(\mathbf{x},t)\hat{\pi}(\mathbf{x})]} f(t). \tag{2}$$

Here $S(t)$ is a real c-number finction of t, $\Phi(\mathbf{x},t)$ and $\Pi(\mathbf{x},t)$ are classical fields and canonucally conjugated momenta, $\hat{\varphi}(\mathbf{x})$ and $\hat{\pi}(\mathbf{x})$ are quantum field and momentum operators, $f(t)$ is a regular as $h \to 0$ state vector.

Superpositions of states (2) are also viewed as semiclassical states.

Presentation of semiclassical form in the form (2) is not manifestly covariant. There are space and time coordinates. It happens that the manifestly covariant form of the state (2) is the following:

$$\Psi \simeq e^{\frac{i}{h}\overline{S}} Texp\{\frac{i}{\sqrt{h}}\int dx J(x)\hat{\varphi}_h(x)\}\overline{f} \equiv e^{\frac{i}{h}\overline{S}} T_J^h \overline{f}. \tag{3}$$

Here $\overline{S}$ is a real number, $J(x)$ is a real function (classical Schwinger source), $\hat{\varphi}_h(x)$ is a Heisenberg field operator, $\overline{f}$ is a state vector being regular as $h \to 0$. The Schwinger source $J(x)$ should be rapidly damping at space and time infinity [**4**].

§2. Investigate properties of the semiclassical state (3). First of all, note that the state $T^h_{J+\sqrt{h}\delta J} f$ can be expressed via the operator T^h_J. To do this, it is necessary to investigate the operator

$$\underline{\Phi}_R(x|J) \equiv -ih(T_J^h)^+ \frac{\delta T_J^h}{\delta J(x)}. \tag{4}$$

It happens to coincide with the well-known LSZ R-function [**5**].

Notice that $\underline{\Phi}_R(x|J)$ is expanded in $\sqrt{h}$; one writes

$$\underline{\Phi}_R(x|J) = \Phi_R(x|J) + \sqrt{h}\Phi_R^{(1)}(x|J) + ... \tag{5}$$

The c-number function $\Phi_R(x|J)$ is called as a retarded classical field generated by the Schwinger source J. It is shown in [**6**] that for the model (1) $\Phi_R(x|J)$ is a solution of the equation

$$\partial_\mu \partial^\mu \Phi_R(x|J) + V'(\Phi_R(x|J)) = J(x), \quad \Phi_R|_{x \underset{\sim}{<} suppJ} = 0. \tag{6}$$

which vanishes as $x^0 \to -\infty$.

The following properties are corollaries of (4).

1. The Hermitian property

$$\underline{\Phi}_R^+(x|J) = \underline{\Phi}_R(x|J). \tag{7}$$

2. The Poincare invariance property

$$\underline{U}_{g^{-1}} \underline{\Phi}_R(x|u_g J) \underline{U}_g = \underline{\Phi}_R(w_g x|J). \tag{8}$$

3. The Bogoliubov causality property [**1**]: R-function $\underline{\Phi}_R(x|J)$ depends only on the source J at the preceeding time moments. Making use of the standard notations $x > y$ iff $x^0 - y^0 \geq |\mathbf{x} - \mathbf{y}|$, $x < y$ iff $x^0 - y^0 \leq |\mathbf{x} - \mathbf{y}|$, $x \sim y$ iff $|x^0 - y^0| < |\mathbf{x} - \mathbf{y}|$, one rewrites the Bogoliubov condition as

$$\frac{\delta \underline{\Phi}_R(x|J)}{\delta J(y)} = 0, \quad y \underset{\sim}{>} x. \tag{9}$$

4. Commutation relation

$$[\underline{\Phi}_R(x|J); \underline{\Phi}_R(y|J)] = -ih \left(\frac{\delta \underline{\Phi}_R(x|J)}{\delta J(y)} - \frac{\delta \underline{\Phi}_R(y|J)}{\delta J(x)} \right). \tag{10}$$

5. Boundary condition at $-\infty$. If $x \underset{\sim}{<} y$ for all $y \in suppJ$, the LSZ R-functiion does not depend on the source:

$$\underline{\Phi}_R(x|J) = \hat{\varphi}_h(x)\sqrt{h}, \quad x \underset{\sim}{<} suppJ. \tag{11}$$

In particular, the classical retarded field vanishes as $x^0 \to -\infty$.

Making use of the operator (4), one can construst the semiclassical field:

$$\underline{\Phi}(x|J) = (T_J^h)^+ \hat{\varphi}_h(x) T_J^h$$

coincides with $\underline{\Phi}_R(x|J)$ at $x^0 \to +\infty$:

$$\underline{\Phi}(x|J) = \underline{\Phi}_R(x|J), \quad x \underset{\sim}{>} suppJ. \tag{12}$$

§3. Another interesting feature of semiclassical states is that some of them are approximately equal each other. We say that $J \sim 0$ iff

$$T_J^h \overline{f} \simeq e^{\frac{i}{h}\overline{I}_J} \underline{W}_J \overline{f} \tag{13}$$

for some number $\overline{I}_J$ and operator $\underline{W}_J$ presented as a formal asymptotic series.

It is shown in [**6**] for the model (1) that the source J is equivalent to zero iff the retarded field generated by J vanishes at $+\infty$.

Analogously to [**6**], one derives the following properties.

1. Poincare invariance.

$$\underline{U}_g \underline{W}_J \underline{U}_{g^{-1}} = \underline{W}_{u_g J}, \qquad \overline{I}_{u_g J} = \overline{I}_J; \tag{14}$$

2. Unitarity

$$\underline{W}_J^+ = \underline{W}_J^{-1}; \tag{15}$$

3. Bogoliubov causality: as $J + \Delta J_2 \sim 0$, $J + \Delta J_1 + \Delta J_2 \sim 0$ and $supp\Delta J_2 \underset{\sim}{>} supp\Delta J_1$, the operator

$$(\underline{W}_{J+\Delta J_2})^+ \underline{W}_{J+\Delta J_1+\Delta J_2}$$

and number

$$-\overline{I}_{J+\Delta J_2} + \overline{I}_{J+\Delta J_1+\Delta J_2}$$

do not depend on ΔJ_2.

4. Variational property:

$$\delta\overline{I}_J - ih\underline{W}_J^+ \delta\underline{W}_J = \int dx \underline{\Phi}_R(x|J)\delta J(x), \tag{16}$$

which is valid as $J \sim 0$ and $J + \delta J \sim 0$.

5. Boundary condition at $+\infty$:

$$\underline{\Phi}_R(x|J) = \underline{W}_J^+ \hat{\varphi}(x)\sqrt{h}\underline{W}_J, \quad x \underset{\sim}{>} suppJ. \tag{17}$$

It follows from (17) that the retarded classical field generated by the source $J \sim 0$ will vanish at $+\infty$. For the model (1), an inverse statement is also valid: *for any field configuration $\Phi_c(x)$ with the compact support one can uniquely choose a source $J \sim 0$ (denoted as $J = J_{\Phi_c} = J(x|\Phi_c)$; for example* (1)*, it is found from the relation* (6)*) generating $\Phi_c(x)$ as a retarded classical field: $\Phi_c(x) = \Phi_R(x|J)$; it satisfies the locality condition $\frac{\delta J(x|\Phi_c)}{\delta\Phi_c(y)} = 0$ as $x \neq y$.*

It is possible to treat this statement as a basic postulate of semiclassical field theory. Then the theory may be developed without additional postulating classical stationary action principle and canonical commutation relation.

Namely, it follows from eq.(16) in the leading order in h that the functional

$$I[\Phi_c] = \overline{I}_{J_{\Phi_c}} - \int dx J_{\Phi_c}(x)\Phi_c(x) \tag{18}$$

satisfies the "classical equation of motion"

$$J_{\Phi_c}(x) = -\frac{\delta I[\Phi_c]}{\delta\Phi_c(x)}. \tag{19}$$

The functional $I[\Phi]$ should satisfy the locality condition

$$\frac{\delta^2 I}{\delta\Phi_c(x)\delta\Phi_c(y)} = 0. \quad x \neq y \tag{20}$$

This means that it is presented as an integral of a local Lagrangian.

Relation (19) allows us to reconstruct the classical retarded field, making use of known source $J \sim 0$, since the boundary condition at $-\infty$ is

known. It follows from the Bogoliubov causality condition that the retarded field depends only on J at the preceeding time moments. If the sourse $J(x)$ is not equivalent to zero, it can be modified at $+\infty$ and transformed to the sourse equivalent to zero. Therefore, the relation

$$\frac{\delta I}{\delta \Phi_c}[\Phi_R(\cdot|J)] = -J(x), \quad \Phi_R|_{x<suppJ} = 0 \tag{21}$$

is valid for all sourses J. For the case $x \underset{\sim}{>} suppJ$, the property (21) is taken to the classical field equation

$$\frac{\delta I}{\delta \Phi_c(x)}[\Phi(\cdot|J)] = 0. \tag{22}$$

This is a classical stationary action principle. It is viewed a coroollory of other general principles of semiclassical field theory.

Thus, we see that classical action $I[\Phi_c]$ in field theory is related with the phase of the state $T_J^h \overline{f}$ as $J \sim 0$ according to eq.(18).

Let us rewrite the properties of the operator $\underline{W}_J$ via the field Φ_c. Denote $\underline{W}[\Phi_c] \equiv \underline{W}_{J_{\Phi_c}}$.

1. Poincare invariance.

$$\underline{U}_g \underline{W}[\Phi_c] \underline{U}_{g^{-1}} = \underline{W}[u_g \Phi_c]. \tag{23}$$

2. Unitarity.

$$\underline{W}^+[\Phi_c] = (\underline{W}[\Phi_c])^{-1}. \tag{24}$$

3. Bogoliubov causality.

$$\frac{\delta}{\delta \Phi_c(y)} \left(\underline{W}^+[\Phi_c] \frac{\delta \underline{W}[\Phi_c]}{\delta \Phi_c(x)} \right) = 0, \quad y \underset{\sim}{>} x; \tag{25}$$

4. Yang-Feldman relation [**7**].

$$\int dy \frac{\delta^2 I}{\delta \Phi_c(x) \delta \Phi_c(y)} [\underline{\Phi}_R(y|J) - \Phi_c(y|J)] = ih \underline{W}^+[\Phi_c] \frac{\delta \underline{W}[\Phi_c]}{\delta \Phi_c(x)}. \tag{26}$$

5. Boundary condition.

$$\underline{W}^+[\Phi_c] \hat{\varphi}_h(x) \sqrt{h} \underline{W}[\Phi_c] = \underline{\Phi}_R(x|J_{\Phi_c}), \quad x \underset{\sim}{>} supp\Phi_c, \tag{27}$$

Here $\hat{\varphi}_h(x) = \underline{\Phi}_R(x|0)$ is the field operator without source.

§4. The covariant axioms of semiclassical field theory are as follows.

C1. *A Hilbert state space $\mathcal{F}$ is given.*

C2. *An unitary represatation of the Poincare group is given. The operators of the representation $\underline{U}_g : \mathcal{F} \to \mathcal{F}$ are asymptoitc series in $\sqrt{h}$.*

C3. *To each classical source $J(x)$ with compact support one assignes a retarded field (LSZ R-function). It is an operator-valued distribution*

$\underline{\Phi}_R(x|J)$ *expanded in* $\sqrt{h}$ *according to* (5). *It satisfies the properties* (7), (8), (9), (10).

C4. *To each classical field configuration* $\Phi_c(x)$ *with compact support one assigns a c-number. It is a classical action* $I[\Phi_c]$ *satisfying the locality condition* (20). *The property* $\Phi_c(x) = \Phi_R(x|J)$ *is valid iff*

$$J(x) = -\frac{\delta I[\Phi_c]}{\delta \Phi_c(x)}. \tag{28}$$

C5. *To each classical field configuration* $\Phi_c(x)$ *with compact support one assigns the operator* $\underline{W}[\Phi_c]$ *expanded in* $\sqrt{h}$. *It satisfies the relations* (23), (24), (25), (26), (27).

It is possible to develop a semiclassical perturbation theory, making use of these properties.

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Convex analysis, transportation and reconstruction of peculiar velocities of galaxies[1]

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We show how the problem of reconstruction of peculiar velocities of galaxies starting from redshift-space catalogues can be rendered as a convex quadratic optimization problem, invoking optimal transport techniques

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for efficient large-scale astrophysical data processing. Connection with tropical algebra is briefly discussed.

The Weyl algebra and quantization of fields

Alexander V. Stoyanovsky

In this talk we present a logically self-consistent procedure of quantization of fields. In more detail our approach is exposed in the papers [1,2] and in the book [3]. As a basic example we use the φ^4 model in 4-dimensional space-time.

1. Difficulties of traditional approaches to quantum field theory

Let us briefly discuss the logical contradictions in the known procedure of quantization of fields. Usually one starts with the action functional

$$J = \int F(x^0, \dots, x^n, u^1, \dots, u^m, u^1_{x^0}, \dots, u^m_{x^n})\, dx^0 \dots dx^n, \tag{1.1}$$

where $x^0 = t, x^1, \dots, x^n$ are the independent variables, $u^1, \dots, u^m$ are the dependent variables, and $u^i_{x^j} = \frac{\partial u^i}{\partial x^j}$. For the φ^4 model the action has the form

$$J = \int \left(\frac{1}{2} \left(u_t^2 - \sum_{j=1}^{3} u^2_{x^j} - m^2 u^2 \right) - \frac{1}{4!} g u^4 \right) dt dx^1 dx^2 dx^3. \tag{1.2}$$

One writes down the quantum field theory Schroedinger equation

$$ih \frac{\partial \Psi}{\partial t} = \int \widehat{H} \left(t, \mathbf{x}, u^i(\mathbf{x}), \frac{\partial u^i}{\partial \mathbf{x}}, -ih \frac{\delta}{\delta u^i(\mathbf{x})} \right) \Psi \, d\mathbf{x}, \tag{1.3}$$

where $\mathbf{x} = (x_1, \dots, x_n)$; the density of the Hamiltonian H is the Legendre transform of the Lagrangian F with respect to the variables u^i_t; $\frac{\delta}{\delta u^i(\mathbf{x})}$ is the variational derivative operator.

Note that the Schroedinger equation is not well defined in quantum field theory even for the free scalar field ($g = 0$ in (2)). For example, if we consider mathematical equation (1.3) literally, then it is not difficult to check that this equation has no nonzero four times differentiable solutions (the expression for the derivative $\frac{\partial^2 \Psi}{\partial t^2}$ has no sense). The traditional approach is to "subtract infinity" from the RHS of equation (1.3) and to solve it in the Fock space of functionals. However, this approach contradicts physical as well as mathematical considerations. Physically, if states were

functionals and energy were finite, then, in principle, we could measure some quantities related with these functionals (such as energy). However, it is known that quantum mechanical quantities like energy and momentum are theoretically non-measurable in relativistic quantum dynamics, and the only measurable quantities are the scattering sections. Mathematically, equation (1.3) in the Fock space does not admit a relativistically invariant generalization (usually called the Tomonaga–Schwinger equation [4]), as shown in the important paper [5]. In this paper it is shown that the evolution operators of the Klein–Gordon equation from one space-like surface to another, which are symplectic transformations of the phase space of the field, do not belong to the version of the infinite dimensional symplectic group which acts on the Fock space.

So quantization of free fields, for example, following the lines of the book [4], meets difficulties of the logical kind and is therefore not completely satisfactory. Due to this fact, the renormalization procedure for interacting fields, defined using the Bogolyubov–Parasyuk theorem, gives us a model in which it is difficult to say how quantum field theory turns into the classical one as $h \to 0$.

2. The infinite dimensional Weyl algebra

The proposed way to overcome these difficulties is to replace the algebra of variational differential operators by the infinite dimensional Weyl algebra defined below, which admits an explicit action of the infinite dimensional group of continuous symplectic transformations of the phase space of a field. This phase space is the Schwartz space of functions $(u^i(s),p^i(s))$, where $p^i(s)$ are the variables conjugate to $u^i(s)$, and $s = (s_1, \dots, s_n)$ are parameters on a spacelike surface, with the Poisson bracket

$$\{\Phi_1, \Phi_2\} = \sum_i \int \left(\frac{\delta \Phi_1}{\delta u^i(s)} \frac{\delta \Phi_2}{\delta p^i(s)} - \frac{\delta \Phi_1}{\delta p^i(s)} \frac{\delta \Phi_2}{\delta u^i(s)} \right) ds \tag{2.1}$$

of two functionals $\Phi_l(u^i(\cdot), p^i(\cdot))$, $l = 1, 2$. Let us write this bracket in the form

$$\{\Phi_1, \Phi_2\} = \int \sum_{i,j} \omega^{ij} \frac{\delta \Phi_1}{\delta y^i(s)} \frac{\delta \Phi_2}{\delta y^j(s)} \, ds, \tag{2.2}$$

where $y^i = u^i$ for $1 \le i \le m$ and $y^i = p^{i-m}$ for $m+1 \le i \le 2m$, and $\omega^{ij} = \delta_{i,j-m} - \delta_{i-m,j}$. The Weyl algebra is defined as the algebra of weakly infinite differentiable functionals $\Phi(u^i(\cdot), p^i(\cdot))$ with respect to the Moyal

$*$-product

(2.3)

$$(\Phi_1 * \Phi_2)(y^i(\cdot)) = \exp\left(-\frac{ih}{2}\int \sum_{i,j} \omega^{ij} \frac{\delta}{\delta y^i(s)} \frac{\delta}{\delta z^j(s)}\, ds\right) \Phi_1(y^i(\cdot))\Phi_2(z^i(\cdot))\Big|_{z^i(\cdot)=y^i(\cdot)}.$$

This product is not everywhere defined: for example, $u^i(s) * p^i(s)$ is undefined. Note only that if all necessary series and integrals are absolutely convergent, then the $*$-product is associative. Below we will be interested only in some concrete computations in the Weyl algebra.

Let us replace the Schrodinger equation (1.3) by the Heisenberg equation in the Weyl algebra

$$ih\frac{\partial \Phi}{\partial t} = \left[\int H(t, \mathbf{x}, u^i(\mathbf{x}), \frac{\partial u^i}{\partial \mathbf{x}}, p^i(\mathbf{x}))d\mathbf{x}, \Phi\right] \tag{2.4}$$

and by its relativistically invariant generalization, where

$$[\Phi_1, \Phi_2] = \Phi_1 * \Phi_2 - \Phi_2 * \Phi_1 \tag{2.5}$$

is the commutator in the Weyl algebra. The classical limits of equation (2.4) are the Hamilton equations

$$\frac{\partial \Phi}{\partial t} = \{\Phi, \int H\, d\mathbf{x}\} \tag{2.6}$$

equivalent to the Euler–Lagrange equations.

3. Quantization of free scalar field

Put $g = 0$ in (1.2). Since the obtained Hamiltonian

$$H_0 = \int \frac{1}{2}(p(\mathbf{x})^2 + (\operatorname{grad} u(\mathbf{x}))^2 + m^2 u(\mathbf{x})^2)d\mathbf{x} \tag{3.1}$$

is quadratic, we have

$$\frac{1}{ih}[H_0, \Phi] = \{\Phi, H_0\}, \tag{3.2}$$

hence, $\Phi(t_1; u(\cdot), p(\cdot))$, subject to the Heisenberg equation, is obtained from $\Phi(t_0; u(\cdot), p(\cdot))$ by the linear symplectic change of variables

$$(u(t_0, \mathbf{x}), p(t_0, \mathbf{x}) = u_t(t_0, \mathbf{x})) \to (u(t_1, \mathbf{x}), p(t_1, \mathbf{x}) = u_t(t_1, \mathbf{x})), \tag{3.3}$$

given by the evolution operator of the Hamilton equations, i. e., of the Klein–Gordon equation, from the Cauchy surface $t = t_0$ to the Cauchy surface $t = t_1$. Hence we can identify the Weyl algebras of various Cauchy

surfaces by means of evolution operators of the Klein–Gordon equation. In other words, we can consider the Weyl algebra W_0 of the symplectic vector space of solutions $u(t,\mathbf{x})$ of the Klein–Gordon equation on the whole space-time. The symplectic form on this vector space is obtained by taking Cauchy data on any spacelike surface (for example, on the surface $t = \text{const}$). Below we will fix this identification of the Weyl algebras of various spacelike surfaces.

Define the *vacuum average* linear functional

$$\Phi \to \langle 0|\Phi|0\rangle \tag{3.4}$$

on the Weyl algebra W_0 as the unique (not everywhere defined) functional with the properties

$$\langle 0|\Phi * u_-(t,\mathbf{x})|0\rangle = \langle 0|u_+(t,\mathbf{x}) * \Phi|0\rangle = 0, \quad \langle 0|1|0\rangle = 1. \tag{3.5}$$

Here $u = u_+ + u_-$ is the decomposition of a solution $u(t,\mathbf{x})$ of the Klein–Gordon equation into the positive and negative frequency parts (we assume $m > 0$ so that this decomposition is unique). For $\Phi \in W_0$, define an operator in the standard Fock space with the matrix elements

$$\langle 0|\tilde{u}_-(-p'_{(1)}) \dots \tilde{u}_-(-p'_{(N')}) * \Phi * \tilde{u}_+(p_{(1)}) \dots \tilde{u}_+(p_{(N)})|0\rangle. \tag{3.6}$$

Here $\tilde{u}_\pm(p)$ is the Fourier transform (the momentum representation) of $u_\pm$, $p = (p_0, \dots, p_n)$.

One can check the following two properties of this correspondence:

i) $*$-product of functionals Φ goes to composition of operators in the Fock space, so that this correspondence is a (not everywhere defined) homomorphism from the algebra W_0 to the algebra of operators in the Fock space;

ii) complex conjugation of functionals Φ goes to Hermitian conjugation of operators in Hilbert space.

4. Quantization of interacting fields

Statement. *There exists a map from the set of smooth functions $g = g(t,\mathbf{x})$ with compact support to the set of functionals $P(g) \in W_0$ with the following properties.*

1) $P(g)$ is a formal series in g with the first two terms

$$P(g) = 1 + \frac{1}{ih}\int g(t,\mathbf{x})u(t,\mathbf{x})^4/4!\, dtd\mathbf{x} + \dots. \tag{4.1}$$

2) Classical limit: $P(g) = a(g,h)\exp(iR(g)/h)$ where $a(g,h)$ is a formal series in h, and conjugation by $\exp(iR(g)/h)$ in the Weyl algebra W_0

up to $O(h)$ yields the perturbation series for the evolution operator of the nonlinear classical field equation

$$\Box u(x) - m^2 u(x) = g(x)u^3(x)/3! \tag{4.2}$$

from $t = -\infty$ to $t = \infty$.

3) The Lorentz invariance condition:

$$LP(L^{-1}g) = P(g) \tag{4.3}$$

for a Lorentz transformation L.

4) The unitarity condition:

$$P(g) * \overline{P(g)} = 1, \tag{4.4}$$

where $\overline{P(g)}$ is complex conjugate to $P(g)$.

5) The causality condition: for two functions g_1, g_2 equal for $t \leq t_0$, the product $P(g_1) * P(g_2)^{-1}$ does not depend on the behavior of the functions g_1, g_2 for $t < t_0$.

6) The quasiclassical dynamical evolution (cf. with the Maslov–Shvedov quantum field theory complex germ [6]): for any spacelike surfaces $\mathcal{C}_1$, $\mathcal{C}_2$ there exists a limit $P_{\mathcal{C}_1,\mathcal{C}_2}$ of $P(g)$ modulo $o(h)$ as the function $g(x)$ tends to 1 *if x belongs to the strip between the spacelike surfaces and to* 0 *otherwise. This limit possesses the property*

$$P_{\mathcal{C}_1,\mathcal{C}_3} = P_{\mathcal{C}_2,\mathcal{C}_3} * P_{\mathcal{C}_1,\mathcal{C}_2} + o(h). \tag{4.5}$$

7) The S-matrix: there exists a limit P of $P(g)$ as $g(x)$ tends to the function $g = \mathrm{const}$. This P is a formal power series in g.

Any other choice of $P(g)$ with the properties 1–7 above is equivalent to some change of parameters $m, g(x)$.

This statement is completely similar to the Bogolyubov–Parasyuk theorem. Moreover, if we denote by $S(g)$ the operator in the Fock space corresponding to $P(g)$ and by S the operator corresponding to P, then $S(g)$ is exactly the Bogolyubov S-matrix and S is the physical S-matrix. The elements $P(g)$ are constructed in the same way as $S(g)$ in [4], using the renormalization procedure, the main difference being that composition of operators is replaced by $*$-product of functionals, and the normally ordered product of operators is replaced by the usual (commutative) product of functionals.

Note that the conditions on $P(g)$ (in particular, the causality condition) are the natural analogs of conditions on dependence of the evolution operator of a partial differential equation on the coefficient functions of this equation. Therefore the above apparatus is similar to the scattering theory in the theory of partial differential equations.

Note also that the presence of interaction cutoff function $g(x)$ is necessary from the physical point of view, since the scattering particles are considered as non-interacting at infinity (which means that $g = 0$ at infinity).

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Polynomial quantization on para-hermitian spaces with pseudo-orthogonal group of translations[1]

Svetlana V. Tsykina

We construct polynomial quantization, which is a variant of quantization in spirit of Berezin, on para-Hermitian symmetric spaces G/H with the pseudo-orthogonal group $G = \mathrm{SO}_0(p, q)$. For all these spaces, the connected component H_e of the subgroup H containing the identity of G is the direct product $\mathrm{SO}_0(p-1, q-1) \times \mathrm{SO}_0(1, 1)$, so that G/H is covered by G/H_e (with multiplicity 1, 2 or 4). The dimension of G/H is equal to $2n - 4$, where $n = p + q$. We restrict ourselves to the spaces G/H that are G-orbits in the adjoint representation of G.

A construction of quantization on arbitrary para-Hermitian symmetric spaces was given in [2]. The term "polynomial quantization" means in particular that both covariant and contravariant symbols are polynomials on G/H. Following the general scheme of [2], we introduce multiplication

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of covariant symbols, establish the correspondence principle, and study the Berezin transform.

The polynomial quantization on rank one para-Hermitian symmetric spaces has been constructed in [3]. In this paper, we consider the spaces G/H with $G = \mathrm{SO}_0(p,q)$. Note that these spaces have rank 2.

1. The pseudo-orthogonal group and its Lie algebra

Consider the space $\mathbb{R}^n$ equipped with the following bilinear form:

$$[x,y] = \sum_{i=1}^{n} \lambda_i x_i y_i,$$

where $\lambda_1 = \ldots = \lambda_p = -1$, $\lambda_{p+1} = \ldots = \lambda_n = 1$, and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are vectors in $\mathbb{R}^n$.

Let G denote the group $\mathrm{SO}_0(p,q)$. This group is the connected component of the identity, in the group of linear transformations of $\mathbb{R}^n$ that preserve $[x,y]$ and have determinant equal to 1. We assume that G acts linearly on $\mathbb{R}^n$ from the right: $x \mapsto xg$. In accordance with that, we write vectors in the row form. We also assume that $p > 1, q > 1$.

Let us write matrices $g \in G$ in the block form corresponding to the partition $n = 1 + (n-2) + 1$. Denote by H the subgroup of G consisting of matrices

$$h = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & v & 0 \\ \beta & 0 & \alpha \end{pmatrix}, \tag{1.1}$$

where $\alpha^2 - \beta^2 = 1$, $v \in \mathrm{SO}(p-1, q-1)$. The subgroup H consists of two connected components. The connected component H_e, containing the unit matrix E of G consists of matrices (1.1), where $\alpha = \mathrm{ch}t$, $\beta = \mathrm{sh}t$. Thus, it is $\mathrm{SO}_0(p-1,q-1) \times \mathrm{SO}_0(1,1)$. The second connected component of H (which does not contain E) contains the matrix $\mathrm{diag}\,\{-1,-1,1,\ldots,1,-1,-1\}$, as a representative.

The Lie algebra $\mathfrak{g}$ of G consists of real matrices X of order n satisfying the condition $X'I + IX = 0$, where $I = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_n\}$, the prime denotes matrix transposition.

Let

$$Z_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

The stabilizer of Z_0 in the adjoint representation is exactly the group H, therefore, the manifold G/H is just the G-orbit of the matrix Z_0 in $\mathfrak{g}$.

The operator $\operatorname{ad} Z_0$ has three eigenvalues: $-1, 0, +1$. Respectively, the Lie algebra $\mathfrak{g}$ is decomposed into the direct sum of eigenspaces

$$\mathfrak{g} = \mathfrak{q}^- + \mathfrak{h} + \mathfrak{q}^+,$$

where $\mathfrak{h}$ is the Lie algebra of H. The subspaces $\mathfrak{q}^-, \mathfrak{q}^+$ consist of matrices

$$X_\xi : \begin{pmatrix} 0 & \xi & 0 \\ \xi^* & 0 & \xi^* \\ 0 & -\xi & 0 \end{pmatrix}, \qquad Y_\eta : \begin{pmatrix} 0 & \eta & 0 \\ \eta^* & 0 & -\eta^* \\ 0 & \eta & 0 \end{pmatrix}$$

respectively, where ξ, η are rows in $\mathbb{R}^{n-2}$. Both spaces $\mathfrak{q}^\pm$ are Abelian subalgebras of $\mathfrak{g}$, they have dimension $n-2$. The subgroup H preserves both subspaces $\mathfrak{q}^-$ and $\mathfrak{q}^+$ in the adjoint action:

$$Z \mapsto h^{-1} Z h, \ h \in H. \tag{1.3}$$

Let $h \in H$ have the form (1.1). For simplicity, we identify matrices X_ξ and Y_η with vectors ξ and η, respectively. Under the action (1.3) vectors $\xi \in \mathfrak{q}^-$ and $\eta \in \mathfrak{q}^+$ are transformed as follows:

$$\xi \mapsto \widetilde{\xi} = (\alpha + \beta)\xi v, \quad \eta \mapsto \widehat{\eta} = (\alpha - \beta)\eta v. \tag{1.4}$$

Consider the space $\mathbb{R}^{n-2}$ with bilinear form defined by the matrix $I_1 = \operatorname{diag}\{\lambda_2, \dots, \lambda_{n-1}\}$:

$$\langle \xi, \eta \rangle = \sum_{i=2}^{n-1} \lambda_i \xi_i \eta_i.$$

2. Representations of G associated with a cone

The group $G = \mathrm{SO}_0(p, q)$ preserves manifolds $[x, x] = c$, $c \in \mathbb{R}$, in $\mathbb{R}^n$. Let $\mathcal{C}$ be the cone $[x, x] = 0$, $x \neq 0$, in $\mathbb{R}^n$. Let us fix two points in the cone: $s^+ = (1, 0, \dots, 0, 1)$, $s^- = (1, 0, \dots, 0, -1)$. Consider the following two sections of the cone:

$$\begin{aligned} \Gamma^+ &= \{x_1 + x_n = 2\} = \{[x, s^-] = -2\}, \\ \Gamma^- &= \{x_1 - x_n = 2\} = \{[x, s^+] = -2\}. \end{aligned}$$

The points s^+, s^- belong to Γ^+, Γ^- respectively. They are eigenvectors of the maximal parabolic subgroups $P^+ = Q^+ H$ and $P^- = Q^- H$ respectively, with eigenvalues $\alpha - \beta$ and $\alpha + \beta$, where α, β are parameters of $h \in H$, see (1.1). Here $Q^- = \exp \mathfrak{q}^-$, $Q^+ = \exp \mathfrak{q}^+$.

The section $\Gamma^{\pm}$ meets almost all generatrices of the cone $\mathcal{C}$. The linear action of G on the cone induces the following actions of G on Γ^- and Γ^+ respectively:

$$x \longmapsto \widetilde{x} = -\frac{2}{[xg, s^+]} \cdot xg, \ x \in \Gamma^-, \tag{2.1}$$

$$x \longmapsto \widehat{x} = -\frac{2}{[xg, s^-]} \cdot xg, \ x \in \Gamma^+, \tag{2.2}$$

defined almost everywhere on $\Gamma^{\pm}$. For the subgroups Q^- and Q^+ respectively, these actions turn out to be linear: $x \mapsto xg$. Moreover, the subgroups $Q^{\pm}$ act on $\Gamma^{\pm}$ simply transitively. This allows to define the coordinates $\xi = (\xi_2, \dots, \xi_{n-1})$ on Γ^- and $\eta = (\eta_2, \dots, \eta_{n-1})$ on Γ^+ transferring them from $\mathfrak{q}^-$ on $\mathfrak{q}^+$ respectively, namely, for $u \in \Gamma^-$ and $v \in \Gamma^+$ we set:

$$u = u(\xi) = s^- e^{X_\xi} = (1 + \langle \xi, \xi \rangle, 2\xi, -1 + \langle \xi, \xi \rangle), \tag{2.3}$$

$$v = v(\eta) = s^+ e^{Y_\eta} = (1 + \langle \eta, \eta \rangle, 2\eta, 1 - \langle \eta, \eta \rangle). \tag{2.4}$$

The stabilisers in G of the points $s^- \in \Gamma^-$ and $s^+ \in \Gamma^+$ under the actions (2.1) and (2.2) are the subgroups $P^+ = Q^+H$ and $P^- = Q^-H$ respectively.

Let $\sigma \in \mathbb{C}$, $\varepsilon = 0, 1$. Let $\mathcal{D}_{\sigma,\varepsilon}(\mathcal{C})$ be the space of C^∞ functions f on the cone $\mathcal{C}$ with homogeneity σ and parity ε, i.e.

$$f(tx) = t^{\sigma,\varepsilon} f(x), \ x \in \mathcal{C}, \ t \in \mathbb{R}^* = \mathbb{R} \setminus \{0\},$$

where we denote $t^{\sigma,\varepsilon} = |t|^\sigma \mathrm{sgn}^\varepsilon t$. Denote by $T_{\sigma,\varepsilon}$ the representation of G which acts on $\mathcal{D}_{\sigma,\varepsilon}(\mathcal{C})$ by translations: $(T_{\sigma,\varepsilon}(g)f)(x) = f(xg)$.

Consider now the restrictions of functions from $\mathcal{D}_{\sigma,\varepsilon}(\mathcal{C})$ to the sections $\Gamma^{\pm}$. Such restrictions form a space $\mathcal{D}_{\sigma,\varepsilon}(\Gamma^{\pm})$ of functions f on $\Gamma^{\pm}$. This space is contained in $C^\infty(\Gamma^{\pm})$ and contains $\mathcal{D}(\Gamma^{\pm})$. In the coordinates ξ, η, the representation $T_{\sigma,\varepsilon}$ of the group G acts on the space of restrictions $\mathcal{D}_{\sigma,\varepsilon}(\mathcal{C})$ by

$$(T_{\sigma,\varepsilon}(g)f)(\xi) = f(\widetilde{\xi}) \left\{ -\frac{1}{2}[ug, s^+] \right\}^{\sigma,\varepsilon}, \tag{2.5}$$

$$(T_{\sigma,\varepsilon}(g)f)(\eta) = f(\widehat{\eta}) \left\{ -\frac{1}{2}[vg, s^-] \right\}^{\sigma,\varepsilon}, \tag{2.6}$$

where $u = u(\xi)$, $v = v(\eta)$ are defined by (2.3), (2.4), actions $\xi \mapsto \widetilde{\xi}$ and $\eta \mapsto \widehat{\eta}$ are defined by (2.1), (2.2).

Define the operator $A_{\sigma,\varepsilon}$ on $\mathcal{D}_{\sigma,\varepsilon}(\Gamma^{\pm})$ by:

$$(A_{\sigma,\varepsilon}f)(\xi) = \int\limits_{\mathbb{R}^{n-2}} N(\xi,\eta)^{2-n-\sigma,\varepsilon} f(\eta)d\eta, \tag{2.7}$$

where

$$N(\xi,\eta) = -\frac{1}{2}[u(\xi),v(\eta)] = 1 - 2\langle\xi,\eta\rangle + \langle\xi,\xi\rangle\langle\eta,\eta\rangle.$$

The function $N(\xi,\eta)$ is a polynomial in ξ,η. The operator $A_{\sigma,\varepsilon}$ intertwines the representations $T_{\sigma,\varepsilon}$ and $T_{2-n-\sigma,\varepsilon}$. These representations act on functions on *different* sections. We can change the position of ξ and η in (2.7). The product $A_{2-n-\sigma,\varepsilon}A_{\sigma,\varepsilon}$ is a scalar operator:

$$A_{2-n-\sigma,\varepsilon}A_{\sigma,\varepsilon} = \omega_0(\sigma,\varepsilon)E,$$

where

$$\begin{aligned}\omega_0(\sigma,\varepsilon) &= 2^3\pi^{n-3}\frac{\Gamma(\sigma+1)\Gamma(3-n-\sigma)}{(2\sigma+n-2)\sin\left(\sigma+\frac{n}{2}\right)\pi}\times\\ &\times \sin\frac{\sigma-\varepsilon}{2}\pi\cdot\sin\frac{\sigma-\varepsilon+p}{2}\pi\cdot\sin\frac{\sigma+\varepsilon+q}{2}\pi\cdot\sin\frac{\sigma+\varepsilon+n}{2}\pi.\end{aligned}$$

3. The space G/H

Consider the following realization of the space G/H. Let Ω be the set of matrices:

$$z = \frac{y^*x}{[x,y]}, \tag{3.1}$$

where $x,y\in\mathcal{C}$, $y^* = Iy'$. For these matrices, rank and trace are equal to 1. The adjoint action $z\mapsto g^{-1}zg$ preserves Ω. The stabilizer of the matrix z^0, corresponding to the pair $x = s^-, y = s^+$, is the subgroup H, so that Ω is just G/H.

Take vectors $u = u(\xi)$ and $v = v(\eta)$ in the sections Γ^- and Γ^+ of the cone $\mathcal{C}$, respectively, for x and y in (3.1) We obtain an embedding $\Gamma^-\times\Gamma^+\to\Omega$ given by

$$z = z(\xi,\eta) = \frac{v^*u}{[u,v]}, \quad u = u(\xi),\ v = v(\eta), \tag{3.2}$$

The map $(u,v)\longmapsto z$ given by formula (3.2) is defined for $\xi,\eta\in\mathbb{R}^{n-2}$ such that $N(\xi,\eta)\neq 0$, since $[u,v] = -2N(\xi,\eta)$. Therefore, vectors $\xi,\eta\in\mathbb{R}^{n-2}$ with the condition $N(\xi,\eta)\neq 0$ are local coordinates on Ω. The adjoint action of the group G on Ω is generated by its actions on ξ and η. For each $g\in G$, this action is defined on a dense set of Ω.

We can identify the tangent space of G/H at the initial point z^0 with the space $\mathfrak{q} = \mathfrak{q}^- + \mathfrak{q}^+$ in the Lie algebra $\mathfrak{g}$. Let $S(\mathfrak{q})$ denote the algebra of polynomials on $\mathfrak{q}$. The action (1.3) of the group H on $\mathfrak{q}$ induces an action of H on $S(\mathfrak{q})$. Let $S(\mathfrak{q})^H$ denote the algebra of polynomials invariant with respect to H. This algebra is generated by two polynomials $\langle \xi, \eta \rangle$ and $\langle \xi, \xi \rangle \langle \eta, \eta \rangle$.

Let $\mathbb{D}(G/H)$ denote the algebra of differential operators on G/H invariant with respect to G. This algebra is in the one-to-one correspondence with the algebra $S(\mathfrak{q})^H$. Let Δ_2 and Δ_4 denote operators in $\mathbb{D}(G/H)$ corresponding to generators $\langle \xi, \eta \rangle$ and $\langle \xi, \xi \rangle \langle \eta, \eta \rangle$ of $S(\mathfrak{q})^H$ respectively. Let us call these operators Δ_2 and Δ_4 the Laplace operators on G/H. The operator Δ_2 is the Laplace-Beltrami operator. These operators are differential operators of the second and the fourth order respectively, they are generators in $\mathbb{D}(G/H)$. Explicit expressions of them are very cumbersome. We write explicit expressions for their radial parts $\overset{0}{\Delta}_2$ and $\overset{0}{\Delta}_4$ in horospherical coordinates.

These coordinates are defined as follows. Let us take in $\mathfrak{q} = \mathfrak{q}^+ + \mathfrak{q}^-$ the Cartan subspace $\mathfrak{a}$, consisting of matrices

$$A_t = \begin{pmatrix} 0 & 0 & 0 & t_1 & 0 \\ 0 & 0 & 0 & 0 & t_2 \\ 0 & 0 & 0 & 0 & 0 \\ t_1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 \end{pmatrix},$$

where $t = (t_1, t_2) \in \mathbb{R}^2$. Introduce in $\mathfrak{a}^*$ the lexicographical order in coordinates. Let $\mathfrak{n}$ denote the subalgebra of $\mathfrak{g}$ formed by the corresponding positive root spaces. Let $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$. Consider the set of points z in Ω obtained from z^0 via the translation by $a = a(t_1, t_2) \in A$ and then by $n \in N$, i.e. $z = n^{-1} a^{-1} z^0 a n$. It is a neighbourhood U of the point z^0. Parameters t_1, t_2 of the subgroup A and also parameters of the subgroup N are coordinates in this neighbourhood (horospherical coordinates).

Let f be a function defined on U that does not depend on $n \in N$. Then it is a function of $t = (t_1, t_2)$: $f(z) = F(t)$. Let D be a differential operator in $\mathbb{D}(G/H)$. Then Df also does not depend on $n \in N$:

$$Df = \overset{0}{D}\, F,$$

where $\overset{0}{D}$ is a differential operator in t_1, t_2, the radial part of D with respect to N. It turns out to be a differential operator with constant coefficients.

Introduce operators

$$\begin{aligned} D_1 &= \left[\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}+n-3\right]^2-(2n-7) \\ D_2 &= \left[\frac{\partial}{\partial t_1}-\frac{\partial}{\partial t_2}+1\right]^2-(2n-7). \end{aligned}$$

THEOREM 3.1. *We have that*

$$\begin{aligned} \overset{0}{\Delta}_2 &= \frac{1}{2}\left\{D_1+D_2-(n-4)(n-6)\right\}, \\ \overset{0}{\Delta}_4 &= D_1D_2+2(n-4)^3. \end{aligned}$$

4. Polynomial quantization on G/H

We follow the scheme from [2]. The role of supercomplete system is played by the kernel $\Phi(\xi,\eta)=\Phi_{\sigma,\varepsilon}(\xi,\eta)=N(\xi,\eta)^{\sigma,\,\varepsilon}$ of the intertwining operator $A_{2-n-\sigma,\,\varepsilon}$. As an analogue of the Fock space, we take the space of functions $\varphi(\xi)$. We start from the algebra of operators $D=T_{\sigma,\,\varepsilon}(X)$, where X belongs to the universal enveloping algebra $\mathrm{Env}(\mathfrak{g})$ for $\mathfrak{g}$. *The covariant symbol* $F(\xi,\eta)$ of the operator D is defined by:

$$F(\xi,\eta)=\frac{1}{\Phi(\xi,\eta)}\,D_\xi\Phi(\xi,\eta),$$

where D_ξ means that the operator D acts on $\Phi(\xi,\eta)$ as on a function of ξ. These covariant symbols are independent of ε. They are functions on G/H. Moreover, they are *polynomials* on G/H (i.e. restrictions on G/H of polynomials on the space of matrices z, see (3.1)).

For generic σ the space $\mathcal{A}_\sigma$ of covariant symbols is the space $S(G/H)$ of all polynomials on G/H.

The map $D\mapsto F$, which assigns to an operator its covariant symbol, is $\mathfrak{g}$–equivariant. For an arbitrary σ the operator D is reconstructed from its covariant symbol F:

$$(D\varphi)(\xi)=c(\sigma,\varepsilon)\int F(\xi,v)\,\frac{\Phi(\xi,v)}{\Phi(u,v)}\,\varphi(u)\,dx(u,v), \tag{4.1}$$

where $c(\sigma,\varepsilon)=\omega_0(\sigma,\varepsilon)^{-1}$.

The multiplication of operators gives rise to a multiplication (denote it by $*$) of covariant symbols. Let F_1, F_2 be the covariant symbols of operators D_1, D_2 respectively. We have that

$$F_1*F_2=\frac{1}{\Phi}(D_1)_\xi(\Phi F_2).$$

This multiplication is given by

$$(F_1 * F_2)(\xi,\eta) = \int F_1(\xi,v)F_2(u,\eta)\mathcal{B}(\xi,\eta;u,v)\,dx(u,v),$$

where $dx(u,v)$ is an invariant measure on G/H, and

$$\mathcal{B}(\xi,\eta;u,v) = c\,\frac{\Phi(\xi,v)\Phi(u,\eta)}{\Phi(\xi,\eta)\Phi(u,v)}.$$

Let us call this kernel $\mathcal{B}$ the *Berezin kernel*.

Thus, the spaces $\mathcal{A}_\sigma$ turn out to be associative algebras with unit (with respect to $*$).

On the other hand, we can define *contravariant symbols* of the operators. A function $F(\xi,\eta)$ can be viewed as the contravariant symbol for the following operator A (acting on functions $\varphi(\xi)$):

$$(A\varphi)(\xi) = c(\sigma,\varepsilon)\int F(u,v)\,\frac{\Phi(\xi,v)}{\Phi(u,v)}\varphi(u)\,dx(u,v).$$

Notice that this expression differs from (4.1) only by the first argument of function F. A contravariant symbol can be reconstructed from the corresponding operator.

Thus we obtain two maps $D \mapsto F$ ("co") and $F \mapsto A$ ("contra"), connecting operators D and A with polynomials F on G/H.

The passage from the contravariant symbol of an operator to its covariant symbol is an integral operator with the Berezin kernal. Let us call $\mathcal{B}$ the *Berezin transform*.

THEOREM 4.1. *The Berezin transform can be expressed in terms of Laplace operators:*

$$\mathcal{B} = \frac{\Gamma(\sigma+n-2+\frac{a+b}{2})\Gamma(\sigma+1-\frac{a+b}{2})\Gamma(\sigma+\frac{n}{2}+\frac{a-b}{2})\Gamma(\sigma+\frac{n}{2}-1-\frac{a-b}{2})}{\Gamma(\sigma+n-2)\Gamma(\sigma+1)\Gamma(\sigma+\frac{n}{2})\Gamma(\sigma+\frac{n}{2}-1)}$$

where a,b are some variables and one has to consider

$$D_1 = (a+b)^2 + 2(n-3)(a+b) + (n-4)^2,$$
$$D_2 = (a-b)^2 + 2(a-b) - 2(n-4).$$

Note that on finite-dimensional subspaces in $S(G/H)$ the Berezin transform is a differential operator.

Now let $\sigma \to -\infty$. The first two terms of the asymptotic expansion of $\mathcal{B}$ are given by:

$$\mathcal{B} \sim 1 - \frac{1}{\sigma}\,\Delta_2. \tag{4.2}$$

The relation (4.2) implies the following *correspondence principle* (as the "Planck constant" one has to take $h = -1/\sigma$):

$$F_1 * F_2 \longrightarrow F_1 F_2, \tag{4.3}$$

$$-\sigma\,(F_1 * F_2 - F_2 * F_1) \longrightarrow \{F_1,\, F_2\}, \tag{4.4}$$

as $\sigma \to -\infty$, In (4.3) and (4.4), $F_1 F_2$ denotes the pointwise multiplication of F_1 and F_2, and $\{F_1, F_2\}$ stands for the Poisson bracket of F_1 and F_2.

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The horofunction boundary[1]

Cormac Walsh

The horofunction boundary (also known as the 'metric' or 'Busemann' boundary) is a means of compactifying metric spaces. Its definition goes back to Gromov [**10**] in the 1970s but it seems not to have received much study until recently, when it has appeared in several different domains [**11, 16, 15, 1, 13**]. To define this boundary for a metric space (X, d), one assigns to each point $z \in X$ the function $\phi_z : X \to \mathbb{R}$,

$$\phi_z(x) := d(x, z) - d(b, z),$$

where b is some basepoint. If X is proper, then the map $\phi : X \to C(X)$, $z \mapsto \phi_z$ defines an embedding of X into $C(X)$, the space of continuous real-valued functions on X endowed with the topology of uniform convergence on compacts. The horofunction boundary is defined to be $X(\infty) := \mathrm{cl}\{\phi_z \mid z \in X\} \backslash \{\phi_z \mid z \in X\}$, and its elements are called horofunctions.

This boundary is not the same as the better known Gromov boundary of a δ-hyperbolic space. For these spaces, it has been shown [**5, 22, 17**]

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that the horoboundary is finer than the Gromov boundary in the sense that there exists a continuous surjection from the former to the latter.

Of particular interest are those horofunctions that are the limits of almost-geodesics. An almost-geodesic, as defined by Rieffel [**16**], is a map γ from an unbounded set $T \subset \mathbb{R}_+$ containing 0 to X, such that for any $\epsilon > 0$,

$$|d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t| < \epsilon$$

for all $t \in T$ and $s \in T$ large enough with $t \geq s$. Rieffel called the limits of such paths Busemann points. See [**1**] for a slightly different definition of almost-geodesic which nevertheless gives rise to the same set of Busemann points.

As noted by Ballmann [**2**], the construction above is an additive analogue of the way the Martin boundary is constructed in Probabilistic Potential Theory. One may pursue the analogy further in the framework of max-plus algebra, where one replaces the usual operations of addition and multiplication by those of maximum and addition. Indeed, this approach has already provided inspiration for many results about the horofunction boundary [**1, 18**]. We mention, for example, the characterisation of Busemann points as the functions in the horoboundary that are extremal generators in the max-plus sense of the set of 1-Lipschitz functions. So the set of Busemann points is seen to be an analogue of the *minimal* Martin boundary. There is also a representation of 1-Lipschitz functions in terms of horofunctions analogous to the Martin representation theorem.

There are few examples of metric spaces where the horofunction boundary or Busemann points are explicitly known. The first cases to be investigated were those of Hadamard manifolds [**3**] and Hadamard spaces [**2**], where the horofunction boundary turns out to be homeomorphic to the ray boundary and all horofunctions are Busemann points. The case of finite-dimensional normed spaces has also received attention. Karlsson *et. al.* determined the horofunction boundary in the case when the norm is polyhedral [**12**]. Other examples of metric spaces where the horofunction boundary has been studied include the Cayley graphs of finitely-generated abelian groups, studied by Develin [**6**], and Finsler p–metrics on $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}_n$, where explicit expressions for the horofunctions were found by Friedland and Freitas [**8, 9**]. Webster and Winchester have some general results on when all horofunctions are Busemann points [**24**], [**23**].

In the following sections, we describe our recent work elucidating the horoboundary of some particular metric spaces.

1. Normed spaces

Rieffel comments that it is an interesting question as to when all boundary points of a metric space are Busemann points and asks whether this is the case for general finite-dimensional normed spaces. In [**19**], we answer this question in the negative and give a necessary and sufficient criterion for it to be the case.

Let V be an arbitrary finite-dimensional normed space with unit ball B. Recall that a convex subset E of a convex set D is said to be an *extreme set* if the endpoints of any line segment in D are contained in E whenever any interior point of the line segment is. For any extreme set E of the dual unit ball B° and point p of V, define the function $f_{E,p}$ from the dual space V^* to $[0,\infty]$ by

$$f_{E,p}(q) := I_E(q) + \langle q|p\rangle - \inf_{y\in E}\langle y|p\rangle \qquad \text{for all } q \in V^*.$$

Here I_E is the indicator function, taking value 0 on E and $+\infty$ everywhere else.

Our first theorem characterises the Busemann points of V as the Legendre-Fenchel transforms of these functions.

THEOREM 1.1. *The set of Busemann points of a finite-dimensional normed space* $(V, ||\cdot||)$ *is*

$$\{f^*_{E,p} \mid E \text{ is a proper extreme set of } B^\circ \text{ and } p \in V\}.$$

We use this knowledge to characterise those norms for which all horofunctions are Busemann points.

THEOREM 1.2. *A necessary and sufficient condition for every horofunction of a finite-dimensional normed space to be a Busemann point is that the set of extreme sets of the dual unit ball be closed in the Painlevé–Kuratowski topology.*

2. The Hilbert metric

Let x and y be distinct points in a bounded open convex subset D of $\mathbb{R}^N$, with $N \geq 1$. Define w and z to be the points in the Euclidean boundary of D such that w, x, y, and z are collinear and arranged in this order along the line in which they lie. The Hilbert distance between x and y is defined to be the logarithm of the cross ratio of these four points:

$$\mathrm{Hil}(x,y) := \log \frac{|zx|\,|wy|}{|zy|\,|wx|}.$$

If D is the open unit disk, then the Hilbert metric is exactly the Klein model of the hyperbolic plane.

As pointed out by Busemann [**4**, p105], the Hilbert geometry is related to hyperbolic geometry in much the same way that normed space geometry is related to Euclidean geometry. It will not be surprising therefore that there are similarities between the horofunction boundaries of Hilbert geometries and of normed spaces.

Define the function

$$\text{Funk}(x,y) := \log \frac{|zx|}{|zy|}, \qquad \text{for all } x \text{ and } y \text{ in } D. \tag{2.1}$$

This function satisfies the usual metric space axioms, apart from that of symmetry.

Hilbert's metric can now be written

$$\text{Hil}(x,y) := \text{Funk}(x,y) + \text{Funk}(y,x), \qquad \text{for all } x \text{ and } y \text{ in } D.$$

This expression of the Hilbert metric as the symmetrisation of the Funk metric plays a crucial role. It turns out that every Hilbert horofunction is the sum of a horofunction in the Funk geometry and a horofunction in the reverse Funk geometry, where the metric in the latter is given by

$$\text{rev}(x,y) := \text{Funk}(y,x).$$

This allows us to simplify the problem by investigating separately the horofunction boundaries of these two geometries and then combining the results. Determining the boundary of the Funk geometry turns out to be very similar to determining that of a normed space, which was done in [**19**].

In [**20**], we characterise those Hilbert geometries for which all horofunctions are Busemann points.

THEOREM 2.1. *A necessary and sufficient condition for every horofunction on a bounded convex open subset of $\mathbb{R}^N$ containing the origin to be a Busemann point in the Hilbert geometry is that the set of extreme sets of its polar be closed in the Painlevé–Kuratowski topology.*

It had previously been shown [**12**] that all horofunctions of the Hilbert geometry on a polytope are Busemann points.

THEOREM 2.2. *Let D be a bounded convex open subset of $\mathbb{R}^N$. If a sequence in D converges to a point in the horofunction boundary of the Hilbert geometry, then the sequence converges in the usual sense to a point in the Euclidean boundary ∂D.*

3. Finitely generated groups

An interesting class of metric spaces are the Cayley graphs of finitely generated groups with their word metric. Here one may hope to have a combinatorial description of the horoboundary.

The first to consider the horoboundary in this setting was Rieffel [**16**] who studied the horoboundary of $\mathbb{Z}^n$ with an arbitrary finite generating set in connection with his work on non-commutative geometry.

In [**21**], we investigate the horofunction boundary of Artin groups of dihedral type. Let $\mathrm{prod}(s,t;n) := ststs\cdots$, with n factors in the product. The Artin groups of dihedral type have the following presentation:

$$A_k = \langle a, b \mid \mathrm{prod}(a,b;k) = \mathrm{prod}(b,a;k)\rangle, \qquad \text{with } k \geq 3.$$

Observe that A_3 is the braid group on three strands. The generators traditionally considered are the Artin generators $S := \{a, b, a^{-1}, b^{-1}\}$.

In what follows, we will have need of the Garside normal form for elements of A_k. The element $\Delta := \mathrm{prod}(a,b;k) = \mathrm{prod}(b,a;k)$ is called the Garside element. Let

$$M^+ := \{a, b, ab, ba, \ldots, \mathrm{prod}(a,b;k-1), \mathrm{prod}(b,a;k-1)\}.$$

It can be shown [**7**] that $w \in A_k$ can be written

$$w = w_1 \cdots w_n \Delta^r$$

for some $r \in \mathbb{Z}$ and $w_1, \ldots, w_n \in M^+$. This decomposition is unique if n is required to be minimal. We call it the right normal form of w. The factors $w_1, \ldots, w_n$ are called the canonical factors of w.

An algorithm was given in [**14**] for finding a geodesic word representing any given element of $A_k; k \geq 3$. We use this algorithm to find a simple formula for the word length metric.

Proposition 3.1. *Let $x = z_1 \cdots z_m \Delta^r$ be an element of A_k written in right normal form. Let $(p_0, \ldots, p_{k-1}) \in \mathbb{N}^k$ be such that $p_0 := r$ and, for each $i \in \{1, \ldots, k-1\}$, $p_i - p_{i-1} = m_{k-i}$, where m_i is the number of canonical factors of x of length i. Then the distance from the identity e to x in the Artin-generator word-length metric is*

$$d(e,x) = \sum_{i=0}^{k-1} |p_i|.$$

Since d is invariant under left multiplication, that is, $d(y,x) = d(e, y^{-1}x)$, we can use this formula to calculate the distance between any pair of elements y and x of A_k. With this knowledge we can find the following description of the horofunction compactification.

Let Z be the set of possibly infinite words of positive generators having no product of consecutive letters equal to Δ. We can write each element z of Z as a concatenation of substrings in such a way that the products of the letters in every substring equals an element of M^+ and the combined product of letters in each consecutive pair of substrings is not in M^+. Because z does not contain Δ, this decomposition is unique. Let $m_i(z)$ denote the number of substrings of length i. Note that if z is an infinite word, then this number will be infinite for some i.

Let Ω' denote the set of (p, z) in $(\mathbb{Z} \cup \{-\infty, +\infty\})^k \times Z$ satisfying the following:

- $p_i - p_{i-1} \geq m_{k-i}(z)$ for all $i \in \{1, \dots, k-1\}$ such that p_i and p_{i-1} are not both $-\infty$ nor both $+\infty$;
- if z is finite, then $p_i - p_{i-1} = m_{k-i}(z)$ for all $i \in \{1, \dots, k-1\}$ such that p_i and p_{i-1} are not both $-\infty$ nor both $+\infty$.

We take the product topology on Ω'.

We now define Ω to be the quotient topological space of Ω' where the elements of $(+\infty, \dots, +\infty) \times Z$ are considered equivalent and so also are those in $(-\infty, \dots, -\infty) \times Z$. We denote these two equivalence classes by $+\hat{\infty}$ and $-\hat{\infty}$, respectively.

We let $\mathcal{M}$ denote the horofunction compactification of A_k with the Artin-generator word metric. The basepoint is taken to be the identity.

THEOREM 3.2. *The sets Ω and $\mathcal{M}$ are homeomorphic.*

Let Z_0 be the set of elements of Z that are finite words. Let Ω_0 denote the set of (p, z) in $\mathbb{Z}^k \times Z_0$ such that $p_i - p_{i-1} = m_{k-i}(z)$ for all $i \in \{1, \dots, k-1\}$. One can show that the elements of Ω_0 are exactly the elements of Ω corresponding to functions of the form $d(\cdot, z) - d(e, z)$ in $\mathcal{M}$.

In the present context, since the metric takes only integer values, the Busemann points are exactly the limits of geodesics (see [**24**]). Develin [**6**], investigated the horoboundary of finitely generated abelian groups with their word metrics and showed that all their horofunctions are Busemann.

We have the following characterisation of the Busemann points of A_k.

THEOREM 3.3. *A function in $\mathcal{M}$ is a Busemann point if and only if the corresponding element (p, z) of Ω is in $\Omega \backslash \Omega_0$ and satisfies the following: $p_i - p_{i-1} = m_{k-i}(z)$ for every $i \in \{1, \dots, k-1\}$ such that p_i and p_{i-1} are not both $-\infty$ nor both $+\infty$.*

The group A_k also has a dual presentation:

$$A_k = \langle \sigma_1, \dots, \sigma_k \mid \sigma_1\sigma_2 = \sigma_2\sigma_3 = \cdots = \sigma_k\sigma_1 \rangle, \qquad \text{with } k \geq 3.$$

The set of dual generators is $\tilde{S} := \{\sigma_1, \ldots, \sigma_k, \sigma_1^{-1}, \ldots, \sigma_k^{-1}\}$.

Again, one can find a formula for the word length metric and use it to determine the horoboundary. This time however, it turns out that there are no non-Busemann points.

THEOREM 3.4. *In the horoboundary of A_k with the dual-generator word metric, all horofunctions are Busemann points.*

In general, one would expect the properties of the horoboundary of a group with its word length metric to depend strongly on the generating set. It would be interesting to know for which groups and for which properties there is not this dependence. As already mentioned, all boundary points of abelian groups are Busemann no matter what the generating set [**6**]. On the other hand, the above results show that for Artin groups of dihedral type the existence of non-Busemann points depends on the generating set.

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Квантование как приближенное описание некоторого диффузионного процесса

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1. Описание и некоторые свойства модели

Рассматривается некоторая математическая модель процесса, состояние которого в каждый момент времени задается волновой функцией – комплекснозначной функцией $\varphi(x,p)$, где $(x,p) \in R^{2n}$, и n — размерность конфигурационного пространства. В отличие от квантовой механики, где волновая функция зависит только от координат или только от импульсов, в нашем случае волновая функция зависит и от координат и и от импульсов. Так же, как в квантовой механике, предполагается, что для волновых функций выполняется принцип суперпозиции, и

плотность вероятности $\rho(x,p)$ на фазовом пространстве, соответствующая волновой функции $\varphi(x,p)$, задается стандартной формулой

$$\rho(x,p) = \varphi^*(x,p)\varphi(x,p) = |\varphi(x,p)|^2. \tag{1.1}$$

В работе рассматривается классическая модель диффузионного процесса для волновой функции $\varphi(x,p)$ на фазовом пространстве. Предполагается, что каждый комплексный вектор волновой функции одновременно находится в 4-х движениях:

точка приложения вектора движется по классической траектории, заданной функцией Гамильтона $H(x,p)$;

точка приложения вектора перемещается случайно по координатам и импульсам, находясь в диффузионном процессе с постоянными коэффициентами диффузий a^2 и b^2 по координатам и импульсам, соответственно;

точка приложения каждого вектора движется по случайной траектории в результате движений, описанных в двух предыдущих пунктах, а сам вектор вращается с постоянной угловой скоростью $\omega = mc^2/\hbar$ в системе координат, связанной с этой точкой, где m – масса частицы, c – скорость света, $\hbar$ – постоянная Планка;

длина всех комплексных векторов волновой функции в момент времени t умножается на $\exp(abnt/\hbar)$ (это чисто техническое требование, которое не сказывается на относительных вероятностях нахождения частицы в фазовом пространстве).

Предполагается, что волновой вектор $\varphi(x,p,t)$ в точке (x,p) в момент времени t по принципу суперпозиции равен сумме волновых векторов, заданных распределением векторов $\varphi^0(x,p)$ в начальный момент времени и попавших в результате описанных выше движений в точку (x,p) в момент времени t.

Процесс описывается дифференциальным уравнением диффузионного типа. Анализ дифференциального уравнения модели показывает, что движение в модели раскладывается на быстрое и медленное. В результате быстрого движения система, начиная с произвольной волновой функции на фазовом пространстве, переходит к функции, принадлежащей некоторому особому подпространству. Элементы этого подпространства параметризуются волновыми функциями, зависящими только от координат. Медленное движение по подпространству описывается уравнением Шредингера.

Исходя из предположений о тепловой причине диффузий и соответствии следствий модели известным физическим экспериментам Лэмба - Резерфорда [**2**] (сдвиг Лэмба в спектре атома водорода), в

работе делается оценка коэффициентов диффузий и времени переходного процесса от классического описания процесса, в котором принцип неопределенности Гейзенберга может не выполняться, к квантовому, в котором принцип Гейзенберга уже выполняется. Время переходного процесса имеет порядок $1/T \cdot 10^{-11}$с, где T — температура среды.

2. Основные результаты

Рассмотрим диффузионный процесс на фазовом пространстве, в котором волновая функция $\varphi(x,p,t)$ в момент времени t удовлетворяет дифференциальному уравнению

$$\frac{\partial\varphi}{\partial t} = \sum_{k=1}^{n}\left(\frac{\partial H}{\partial x_k}\frac{\partial\varphi}{\partial p_k} - \frac{\partial H}{\partial p_k}\frac{\partial\varphi}{\partial x_k}\right) - \frac{i}{\hbar}\left(H - \sum_{k=1}^{n}\frac{\partial H}{\partial p_k}p_k\right)\varphi + \Delta_{a,b}\varphi, \tag{2.1}$$

$$\text{где} \qquad \Delta_{a,b}\varphi = a^2\sum_{k=1}^{n}\left(\frac{\partial}{\partial x_k} - \frac{ip_k}{\hbar}\right)^2\varphi + b^2\sum_{k=1}^{n}\frac{\partial^2}{\partial p_k^2}\varphi + \frac{abn}{\hbar}\varphi, \tag{2.2}$$

где $H(x,p)$ — функция Гамильтона; a^2 и b^2 — коэффициенты диффузий по координатам и импульсам, соответственно.

Если в уравнении (2.1) отбросить последнее слагаемое, то получим дифференциальное уравнение в частных производных первого порядка. Эта часть уравнения (2.1) описывает детерминированную составляющую движения комплексных векторов $\varphi(x,p,t)$. Согласно уравнению, в этом движении точка приложения каждого вектора движется по классической траектории, заданной гамильтонианом $H(x,p)$, а сам вектор при этом вращается в каждой точке траектории с угловой скоростью

$$\omega' = \frac{1}{\hbar}\left(H - \sum_{k=1}^{n}\frac{\partial H}{\partial p_k}p_k\right). \tag{2.3}$$

Заметим, что в случае, когда конфигурационное пространство трехмерно и $H = c\sqrt{m^2c^2+p^2}$, то $\omega' dt = \frac{mc^2}{\hbar}\frac{mc^2 dt}{H} = \frac{mc^2}{\hbar}d\tau$, где τ — собственное время в системе координат, связанной с частицей, движущейся с импульсом p. То есть в этом случае, вектор, точка приложения которого движется по классической траектории, вращается с постоянной угловой скоростью $\omega = mc^2/\hbar$ в системе координат, связанной с этой точкой.

Наоборот, если в правой части уравнения (2.1) оставить только последнее слагаемое вида (2.2), то получим уравнение

$$\frac{\partial\varphi}{\partial t} = a^2\sum_{k=1}^{n}\left(\frac{\partial}{\partial x_k} - \frac{ip_k}{\hbar}\right)^2\varphi + b^2\sum_{k=1}^{n}\frac{\partial^2}{\partial p_k^2}\varphi + \frac{abn}{\hbar}\varphi. \tag{2.4}$$

Это уравнение описывает диффузионную составляющую движения векторов $\varphi(x,p,t)$ на фазовом пространстве. В этом движении точки приложения векторов перемещаются в соответствии с классическим однородным диффузионным процессом с коэффициентами диффузий по координатам и импульсам равными a^2 и b^2, соответственно. При этом сам вектор при малых случайных перемещениях из точки (x,p) в точку $(x+dx, p+dp)$ переносится параллельно, а его длина в момент времени t умножается на $\exp(abnt/\hbar)$. Заметим, что параллельный перенос векторов на фазовом пространстве задается связностью, которая выражается формулой: $L_{(dx,dp)}\varphi(x,p) - \varphi(x,p) \approx -(i/\hbar)\varphi(x,p)pdq$, где $L_{(dx,dp)}\varphi(x,p)$ — параллельный перенос вектора $\varphi(x,p)$ из точки (x,p) по бесконечно малому вектору (dx,dp). В частном случае, когда конфигурационное пространство трехмерно, такая связность на фазовом пространстве вызвана синхронизацией движущихся часов в точках фазового пространства.

Правая часть уравненния (2.4) — самосопряженный оператор. Задача на собственные значения для этого оператора преобразованием Фурье по координатам сводится к стационарному уравнению Шредингера для гармонических колебаний. Отсюда показывается, что собственные значения оператора уравнения (2.4) неположительны, и верна следующая теорема.

Теорема 2.1. *Пусть $\varphi(x,p,0)$ — произвольная функция, преобразование Фурье которой по p стремится к нулю при $x\to\infty$. Тогда решение $\varphi(x,p,t)$ диффузионного уравнения (2.4) экспоненциально по времени (с показателем равным $-abt/\hbar$) стремится к стационарному решению вида:*

$$\varphi(x,p) = \lim_{t\to\infty}\varphi(x,p,t) = \frac{1}{(2\pi\hbar)^{n/2}}\int\limits_{R^n}\psi(y)\chi(x,y)e^{-i(y-x)p/\hbar}dy, \tag{2.5}$$

где

$$\psi(y) = \frac{1}{(2\pi\hbar)^{n/2}} \int\limits_{R^{2n}} \varphi(x,p,0) e^{i(y-x)p/\hbar} \chi(x,y) dp dx, \tag{2.6}$$

$$\chi(x,y) = \left(\frac{b}{a\pi\hbar}\right)^{n/4} e^{-b(x-y)^2/(2a\hbar)}. \tag{2.7}$$

Заметим, что $\chi^2(x,y)$ представляет собой плотность вероятностей нормального распределения по x с математическим ожиданием y, и дисперсией $a\hbar/(2b)$. Если величина $a\hbar/(2b)$ мала, то функция $\chi^2(x,y)$ близка к дельта-функции от $x-y$.

Композиция выражений (2.6) и (2.5) строит проектор из пространства всех волновых функций, заданных на фазовом пространстве, на некоторое подпространство. Элементы этого подпространства параметризуются функциями вида $\psi(y)$, где $y \in R^n$, т. е. волновыми функциями на конфигурационном пространстве.

Если же предполагать, что диффузия вызывается тепловыми воздействиями на электрон, то коэффициенты диффузий по координатам и импульсам выражаются в статистической физике (см., например [**4**], гл.7, §4 и §9) через температуру T по формулам: $a^2 = kT/(m\gamma)$ и $b^2 = \gamma kTm$, где k — постоянная Больцмана, m — масса электрона, γ — коэффициент трения среды на единицу массы. Отсюда, $a/b = (\gamma m)^{-1}$ и $ab = kT$. То есть, в этом случае, величина a/b, которая входит в выражение (2.7), не зависит от температуры. С другой стороны, t — время переходного процесса, определенное в теореме 1, имет вид: $t \sim \hbar/(ab) = \hbar/(kT) = T^{-1} \cdot 7.638 \cdot 10^{-12}$с.

С учетом этой оценки, будем считать в уравнении (2.1) величину $\hbar/(ab)$ малым параметром и предполагать, что координаты и импульсы мало меняются за это время при классическом движении, определенном гамильтонианом $H(x,p)$.

Теорема 2.2. *Движение, описываемое уравнением (2.1), асимптотически распадается при $\hbar/(ab) \to 0$ на быстрое движение и медленное движение. В результате быстрого движения произвольная волновая функция $\varphi(x,p,0)$ переходит за время порядка $\hbar/(ab)$ к виду (2.5). Волновые функции вида (2.5) образуют линейное подпространство. Элементы этого подпространства параметризуются волновыми функциями $\psi(y)$, зависящими только от координат $y \in R^n$. Медленное движение, начинающееся с ненулевой волновой функции из этого подпространства, происходит по подпространству и параметризуется волновой функцией $\psi(y,t)$, зависящей от*

времени. Функция $\psi(y,t)$ *удовлетворяет уравнению Шредингера вида* $i\hbar\partial\psi/\partial t = \hat{H}\psi$, *где*

$$\begin{aligned}\hat{H}\psi &= \frac{1}{(2\pi\hbar)^n}\int\limits_{R^{3n}}\left(H(x,p)-\sum_{k=1}^{n}\left(\frac{\partial H}{\partial x_k}+\frac{ib}{a}\frac{\partial H}{\partial p_k}\right)(x_k-y'_k)\right)\times\\ &\times\chi(x,y)\chi(x,y')e^{\frac{i}{\hbar}(y-y')p}\psi(y',t)dy'dxdp,\end{aligned}$$

и $\chi(x,y)$ *задается формулой (2.7).*

Теорема 2.3. *Если* $\frac{a\hbar}{b}$ *— малая величина и* $H(x,p)=\frac{p^2}{2m}+V(x)$, *то оператор* $\hat{H}$ *с точностью до членов порядка* $a\hbar/b$ *имеет вид:*

$$\hat{H}\approx -\frac{\hbar^2}{2m}\left(\sum_{k=1}^{n}\frac{\partial^2}{\partial y_k^2}\right)+V(y)-\frac{a\hbar}{4b}\sum_{k=1}^{n}\frac{\partial^2 V}{\partial y_k^2}+\frac{3nb\hbar}{4ma}. \tag{2.8}$$

Первые два слагаемые в формуле (2.8) дают стандартный оператор Гамильтона. Последнее слагаемое — константа, и ею можно пренебречь. Предпоследнее слагаемое рассмотрим (ввиду малости $a\hbar/b$) как возмущение к оператору Гамильтона.

Считая, что отклонения в спектре атома водорода (сдвиг Лэмба), наблюдаемые в экспериментах Лэмба-Резерфода [**2**], вызываются предпоследним слагаемым в формуле (2.8), можно оценить величину a/b. Расчеты стандартным методом возмущений, аналогичные расчетам, выполненным в [**5**], дают следующую оценку: $a/b = 3.41\cdot 10^4$с/г. Отсюда, стандартное отклонение для нормального распределения χ^2, по которому производится сглаживание волновых функций, имеет вид $\sqrt{a\hbar/(2b)} = 4.24\cdot 10^{-12}$см. Эта величина существенно меньше радиуса атома водорода и близка к комптоновской длине волны электрона $\hbar/(mc) = 3.86\cdot 10^{-11}$см.

Таким образом, расчеты показывают, что предложенная модель в виде дифференциального уравнения (2.1) достаточно адекватно описывает физические процессы в стандартных случаях для стандартного гамильтониана. Но эту модель можно применить и для расчетов процессов с нестандартным гамильтонианом или с гамильтонианом, быстро меняющимся во времени, как при внезапных возмущениях или для периодически меняющегося потенциала с частотой порядка $ab/\hbar$, и сравнить с экспериментальными данными.

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Идемпотентные системы нелинейных уравнений и задачи расчета электроэнергетических сетей[1]

А.М. Гельфанд и Б.Х. Кирштейн

Электроэнергетическую сеть можно рассматривать как граф с n вершинами, каждой вершине (узлу) которого сопоставлены два вещественных числа - активная P_k и реактивная Q_k составляющие инъекции мощности в узле k ($k = 1, ..., n$), и каждому ребру (линий электропередач), соединяющих k-ый и j-ый узлы - активная Y^a_{kj} и реактивная Y^r_{kj} составляющие проводимости.

Установившиеся режимы электроэнергетических сетей характеризуются значениями активных U^a_k и реактивных U^r_k составляющих напряжений в узлах, которые должны удовлетворять системе $2n$ алгебраических уравнений [**1**] - узловых уравнений балансов активной и реативных мощностей.

Такие уравнения можно рассматривать как вещественную и комплексную составляющую системы n комплексных уравнений вида

$$W_k = E_k(\sum_{j\in(k)} a_{kj}\overline{E}_j), \tag{1}$$

где $W_k = P_k + iQ_k$, $E_k = U^a_k + iU^r_k$, $\overline{E}_k = U^a_k - iU^r_k$, суммирование идет по всем узлам j связанных линией с узлом k, a_{kj} комплексные числа, которые определяется по комплексным проводимостям $Z_j = U^a_j + iU^r_j$.

[1]Работа выполнена при частичной финансовой поддержке гранта РФФИ 05-01-02807-НЦНИЛ_а.

Эти уравнения дают $2n$ вещественных алгебраических уравнений относительно $2n$ неизвестных $U_k^a, U_k^r, k = (1, ..., n)$.
Важной задачей анализа электроэнергетических сетей является анализ устойчивости (определение запаса устойчивости) установившегося режима. На практике, такой анализ сводится к проверке существовании вещественной деформации решения при деформациях коэффициентов уравнений, входящих в систему. Такие деформации отвечают изменениям инъекций мощности в узлах или измению структуры графа сети (отключение линий).
При этом в качестве критерия потери устойчивости при деформациях уравнений системы обычно принимается либо расходимость итерационного процесса метода Ньютона для нахождении решения, либо вырождение матрицы Якоби системы уравнений. Такой подход не всегда математически корректно отвечает поставленной залаче, но, главное, не позволяет быстро и наглядно получить границу области устойчивости установившегося режима.
Мы рассматриваем некоторую процедуру простого макетирования задачи анализа устойчивости электроэнергетических систем с помощью анализа решений идемпотентной системы уравнений, полученной в результате деквантования по Маслову [**2**] исходной системы (1).
Запишем систему уравнений установившегося режима более симметрично в виде системы $2n$ комплексных алгебраических уравнений [**3**], относительно $2n$ комплексных переменных. Для этого введем новые комплексные переменные S_k вместо $\overline{E}_k$ и новые уравнения, полученные с помощью комплексного сопряженния уравнений (1) и последующей аналогичной заменной переменных.
Получим систему из $2n$ комплексных уравнений

$$W_k = E_k(\sum_{j\in(k)} a_{kj}S_j), \quad \overline{W}_k = S_k(\sum_{j\in(k)} \overline{a}_{kj}E_j) \tag{2}$$

относительно $2n$ комплексных переменных $E_k, S_k, k = 1, ..., n$. Решение системы (2) удовлятворяет системе (1) тогда и только тогда когда

$$\overline{E}_k = S_k \tag{3}$$

для всех $k = 1, ..., n$.
В [**4**] для всякой алгебраической поверхности $f(z) = 0$ в C^{2n} определена ее амеба $\mathcal{A}_f$ в R^{2n}, образ пересечения этой поверхности с комплексным тором относительно отображения

$$\mathrm{Log} : (z_1, \ldots, z_{2n}) \in (\mathbf{C}\backslash\{0\})^{2n} \to (\log|z_1|, \ldots, \log|z_{2n}|) \in \mathrm{R}^{2n}.$$

Условие (3) в этих терминах означает, что пересечение амеб всех уравнений системы (2) содержит точки, удовлетворяющие условиям

$$\log|E_k| = \log|S_k| \tag{4}$$

для всех $k = 1, ..., n$.
Известно [**4**], что амеба $\mathcal{A}_f$ совпадает с дополненнием конечного числа открытых выпуклых подмножеств E_ν в R^{2n}:

$$\mathrm{R}^{2n} \backslash \mathcal{A}_f = \cup\{\mathrm{E}_\nu\}.$$

В [**5**] определены наборы линейных функций на этих выпуклых множествах, нижняя грань которых определяет кусочно-линейное подмножество R^{2n}, которое называется спайном амебы, лежит внутри $\mathcal{A}_f$ и является ее гомотопическим ретрактом.
Чтобы определить идемпотентную систему уравнений определим тропические аналоги операций сложения и умножения в R обычным образом: тропическое сложение как $x \oplus y = \max\{x, y\}$ и тропическое умножение как $x \otimes y = x + y$.
Рассмотрим идемпотентную систему уравнений, полученную из уравнений (2) заменой обычных операций на их тропические аналоги и комплексных коэффициентов на логарифмы их модулей:

$$\log|W_k| = \log|E_k| \otimes (\bigoplus_{j\in(k)} \log|a_{kj}| \otimes \log|S_j|), \tag{5}$$

и

$$\log|W_k| = \log|S_k| \otimes (\bigoplus_{j\in(k)} \log|a_{kj}| \otimes \log|E_j|), \tag{6}$$

Заметим, что многогранники Ньютона для уравнений (2) совпадают с выпуклой оболочкой подмножества вершин единичного куба в R^{2n} и не содержат поэтому внутри себя точек целочисленной решетки.
Как доказано в [**6**], в этом случае уравнение спайна амебы совпадают с уравнениями идемпотентной системы, результата деквантования f. Отсюда легко получается следующая

Теорема. *Решение идемпотентной системы уравнений* (5), (6) *совпадает с множеством точек пересечения пределов амеб уравнений* (2) *при ретракции их на свои спайны.*

Используя эту теорему можно следующим образом получить простую модель для анализа запаса устойчивости электроэнергетической

системы.
Пусть

$$E^* = (E_1, ..., E_n) \in (\mathbf{C}\backslash\{0\})^n$$

решение системы (1). Найдем решение идемпотентной системы уравнений (6), (7), ближайшее к вектору $\mathrm{Log}((E^*, \overline{E}^*))$ в R^{2n}. Будем говорить, что область параметров деформации является областью притяжения (отталкивания), если при соответствующей деформации решение идемпотентной системы приближается (удаляется) от подпространства, определяемого уравнениями (4) в R^{2n}.
Задача нахождения границы, разделяющих области притяжения и отталкивания в случае идемпотентой системы может рассматриваться, как естественный модельный аналог задачи нахождения запаса устойчивости. Такая задача, по сравнению с анализом зависимости от параметров решений вещественных многомерных систем алгебраических уравнений, решается существенно проще в идемпотентном анализе, где она, по существу, сводится к анализу систем линейных уравнений зависящих от параметров.

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Классические и неархимедовы амебы в вопросах расширения полей

О.В. Знаменская

Поле $\mathbb{P}$ называется алгебраическим расширением, или расширением Галуа поля $\mathbb{K}$, если существует алгебраическое уравнение

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0 \tag{1}$$

с коэффициентами в поле $\mathbb{K}$, такое, что поле $\mathbb{P}$ получается присоединением к $\mathbb{K}$ всех корней этого уравнения. Известно, что все такие расширения конечномерны.

Таким образом, в классической теории конечные расширения строятся при помощи присоединения к исходному полю нулей полиномов от одного переменного. Наша цель — изучение бесконечных аналогов этих расширений, определяемых полиномами от нескольких переменных, для случая неархимедовых полей. Более точно, наша задача состоит в описании бесконечных расширений подполей неархимедова поля $\mathbb{K}$ рядов Пюизо.

Напомним, что в поле $\mathbb{K}$ с неархимедовым нормированием норма элемента $a \in \mathbb{K}$ может быть определена через показатель нормирования $\mathrm{val}(a)$ поля $\mathbb{K}$ при помощи соотношения $|a| = e^{-\mathrm{val}(a)}$. Здесь $\mathrm{val}(a)$ есть отображение $\mathbb{K} \to \mathbb{R} \cup \{\infty\}$, определенное на элементах $\mathbb{K}$ и удовлетворяющее следующим условиям [**1**]:

a) $\mathrm{val}(a) = \infty$ тогда и только тогда, когда $a = 0$;
b) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$;
c) $\mathrm{val}(a + b) \geqslant \min(\mathrm{val}(a), \mathrm{val}(b))$.

Пусть $\mathbb{K}$ — поле рядов Пюизо с коэффициентами в произвольном поле k, т.е. рядов $a(t)$ вида

$$a(t) = \sum_{q_j \in A_a} \xi_j t^{q_j}$$

по дробным степеням q_j переменного t, где $A_a \subset \mathbb{Q}$ — вполне упорядоченное множество. Показатель неархимедова нормирования val в этом случае полагается равным $\min A_a$.

Бесконечные расширения поля $\mathbb{K}$ будем строить следующим образом. Рассмотрим полином

$$f = \sum_{\alpha \in A} a_\alpha(t) z^\alpha \tag{2}$$

из $\mathbb{K}[z_1, \ldots, z_n]$ с коэффициентами $a_\alpha(t) \in L \subset \mathbb{K}$, где L — подполе $\mathbb{K}$. Определим расширение $\mathbb{P}/L$ как множество всевозможных значений полиномов

$$\sum_{\beta \in B} b_\beta z(t)^\beta,$$

где $z(t) = \big(z_1(t), \ldots, z_n(t)\big)$ является решением уравнения $f = 0$ для полинома f вида (2).

Отметим, что самый простой случай расширений Галуа — циклические расширения, получаются присоединением к исходному полю всех корней из единицы, т.е. решений двучленного уравнения $x^m - a = 0$. Очевидно, все корни полинома Галуа $f(x) = x^m - a$:

- лежат на окружности;
- на окружности они равномерно распределены.

Многомерный аналог первой из указанных геометрических характеристик решений полиномов Галуа может быть сформулирован на языке амеб.

Амебой $\mathcal{A}_f$ комплексной гиперповерхности $V \in (\setminus\{0\})^n$, задаваемой полиномом f (см. [**2**]), называется ее образ при отображении

$$\mathrm{Log} \ : (\mathbb{C} \setminus \{0\})^n \longrightarrow \mathbb{R}^n,$$

действующем по правилу

$$(z_1, \ldots, z_n) \longrightarrow (\log|z_1|, \ldots, \log|z_n|).$$

Амебу, определяемую таким образом, назовем классической.

Определение 1 (см. [**3**]). Классическая амеба $\mathcal{A}_f$ называется солидной, если число связных компонент дополнения к ней минимально.

Многомерное обобщение того факта, что все корни $f(x) = x^m - a$ лежат на единичной окружности на языке амеб выражается в том, что *амеба $\mathcal{A}_f$ полинома f солидна.*

Заметим, что если $n = 1$, то амеба произвольного полинома от одного переменного есть конечное множество точек в $\mathbb{R}^1$. Солидными в этом случае будут только амебы, состоящие из одной точки и имеющие лишь две связные компоненты в дополнении, а это и есть в точности амебы полиномов Галуа.

Напомним, что многогранником Ньютона $\mathcal{N}_f$ полинома f от n переменных называется выпуклая оболочка показателей его мономов в $\mathbb{R}^n$. Конусом рецессии выпуклого множества $E \subset \mathbb{R}^n$ называется максимальный конус среди тех, которые сдвигом можно поместить в E.

Согласно результатам М. Форсберга, М. Пассаре и А.К. Циха [**4**], справедлива

Теорема 1. *Существует естественная инъективная функция порядка ν на множестве $\{E\}$ связных компонент дополнения $\mathbb{R}^n \setminus \mathcal{A}_f$ амебы гиперповерхности $f = 0$, сопоставляющая каждой компоненте E некоторую целочисленную точку $\nu(E)$ из многогранника Ньютона $\mathcal{N}_f$. Конус рецессии компоненты E совпадает с конусом, двойственным к $\mathcal{N}_f$ в точке $\nu(E)$.*

Таким образом, многогранник Ньютона отражает структуру классической амебы. В частности, число связных компонент дополнения $\mathbb{R}^n \setminus \mathcal{A}_a$ не меньше числа вершин и не больше числа всех целых точек многогранника Ньютона $\mathcal{N}_f$. Ясно, что классическая амеба солидна, если число компонент дополнения строго равно числу вершин $\mathcal{N}_f$.

С точки зрения многогранников Ньютона, многомерным аналогом полиномов Галуа являются, так называемые, *максимально разреженные полиномы* [**3**], т.е. полиномы вида:

$$f = \sum_{\alpha \in A} a_\alpha z^\alpha,$$

где с ненулевыми коэффициентами входят только мономы, соответствующие вершинам $\mathcal{N}_f$. М. Ниссе был заявлен результат, что классическая амеба любого максимально разреженного полинома солидна.

Далее нас будет интересовать вопрос солидности неархимедовых амеб нулевого множества максимально разреженных полиномов (2), при помощи которых строятся бесконечные расширения $\mathbb{P}/L$ неархимедова поля рядов Пюизо.

Определим по аналогии свойство солидности для неархимедовых амеб. Пусть $\mathbb{K}$ — произвольное неархимедово поле и $\mathrm{val}(a)$ — его показатель нормирования.

Определение 2 (см. [**5, 6**]). Амебой $\mathcal{A}(V)$ алгебраической гиперповерхности $V \subset (\mathbb{K}^*)^n$ называется замыкание образа V при отображении

$$\mathrm{Log} : (z_1, \ldots, z_n) \to (-\mathrm{val}(z_1), \ldots, -\mathrm{val}(z_n)).$$

Из теоремы 1 следует, что конусы рецессии всех компонент дополнения классической солидной амебы полномерны. Соответственно, для неархимедова случая можно дать следующее

Определение 3. Неархимедова амеба $\mathcal{A}(V)$ называется солидной, если любая связная компонента дополнения к ней имеет полномерный конус рецессии.

Пусть $\mathbb{K}$ — поле рядов Пюизо.

Теорема 2. *Неархимедова амеба максимально разреженного полинома, определяющего многомерное расширение Галуа поля* $\mathbb{K}$*, солидна.*

Согласно [**6**] существует двойственность между неархимедовой амебой $\mathcal{A}(V)$ и подразбиением многогранника Ньютона полинома, определяющего V. С учетом этого справедлива

Теорема 3. *В случае* $n = 2$ *если неархимедова амеба солидна, то, как граф, она не имеет циклов.*

Пусть $f(z) = \sum\limits_{\alpha \in A} c_\alpha z^\alpha$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ — полином, все коэффициенты которого имеют рациональные модули, т.е. все $|c_\alpha| \in \mathbb{Q}$. Определим полином F следующим образом:

$$F = \sum_{\alpha \in A} c_\alpha(t) \xi^\alpha,$$

где $c_\alpha(t)$ таковы, что $|c_\alpha(t)| = e^{-|c_\alpha|}$. В указанных предположениях справедлива

Теорема 4. *Если классическая амеба* $\mathcal{A}_f$ *солидна, то и неархимедова амеба* $\mathcal{A}(V)$ *солидна.*

В доказательстве теоремы используются понятия хребта амебы и тропического многообразия, определяемого при помощи тропикализации полинома F.

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Обобщение ультравторичного квантования для фермионов при ненулевой температуре[1]

Г.В. Коваль и В.П. Маслов

В работах В.П. Маслова развит метод ультравторичного квантования и концепция истинного символа [**1, 2, 3**]. Этот метод позволяет находить асимптотические серии систем большого числа частиц при нулевой температуре. В частности, некоторые серии определяются периодическими решениями системы уравнений Гамильтона, соответствующей истинному символу [**2, 3**] рассматриваемой физической системы. В данной работе найдено соответствие между уравнениями метода ультравторичного квантования по парам для фермионов и уравнениями вариационного метода Боголюбова. Так как вариационный метод Боголюбова применим для случая ненулевой температуры, из принципа соответствия получено обобщение уравнений метода ультравторичного квантования фермионов на температурный случай.

В статье [**3**] показано, что асимптотика серий собственных значений системы N тождественных фермионов в пределе при $N \to \infty$ определяется решениями следующей системы уравнений

(1)

$$\Omega\Phi(x,y) = \left(-\frac{\hbar^2}{2m}\left(\Delta_x + \Delta_y\right) + U(x) + U(y) + V(x,y)\right)\Phi(x,y) + \\ + 2\iint dzdw\ \left(V(x,y) + V(z,w)\right)\Phi^+(z,w)\Phi(x,z)\Phi(w,y),$$

$$\Omega\Phi^+(x,y) = \left(-\frac{\hbar^2}{2m}\left(\Delta_x + \Delta_y\right) + U(x) + U(y) + V(x,y)\right)\Phi^+(x,y) + \\ + 2\iint dzdw\ \left(V(x,z) + V(y,w)\right)\Phi^+(x,z)\Phi^+(w,y)\Phi(z,w),$$

где $x, y \in \mathcal{M}$ — координаты частиц, пространство $\mathcal{M}$ определяется задачей, например, это может быть $\mathbf{R}^3$, или трехмерный тор, Δ_x, Δ_y — операторы Лапласа, действующий по соответствующей переменной x или y, $U(x)$ — потенциал внешнего поля, $V(x,y)$ — потенциал взаимодействия, симметричный относительно перестановки переменных x и y, m — масса частиц, $\hbar$ — постоянная Планка, Ω — действительное

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число. Функции $\Phi(x,y), \Phi^+(x,y) \in L_2(\mathcal{M})$ — антисимметричны относительно перестановок переменных x и y и удовлетворяют условию

$$\iint dxdy\ \Phi^+(x,y)\Phi(x,y) = \frac{N}{2}. \tag{2}$$

Уравнения (1) получаются при ультравторичном квантовании по парам. В таком квантовании рассматриваемой системе фермионов отвечает истинный символ вида

(3)

$$A[\Phi^+,\Phi] = \iint dxdy\Phi^+(x,y)\left(-\frac{\hbar^2}{2m}(\Delta_x+\Delta_y)+U(x)+U(y)+V(x,y)\right)\Phi(x,y)+$$
$$+2\iint dxdydzdwV(x,y)\Phi^+(x,y)\Phi^+(z,w)\Phi(x,z)\Phi(w,y),$$

который является функционалом от двух антисимметричных функций $\Phi^+(x,y)$ и $\Phi(x,y)$ из $L_2(\mathcal{M}^2)$. Этому символу соответствует система уравнений Гамильтона

$$i\frac{\partial\Phi}{\partial t}(x,y,t) = \frac{\delta A}{\delta\Phi^+(x,y,t)}, \qquad -i\frac{\partial\Phi^+}{\partial t}(x,y,t) = \frac{\delta A}{\delta\Phi(x,y,t)}, \tag{4}$$

где в правой части уравнений стоит вариационная производная функционала (3). Система уравнений (4) имеет интеграл движения, вид которого совпадает с выражением в левой части равенства (2). Уравнения (1) получаются из (4) в частном случае, когда

$$\Phi(x,y,t) = \Phi(x,y)e^{-i\Omega t}, \qquad \Phi^+(x,y,t) = \Phi^+(x,y)e^{i\Omega t}. \tag{5}$$

Запишем (1) в другом виде. Введем функции $G(x,y)$, $R(x,y)$ и $\widetilde{R}(x,y)$

$$\begin{aligned} &\widetilde{R}(x,y) = \Phi^+(x,y), \qquad G(x,y) = 2\int dz\ \Phi^+(x,z)\Phi(y,z), \\ &R(x,y) = 2\left(\Phi(x,y) - \int dz\ \Phi(x,z)G(z,y)\right). \end{aligned} \tag{6}$$

В силу антисимметрии функций $\Phi^+(x,y)$, $\Phi(x,y)$ для функций (6) выполняются равенства

$$R(x,y) = -R(y,x), \qquad \widetilde{R}(x,y) = -\widetilde{R}(y,x), \tag{7}$$

$$G(x,y) = \int dz\ G(x,z)G(z,y) + \int dz\ \widetilde{R}(z,x)R(z,y). \tag{8}$$

Из (2) следует

$$\int dx\ G(x,x) = N. \tag{9}$$

Предложение 1. *Функции $G(x,y)$, $\widetilde{R}(x,y)$, $R(x,y)$ удовлетворяют системе уравнений*

(10)

$$\left(-\frac{\hbar^2}{2m}\left(\Delta_x - \Delta_y\right) + U(x) - U(y)\right) G(x,y) -$$
$$- \int dz\ \left(V(x,z) - V(y,z)\right) \widetilde{R}(x,z) R(z,y) = 0,$$

$$\left(-\frac{\hbar^2}{2m}\left(\Delta_x + \Delta_y\right) + U(x) + U(y) + V(x,y)\right) R(x,y) -$$
$$- \int dz\ \left(V(x,z) G(z,y) R(x,z) + V(y,z) G(z,x) R(z,y)\right) = \Omega R(x,y),$$

$$\left(-\frac{\hbar^2}{2m}\left(\Delta_x + \Delta_y\right) + U(x) + U(y) + V(x,y)\right) \widetilde{R}(x,y) -$$
$$- \int dz\ \left(V(x,z) G(y,z) \widetilde{R}(x,z) + V(y,z) G(x,z) \widetilde{R}(z,y)\right) = \Omega \widetilde{R}(x,y).$$

Действительно, непосредственной проверкой удостоверяется, что функции (6) удовлетворяют уравнениям (10), если функции $\Phi^+(x,y)$, $\Phi(x,y)$ удовлетворяют (1).

Чтобы обобщить уравнения (1) на случай ненулевой температуры, применим принцип соответствия между этими уравнениями и уравнениями вариационного принципа Боголюбова при ненулевой температуре. То есть исходя из температурных уравнений вариационного принципа Боголюбова, по принципу соответствия найдем обобщение уравнений (1) на температурный случай.

Рассмотрим уравнения вариационного метода Боголюбова. При температуре $\theta \geq 0$ для рассматриваемой системы фермионов из вариационного метода Боголюбова [**4**] получаются уравнения

(11)

$$\lambda_\alpha u_\alpha(x) = \left(-\frac{\hbar^2}{2m}\Delta + U(x) - \mu\right) u_\alpha(x) + \int dy\ V(x,y) R_B(x,y) v_\alpha^*(y) + \\ + \int dy\ V(x,y)\left(G_B(y,y)u_\alpha(x) - G_B(y,x)u_\alpha(y)\right),$$

$$-\lambda_\alpha v_\alpha(x) = \left(-\frac{\hbar^2}{2m}\Delta + U(x) - \mu\right) v_\alpha(x) + \int dy\ V(x,y) R_B(x,y) u_\alpha^*(y) + \\ + \int dy\ V(x,y)\left(G_B(y,y)v_\alpha(x) - G_B(y,x)v_\alpha(y)\right),$$

где $\alpha = 1, 2, \ldots$, функции $u_\alpha(x)$ и $u_\alpha^*(x)$, а также $v_\alpha(x)$ и $v_\alpha^*(x)$, комплексно сопряжены друг другу и удовлетворяют условиям

(12)

$$\int dx\ \left(u_\alpha^*(x)v_\beta(x) + v_\alpha(x)u_\beta^*(x)\right) = \int dx\ \left(u_\alpha(x)v_\beta^*(x) + v_\alpha^*(x)u_\beta(x)\right) = 0,$$

$$\int dx\ \left(u_\alpha^*(x)u_\beta(x) + v_\alpha(x)v_\beta^*(x)\right) = \delta_{\alpha\beta},\ \forall \alpha, \beta = 1, 2, \ldots,$$

где $\delta_{\alpha\beta}$ — символ Кронекера. Кроме того в уравнениях (12) функции $R_B(x,y)$ и $G_B(x,y)$ имеют вид

(13)

$$R_B(x,y) = \sum_{\alpha=1}^{\infty}\left(\frac{1}{2} - n_\alpha\right)\left(v_\alpha(x)u_\alpha(y) - v_\alpha(y)u_\alpha(x)\right),$$

$$G_B(x,y) = \sum_{\alpha=1}^{\infty}\left(v_\alpha^*(x)v_\alpha(y)\left(1 - n_\alpha\right) + u_\alpha^*(x)u_\alpha(y)n_\alpha\right),$$

где

(14)
$$n_\alpha = \frac{1}{\exp(\lambda_\alpha/\theta) + 1},$$

а μ определяется из условия, что функция $G_B(x,y)$ (13) удовлетворяет равенству

(15)
$$\int dx\ G_B(x,x) = N.$$

Функции (13) и комплексно сопряженная к $R_B(x,y)$ функция $R_B^*(x,y)$ при любой температуре θ удовлетворяют равенствам

(16) $$R_B(x,y) = -R_B(y,x), \qquad R_B^*(x,y) = -R_B^*(y,x),$$

(17) $$G_B(x,y) = G_B^*(y,x),$$

а из уравнений (11) следует, что эти функции также удовлетворяют системе уравнений

(18)
$$\left(-\frac{\hbar^2}{2m}\left(\Delta_x - \Delta_y\right) + U(x) - U(y)\right) G_B(x,y) +$$
$$+ \int dz \left(V(x,z) - V(y,z)\right) R_B^*(x,z) R_B(z,y) +$$
$$+ \int dz \left(V(x,z) - V(y,z)\right) \left(G_B(z,z) G_B(x,y) - G_B(x,z) G_B(z,y)\right) = 0,$$

$$\left(-\frac{\hbar^2}{2m}\left(\Delta_x + \Delta_y\right) + U(x) + U(y) + V(x,y)\right) R_B(x,y) -$$
$$- \int dz \left(V(x,z) G_B(z,y) R_B(x,z) + V(y,z) G_B(z,x) R_B(z,y)\right) +$$
$$+ \int dz V(x,z) \left(G_B(z,z) R_B(x,y) - G_B(z,x) R_B(z,y)\right) +$$
$$+ \int dz V(y,z) \left(G_B(z,z) R_B(x,y) - G_B(z,y) R_B(x,z)\right) = 2\mu R_B(x,y),$$

где дополнительное уравнение получается комплексным сопряжением второго уравнения формулы (18).

Если $\theta = 0$, то из (14) следует, что n_α принимает значение 0 или 1 для всех $\alpha = 1, 2, \ldots$. Тогда из (12) следует, что функции (13) при нулевой температуре удовлетворяют условию

(19) $$G_B(x,y) = \int dz\; G_B(x,z) G_B(z,y) + \int dz\; R_B^*(z,x) R_B(z,y).$$

Равенства (19), (16) и (15) совпадают соответственно с равенствами (8), (7) и (9), если по принципу соответствия заменить $R_B^*(x,y)$ на $\widetilde{R}(x,y)$, $R_B(x,y)$ на $R(x,y)$, $G_B(x,y)$ на $G(x,y)$. Уравнения (18) при такой замене не переходят в уравнения (10), однако, если 2μ в (18) заменить на Ω, то очевидно соответствие между одной системой и другой, соответствующие друг другу уравнения отличаются несколькими слагаемыми в левой части. Кроме того, для частного вида взаимодействия, например, такого как в модели БКШ [**5**], уравнения (18) после замены совпадают с (10).

Отметим, что функции $G_B(x,y)$, $R_B(x,y)$, $R^*_B(x,y)$ удовлетворяют большему числу условий, чем функции $G(x,y)$, $R(x,y)$, $\widetilde{R}(x,y)$. Функции $\Phi(x,y)$, $\Phi^+(x,y)$, удовлетворяющие уравнениям (1), в общем случае не являются комплексно сопряженными друг другу [**2**]. Поэтому из формул (6) следует, что функция $\widetilde{R}(x,y)$ не должна быть комплексно сопряженной к $R(x,y)$, а функция $G(x,y)$ не должна удовлетворять условию (17).

Из соответствия между функциями (6) и (13), получим, что температурным аналогом системы уравнений (1) является следующая система уравнений:

(20)
$$\begin{aligned}
\lambda_\alpha u_\alpha(x) &= \left(-\frac{\hbar^2}{2m}\Delta + U(x) - \mu\right) u_\alpha(x) + \int dy V(x,y) R(x,y) \widetilde{v}_\alpha(y),\\
-\lambda_\alpha v_\alpha(x) &= \left(-\frac{\hbar^2}{2m}\Delta + U(x) - \mu\right) v_\alpha(x) + \int dy V(x,y) R(x,y) \widetilde{u}_\alpha(y),\\
\lambda_\alpha \widetilde{u}_\alpha(x) &= \left(-\frac{\hbar^2}{2m}\Delta + U(x) - \mu\right) \widetilde{u}_\alpha(x) + \int dy V(x,y) \widetilde{R}(x,y) v_\alpha(y),\\
-\lambda_\alpha \widetilde{v}_\alpha(x) &= \left(-\frac{\hbar^2}{2m}\Delta + U(x) - \mu\right) \widetilde{v}_\alpha(x) + \int dy V(x,y) \widetilde{R}(x,y) u_\alpha(y),
\end{aligned}$$

где $G(x,y)$, $R(x,y)$ и $\widetilde{R}(x,y)$ выражаются следующим образом:

(21)
$$\begin{aligned}
G(x,y) &= \sum_{\alpha=1}^{\infty} \left(\widetilde{v}_\alpha(x) v_\alpha(y) \left(1 - n_\alpha\right) + \widetilde{u}_\alpha(x) u_\alpha(y) n_\alpha\right),\\
R(x,y) &= \sum_{\alpha=1}^{\infty} \left(\frac{1}{2} - n_\alpha\right) \left(v_\alpha(x) u_\alpha(y) - v_\alpha(y) u_\alpha(x)\right),\\
\widetilde{R}(x,y) &= \sum_{\alpha=1}^{\infty} \left(\frac{1}{2} - n_\alpha\right) \left(\widetilde{v}_\alpha(x) \widetilde{u}_\alpha(y) - \widetilde{v}_\alpha(y) \widetilde{u}_\alpha(x)\right),
\end{aligned}$$

n_α выражается через λ_α и θ формулой (14), а функции $u_\alpha(x)$, $v_\alpha(x)$, $\widetilde{u}_\alpha(x)$, $\widetilde{v}_\alpha(x)$, $\alpha = 1, 2, \ldots$ кроме уравнений (20) еще удовлетворяют условиям

(22)
$$\begin{aligned}
&\int dx \left(\widetilde{u}_\alpha(x) v_\beta(x) + v_\alpha(x) \widetilde{u}_\beta(x)\right) = \int dx \left(u_\alpha(x) \widetilde{v}_\beta(x) + \widetilde{v}_\alpha(x) u_\beta(x)\right) = 0,\\
&\int dx \left(\widetilde{u}_\alpha(x) u_\beta(x) + v_\alpha(x) \widetilde{v}_\beta(x)\right) = \delta_{\alpha\beta}, \quad \forall \alpha, \beta = 1, 2, \ldots.
\end{aligned}$$

Параметр μ в уравнениях (20) определяется из условия, что функция $G(x,y)$ из (21) удовлетворяет условию (9).

Предложение 2. *Если функции $u_\alpha(x)$, $\widetilde{u}_\alpha(x)$, $v_\alpha(x)$, $\widetilde{v}_\alpha(x)$, $\alpha = 1,2,\ldots$ удовлетворяют системе уравнений (20) и условиям (22), то функции (21) при любом $\theta \geq 0$ удовлетворяют системе уравнений (10) с $\Omega = 2\mu$ и условиям (7,9), а при $\theta = 0$ еще удовлетворяют условию (8).*

В силу уравнений (20) и условий (22) это утверждение доказывается прямой подстановкой функций (21) в формулы (7-10).

Множество решений уравнений (18) шире, чем множество функций (13), выраженных через решения системы уравнений (11). В [**2**] показано, что система уравнений (18) может быть записана в виде пары:

$$\left[\widehat{A}, \widehat{L}\right] = 0, \tag{23}$$

а множеству решений температурных уравнений (11) соответствует такое решение уравнения (23), для которого

$$\widehat{A} = f_\theta(\widehat{L}), \tag{24}$$

где

$$f_\theta(\xi) = \frac{1}{\exp(\xi/\theta)+1} - \frac{1}{2}. \tag{25}$$

Для уравнений (20) и (20) справедливо аналогичное утверждение. Рассмотрим матрицы $\widehat{A}$ и $\widehat{L}$ вида

$$\widehat{A} = \begin{pmatrix} \widehat{G} - \frac{1}{2} & -\widehat{\widetilde{R}} \\ \widehat{R} & \frac{1}{2} - \widehat{G}^t \end{pmatrix}, \qquad \widehat{L} = \begin{pmatrix} \widehat{T} & -\widehat{\widetilde{B}} \\ \widehat{B} & -\widehat{T} \end{pmatrix}, \tag{26}$$

где $\widehat{G}$, $\widehat{R}$, $\widehat{\widetilde{R}}$ — операторы в пространстве $L_2(\mathcal{M})$, задаваемые интегральными ядрами $G(x,y)$, $R(x,y)$, $\widetilde{R}(x,y)$ соответственно, $\widehat{G}^t$ — оператор, задаваемый в $L_2(\mathcal{M})$ ядром $G^t(x,y) = G(y,x)$, $\widehat{B}$ и $\widehat{\widetilde{B}}$ — ядрами $B(x,y) = V(x,y)R(x,y)$ и $\widetilde{B}(x,y) = V(x,y)\widetilde{R}(x,y)$ соответственно, а оператор $\widehat{T}$ — оператор Гамильтона для одной частицы, то есть оператор вида

$$\widehat{T} = -\frac{\hbar^2}{2m}\Delta + U(x).$$

Подстановка (26) в (23) приводит к четырем уравнениям, из которых два совпадают, а три независимых приводятся к виду (10). Поэтому справедливо следующее утверждение.

Предложение 3. *Система уравнений (4) может быть записана в виде (23), где* $\widehat{A}$ *и* $\widehat{L}$ *имеют вид (26), а* $2\mu = \Omega$.

Кроме того, решениям системы уравнений (10), вида (21), которые получены из решений уравнений (20,22), соответствуют такие $\widehat{A}$ и $\widehat{L}$, что для них справедливо равенство (24). Это является следствием уравнений (20,22).

В заключение отметим, что уравнения (10) в температурном случае, полученные здесь из принципа соответствия, могут быть строго получены из истинного символа [**2, 3**] для ультравторично квантованного уравнения, отвечающего матрице плотности [**1**].

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Контактная классификация уравнений Монжа-Ампера

А.Г. Кушнер

1. Геометрические структуры, ассоциированные с уравнениями Монжа-Ампера

Класс уравнений Монжа-Ампера выделяется из многообразия уравнений второго порядка тем, что он замкнут относительно контактных преобразований. Это обстоятельство было известно еще Софусу Ли, изучавшему уравнения Монжа-Ампера методами созданной им контактной геометрии. В 1870-х и 1880-х он поставил проблемы классификации уравнений Монжа-Ампера относительно (псевдо)группы контактных преобразований, в частности, о приведении уравнений Монжа-Ампера к квазилинейной форме и наиболее простом координатном представлении таких уравнений [**5**].

В 1979 г. в работе [**9**] В. В. Лычагин показал,что уравнения Монжа-Ампера допускают эффективное описание в терминах дифференциальных форм на многообразии 1-джетов гладких функций. Отправной точкой является следующее наблюдение.

Пусть M — n-мерное гладкое многообразие, J^1M — многообразие 1-джетов гладких функций на M. На J^1M естественным образом определена контактная структура — распределение Картана C, в канонических локальных координатах Дарбу $(q,u,p) = (q_1,\dots,q_n,u,p_1,\dots,p_n)$ задаваемое дифференциальной 1-формой Картана $U = du - pdq$. Ограничение дифференциала формы Картана на подпространство Картана не вырождено на нем и определяет симплектическую структуру $\Omega_a = dU|_{C(a)} \in \Lambda^2\,(C^*(a))$.

Со всякой дифференциальной n-формой $\omega \in \Omega^n(J^1M)$ свяжем нелинейный дифференциальный оператор $\Delta_\omega : C^\infty(M) \to \Omega^n(M)$, действующий на гладкую функцию v следующим образом:

$$\Delta_\omega(v) = j_1(v)^*(\omega). \tag{1.1}$$

Здесь $j_1(v) : M \to J^1M$ — 1-джет функции $v \in C^\infty(M)$.

Операторы Δ_ω называются операторами *Монжа-Ампера*, а уравнение $E_\omega = \{\Delta_\omega(v) = 0\} \subset J^2M$ — *уравнением Монжа-Ампера*. Следующее обстоятельство оправдывает эти названия: будучи записанным в локальных канонических координатах на J^1M, оператор Δ_ω имеет тот же самый тип нелинейности по производным второго порядка, что и классические опреаторы Монжа-Ампера, а именно, нелинейности типа определителя матрицы Гессе и ее миноров. При $n = 2$ мы получаем классическое уравнение Монжа-Ампера:

$$Av_{xx} + 2Bv_{xy} + Cv_{yy} + D(v_{xx}v_{yy} - v_{xy}^2) + E = 0, \tag{1.2}$$

где A, B, C, D, E — функции от независимых переменных x, y, функции $v = v(x,y)$ и ее первых производных v_x, v_y.

Преимуществом такого подхода перед классическим является редукция порядка пространства джетов: мы используем более простое пространство 1-джетов J^1M вместо пространства 2-джетов J^2M, в котором, будучи уравнениями второго порядка, *ad hoc* должны лежать уравнения Монжа-Ампера.

В случае, когда коэффициенты уравнения (1.2) не зависят явно от функции v ситуация еще более упрощается: в определении оператора (1.1) вместо пространства 1-джетов можно рассматривать кокасательное расслоение T^*M многообразия M, а вместо контактной геометрии — симплектическую. Такие уравнения Монжа-Ампера будем называть *симплектическими*.

Заметим, что соответствие между дифференциальными n-формами на J^1M и операторами Монжа-Ампера не является взаимно-однозначным. Дифференциальные формы, аннулирующимся на любом интегральном многообразии распределении Картана, образуют идеал $\mathcal{C}$ во внешней алгебре $\Omega^*(J^1M)$, который называется *идеалом Картана*. Им отвечает нулевой дифференциальный оператор. Элементы фактор-алгебры $\Omega^*(J^1M)/\mathcal{C}$ по этому идеалу называются *эффективными* формами.

Далее мы будем рассматривать случай когда M — двумерное гладкое многообразие. В терминах эффективных форм можно определить тип уравнения — эллиптический, параболический, гиперболический или переменный. Функция $\mathrm{Pf}\,(\omega) \in C^\infty\left(J^1M\right)$, определяемая поточечно равенством $\mathrm{Pf}(\omega_a)\Omega_a \wedge \Omega_a = \omega_a \wedge \omega_a$, называется *пфаффианом* формы ω. Уравнение E_ω называется *гиперболическим, параболическим* или *эллиптическим* в точке $a \in J^1M$, если пфаффиан $\mathrm{Pf}(\omega)$ отрицательный, нулевой или положительный в этой точке. Если пфаффиан не равен нулю в точке, то уравнение называется *невырожденным*. Очевидным образом понятие типа распространяется на область.

В силу невырожденности симплектической структуры на подпространстве Картана $C(a)$, формула $X_a \rfloor \omega_a = A_{\omega_a} X_a \rfloor\ \Omega_a$, определяет на $C(a)$ ассоциированный с эффективной дифференциальной 2-формой ω линейный оператор A_{ω_a}. Здесь $X_a \in C(a)$. Этот оператор симметричен относительно симплектической структуры, а его квадрат скалярен и $A^2_\omega + \mathrm{Pf}(\omega) = 0$. Заметим, что операторы A_{ω_a} не образуют поля эндоморфизмов на J^1M, ибо они определены только на подпространствах Картана.

Если в точке $a \in J^1M$ пфаффиан формы ω не обращается в нуль, то в некоторой ее окрестости этой точки форму ω можно нормировать так чтобы $\mathrm{Pf}(\omega) = \pm 1$. В этом случае на подпространсве Картана определена либо структура почти произведения (для гиперболических уравнений), либо комплексная структура (для эллиптических уравнений). В первом случае мы получаем два вещественных, а во втором — два комплексных 2-мерных распределения на J^1M, которые будем называть *характеристическими* и обозначить через C_+ и C_-. Эти распределения косоортогональны друг другу и на каждой из плоскостей $C_\pm(a)$ 2-форма Ω_a не вырождена.

Характеристические распределения порождают еще одно распределение — вещественное одномерное распределение

$$l = [C_+, C_+] \bigcap [C_-, C_-],$$

трансверсальное распределению Картана [**6**].

2. Невырожденные уравнения и инварианты Лапласа

Пусть ω — невырожденная нормированная эффективная дифференциальная 2-форма на J^1M. В каждой точке $a \in J^1M$ комплексификация касательного пространства $T_a(J^1M)$ распадается в прямую сумму

$$T_a(J^1M)^{\mathbb{C}} = C_+(a) \oplus l(a) \oplus C_-(a). \tag{2.1}$$

Обозначим распределения C_+, l, C_- через P_1, P_2 и P_3 соответственно. Формула (2.1) порождает разложение в прямую сумму комплекса де Рама многообразия J^1M, что позволяет найти дифференциальные инварианты уравнения [**3**].

Определим тензорные поля $q^s_{j,k} : D(J^1M)^{\mathbb{C}} \times D(J^1M)^{\mathbb{C}} \to D(J^1M)^{\mathbb{C}}$ на J^1M:

$$q^s_{j,k}(X,Y) = -\mathbf{P}_s\left[\mathbf{P}_j X, \mathbf{P}_k Y\right]. \tag{2.2}$$

Здесь $\mathbf{P}_j$ — проектор на распределение P_j, $D(J^1M)$ — модуль векторных полей на J^1M.

Мы получаем всего 4 нетривиальных тензорных поля: $q^3_{1,2}$, $q^1_{2,3}$, $q^2_{1,1}$, $q^2_{3,3}$. Остальные тензоры (2.2) равны нулю. Определим две дифференциальные 2-формы как свертки тензоров:

$$\xi_+ = \left\langle q^2_{1,1}, q^1_{3,2} \right\rangle, \quad \xi_- = \left\langle q^2_{3,3}, q^3_{1,2} \right\rangle.$$

Эти формы мы будем называть *формами Лапласа*, поскольку они являются обобщением инвариантов Лапласа для случая линейных уравнений [**2**]. Заметим, что классические инварианты Лапласа определены только для гиперболических уравнений.

Формы Лапласа играют важную роль при решении вопроса о контактной линеаризации уравнений Монжа-Ампера. Так, например, если обе формы Лапласа нулевые, то уравнение Монжа-Ампера локально контактно эквивалентно либо волновому уравнению $v_{xx} - v_{yy} = 0$, либо уравнению Лапласа $v_{xx} + v_{yy} = 0$ (см. также [**10**]).

В терминах форм Лапласа формулируется решение проблемы эквивалентности уравнений Монжа-Ампера линейным уравнениям вида

$$av_{xx} + 2bv_{xy} + cv_{yy} + rv_x + sv_y + kv + w = 0,$$

где a, b, c, r, s, k, w — функции только от независимых переменных x, y [**4**]. В частности, для таких уравнений формы Лапласа замкнуты. Например, для уравнения Хантора-Сакстона

$$v_{tx} = vv_{xx} + \kappa u_x^2,$$

возникающего в теории жидких кристаллов, формы Лапласа имеют вид:

$$\xi_1 = -dq_2 \wedge dp_1, \quad \xi_2 = 2\,(1-\kappa)\,dq_2 \wedge dp_1.$$

Это уравнение контактно эквивалентно линейному уравнению Эйлера-Пуассона [**8**]

$$v_{tx} = \frac{1}{\kappa\,(t+x)} v_t + \frac{2\,(1-\kappa)}{\kappa\,(t+x)} v_x - \frac{2\,(1-\kappa)}{(\kappa\,(t+x))^2} u.$$

Если выполняется условие $\xi_- \wedge \xi_- = \xi_+ \wedge \xi_+ = 0$, то формы Лапласа разложимы: $\xi_\pm = \eta_\pm \wedge \vartheta_\mp$ для некоторых дифференциальных 1-форм $\eta_\pm, \vartheta_\pm \in \Omega^1(C_\pm)$. Рассмотрим следующие 1-мерные подраспределения распределения Картана: $X_{\eta_\pm} = C_\pm \cap \ker \eta_\pm$ и $X_{\vartheta_\pm} = C_\pm \cap \ker \vartheta_\pm$.

Для уравнений общего положения эти распределения различны. Это позволяет построить e-структуру для таких уравнений и найти полную систему их скалярных дифференциальных инвариантов.

Заметим, что для уравнений Монжа-Ампера, контактно эквивалентных уравнению, линейному относительно первых производных (т.е. уравнению вида

$$av_{xx} + 2bv_{xy} + cv_{yy} + rv_x + sv_y + w = 0,$$

где a, b, c, r, s, w — функции от x, y, v), посторенные 1-мерные распределения попарно совпадают: $X_{\eta_+} = X_{\vartheta_+}$ и $X_{\eta_-} = X_{\vartheta_-}$.

Подробное изложение геометрии уравнений Монжа-Ампера (и не только двумерных!) можно найти в [**4**].

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Об амебе дискриминанта алгебраического уравнения

Е.Н. Михалкин

Рассмотрим общее алгебраическое уравнение n-ой степени

$$z^n + x_{n-1}z^{n-1} + \ldots + x_1 z - 1 = 0. \tag{1}$$

Нас будет интересовать дискриминантное множество $\nabla = \{\Delta = 0\}$ уравнения (1) (здесь Δ – дискриминант этого уравнения). Запишем ∇, используя параметризацию Пассаре-Циха [**1**]:

$$x_k(s) = \frac{ns_k}{\langle \alpha, s\rangle}\left(-\frac{\langle \alpha, s\rangle}{\langle \beta, s\rangle}\right)^{\frac{k}{n}}, \quad k = 1, \ldots, n-1; \quad s \in \mathbb{CP}_{n-2}, \tag{2}$$

где

$$\alpha = (n-1, \ldots, 2, 1), \quad \beta = (1, 2, \ldots, n-1) \tag{3}$$

– целочисленные векторы, $\langle\, , \rangle$ – знак скалярного произведения.

В случае, когда уравнение (1) содержит лишь один параметр x_i (в этом случае рассматриваемое уравнение называется триномиальным), дискриминантное множество ∇ представляет собой некоторое подмножество точек комплексной плоскости $\mathbb{C}$ (см. [**1**]). Но если уравнение (1) содержит два параметра и более, то дискриминантное множество удобно исследовать, используя амебу дискриминанта (определение амебы было дано Гельфандом-Капрановым-Зелевинским [**2**]). Отметим, что когда интересующее нас уравнение содержит лишь два параметра x_i, то амеба дискриминанта $\mathcal{A}_\nabla$ лежит в $\mathbb{R}^2$.

В статье [**3**] из интегральной формулы Меллина [**4**] было получено интегральное представление для одного из решений уравнения (1) с интегрированием по компакту. Была найдена и область сходимости полученного интеграла. А именно, доказана следующая

Теорема 1. *Ветвь алгебраической функции $z_0(x)$ решения уравнения (1) с условием $z(0)=1$, допускает представление в виде интеграла*

$$z_0(x) = 1+\frac{1}{2\pi i n}\int\limits_0^1 t^{\frac{1-n}{n}}(1-t)^{-\frac{1+n}{n}}\left[e^{\frac{\pi i}{n}}\ln\left(1-\sum_{k=1}^{n-1}e^{\frac{k}{n}\pi i}x_k t^{\frac{k}{n}}(1-t)^{\frac{n-k}{n}}\right)-\right.$$
$$\left.-e^{-\frac{\pi i}{n}}\ln\left(1-\sum_{k=1}^{n-1}e^{-\frac{k}{n}\pi i}x_k t^{\frac{k}{n}}(1-t)^{\frac{n-k}{n}}\right)\right]dt,$$

где ветви логарифма определены в области пространства $\mathbb{C}^{n-1}$ переменного
$x=(x_1,\ldots,x_{n-1})$, *полученной удалением из $\mathbb{C}^{n-1}$ двух семейств комплексных гиперплоскостей*

$$\Sigma_- = \bigcup_{t\in[0;1]}\left\{\sum_{k=1}^{n-1}x_k t^{\frac{k}{n}}(1-t)^{\frac{n-k}{n}}e^{-\frac{k}{n}\pi i}=1\right\},$$
$$\Sigma_+ = \bigcup_{t\in[0;1]}\left\{\sum_{k=1}^{n-1}x_k t^{\frac{k}{n}}(1-t)^{\frac{n-k}{n}}e^{\frac{k}{n}\pi i}=1\right\},$$

и выбираются условием $\ln 1=0$.

Поставим задачу исследовать взаимное расположение дискриминантного множества ∇ уравнения (1) и семейства гиперплоскостей Σ_+ (Σ_-). Решение задачи будет более наглядным, если перейти к логарифмической шкале

$$Log:(x_1,x_2,\ldots,x_{n-1})\longrightarrow(log|x_1|,log|x_2|,\ldots,log|x_{n-1}|).$$

Обозначим через

$$F_\pm(x;t)=\sum_{k=0}^{n-1}x_k t^{\frac{k}{n}}(1-t)^{\frac{n-k}{n}}e^{\pm\frac{k}{n}\pi i}-1$$

– пару функций, линейных относительно x.

Использовав параметризацию (2) дискриминантного множества ∇ уравнения (1), а также параметризацию нулевого множества функций $F_\pm(x;t)$

$$x_l(\tau)=\frac{\tau_l}{a^\pm_{n-1}\tau_{n-1}+\ldots+a^\pm_1\tau_1},\ \tau_l\in\mathbb{C},$$

при

$$a^\pm_l=t^{\frac{l}{n}}(1-t)^{\frac{n-l}{n}}e^{\pm\frac{l}{n}\pi i},\quad l=1,\ldots,n-1,$$

можно показать справедливость следующего утверждения.

Теорема 2. *Контур амебы дискриминанта уравнения (1) при* $s \in \mathbb{R}^+_{n-1}$ *является огибающей для семейства амеб гиперплоскостей* $\Sigma_\pm$ *при* $\arg \tau_l = \mp \frac{l}{n}\pi$. *Более того, в случае* $n = 3$ *для указанного семейства гиперплоскостей является огибающей контур амебы дискриминанта уравнения (1) и при* $\frac{\langle \alpha, s\rangle}{\langle \beta, s\rangle} > 0$. *Значения* $\langle \alpha, s\rangle$, $\langle \beta, s\rangle$ *находятся из равенств (3).*

В дополнение к вышеизложенному, в докладе будет приведена геометрическая иллюстрация Теоремы 2 для дискриминанта

$$\Delta(x) = 27 + 4x_1^3 - 4x_2^3 + 18x_1x_2 - x_1^2x_2^2$$

кубического уравнения

$$z^3 + x_2z^2 + x_1z - 1 = 0. \tag{4}$$

В дополнение к этому, для уравнения (4), в логарифмической шкале

$$Log : (x_1, x_2) \longrightarrow (log|x_1|, log|x_2|)$$

будет найдено уравнение кривой, которая соответствует пересечению дискриминантного множества с комплексными прямыми $\Sigma_\pm$. Отметим некоторые ее свойства: это петля, проходящая вокруг каспидальной точки, симметричная относительно прямой

$$u = v, \text{ где } u = log|x_1|,\ v = log|x_2|.$$

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Программа для демонстрации универсальных алгоритмов решения дискретного уравнения Беллмана в различных полукольцах[1]

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Назначение программы. Программа предназначена для демонстрации некоторых универсальных алгоритмов обращения матрицы и решения уравнения Беллмана в различных полукольцах. В зависимости от выбора полукольца, программа может либо найти обратную матрицу и решить уравнение $Ax = B$, где A и B - пользовательская матрица и вектор-столбец соответственно, либо найти матрицу A^* и решить уравнения Беллмана $x = A \bigotimes x \bigoplus B$ Перед запуском программы пользователь выбирает одно из полуколец, требуемую задачу и алгоритм расчета. Затем исходные данные заносятся в матрицу (для наглядности максимальный размер ограничен величиной 10x10). Результат расчета выводится либо в виде матрицы, либо в виде вектора столбца, в зависимости от поставленной задачи. В процессе разработки программы использовался объектно - ориентированный подход, позволяющий в полной мере использовать универсальность предложенных алгоритмов, подключая в качестве объектов полукольца с определенной арифметикой, актуальной для решения конкретной задачи.

Примеры полуколец. Использование идемпотентных операций $\bigoplus$ и $\bigotimes$ позволяет записать ряд важных алгоритмов обращения матрицы в универсальном виде. Выбор пользователем требуемого полукольца определяет тип данных, с которыми будет работать универсальный вычислительный алгоритм. В программе реализована возможность выбора из следующих полуколец:

1) $\bigoplus =$ " $+$ " и $\bigotimes =$ " $\times$ " - обычная арифметика.

2) $\bigoplus =$ "max" и $\bigotimes =$ " $+$ " - арифметика max-plus, в которой операция взятия максимума используется вместо сложения, а сложение - вместо умножения. Такая арифметика часто используется в задачах максимизации, системах автоматического управления и др.

3) $\bigoplus =$ "min" и $\bigotimes =$ " $+$ " - арифметика min-plus, в которой операция взятия минимума используется вместо сложения, а сложение - вместо умножения. Используется в задачах нахождения кратчайшего пути, задачах оптимизации.

[1]Работа выполнена при поддержке грантов РФФИ 05-01-00824 и 05-01-02807-НЦНИЛ_а.

4) $\bigoplus = "max"$ и $\bigotimes = " \times "$ - арифметика, в которой вместо сложения используется операция взятия максимума.

5) $\bigoplus = "max"$ и $\bigotimes = "min"$ - арифметика max-min, в которой вместо сложения используется операция взятия максимума, вместо умножения - операция взятия минимума. Используется в задачах многокритериальной оптимизации.

6) $\bigoplus = "or"$ и $\bigotimes = "and"$ - логическая арифметика над булевскими переменными.

Каждому полукольцу в программе соответствует пользовательский тип данных, определяемый отдельным классом. Кроме описаний правил сложения и умножения, внутри класса определяются также вид нуля, единицы и правило взятия операции "*". Такой подход дает возможность дополнять программу новыми типами полуколец, не меняя структуру основной программы и не внося никаких изменений в ту ее часть, которая занимается вычислительными алгоритмами.

Универсальные алгоритмы. В последнее время большое количество работ (например, [**1**]-[**5**]) посвящено разработке универсальных версий алгоритмов линейной алгебры и численного анализа. Рассматриваемая программа использует универсальные версии ряда классических алгоритмов обращения матриц и решения систем линейных уравнений. На выбор пользователю предлагаются ряд алгоритмов, как точных, так и с использованием метода последовательных приближений:

1) Метод исключения по схеме Гаусса;

2) Метод окаймления;

3) Итерационный метод Якоби;

4) Итерационный метод Гаусса-Зейделя;

5) Алгоритмы для матриц специального вида: симметричных, треугольных, теплицевых и др., в том числе предложенные в [**4**] и [**5**].

В случае выбора обычной арифметики эти алгоритмы ведут себя классическим образом, однако для случая идемпотентного полукольца они позволяют найти матрицу A^* или решить соответствующее уравнение Беллмана. Для полуколец max-plus или min-plus уравнение Беллмана представляет собой основное функциональное уравнение динамического программирования и выражает принцип оптимальности Беллмана: управление на каждом шаге должно быть оптимальным с точки зрения процесса в целом.

Возможности визуализации. Для наглядного представления информации в программе заложена возможность визуализации исходной матрицы в виде графа с соответствующими весами. На отдельной

вкладке диалогового окна программы отображается введенная пользователем информация об исходной матрице и результат соответствующего расчета. В таком режиме работы программы, например, задача о решении уравнения Беллмана в полукольце min-plus будет отображаться как кратчайший путь между узлами заданного пользователем графа.

Использование различных арифметик для контроля точности. Применение при разработке программы объектно ориентированного подхода позволяет не только независимо менять полукольца и алгоритмы вычисления, но и управлять базовым типом числовых данных для контроля за точностью вычислений. В следующей версии программы предполагается реализовать механизм выбора одной из числовых арифметик. Среди них арифметика целых чисел, арифметика чисел с плавающей точкой, дробно-рациональная арифметика с использованием цепных дробей, в том числе с контролируемой точностью. Это позволит сравнить ошибку округления, накопленную в ходе применения того или иного вычислительного алгоритма с ошибкой самого метода (для итерационных алгоритмов), что позволит судить об их итоговой эффективности.

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Кривые в $\mathbb{C}^2$, амебы которых определяют фундаментальную группу дополнения

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В классической работе ван Кампена [**1**], продолжающей исследования многих математиков, начиная от Зарисского, был предъявлен метод вычисления фундаментальной группы дополнения к плоской комплексной кривой. Результат ван Кампена впоследствии был переформулирован Б. Мойшезоном и М. Тайхер с использованием понятия брэйд-монодромии, рассмотренном в [**2**].

Дадим краткое описание гомоморфизма брэйд-монодромии. Пусть алгебраическая кривая C задана множеством нулей полинома $f(x,y) \in \mathbb{C}[x,y]$. Будем смотреть на f как на полином Вейерштрасса:

$$f = \alpha_0(y)x^d + \alpha_1(y)x^{d-1} + \cdots + \alpha_d(y).$$

Обозначим $B := \mathbb{C} \setminus \Delta$, где $\Delta = \{y_1, \ldots, y_s\}$ — дискриминант полинома f. Ограничение проекции $(x,y) \mapsto y$ на множество $E := \mathbb{C} \times B \setminus C$ определяет локально тривиальное расслоение $p\colon E \to B$. Вычисление фундаментальной группы $\pi_1(\mathbb{C}^2 \setminus C)$ сводится к вычислению группы $\pi_1(E)$. Выберем $(x_0, y_0) \in E$ и обозначим через F слой над точкой y_0. База и слой расслоения p представляют собой дополнения к конечным наборам точек в $\mathbb{C}$, поэтому группы $\pi_1(B, y_0)$ и $\pi_1(F, x_0)$ являются свободными группами и точная последовательность расслоения

$$1 \longrightarrow \pi_1(F, x_0) \longrightarrow \pi_1(E, (x_0, y_0)) \longrightarrow \pi_1(B, y_0) \longrightarrow 1.$$

расщепляется, так что группа $\pi_1(E, (x_0, y_0))$ есть полупрямое произведение групп $\pi_1(B, y_0)$ и $\pi_1(F, x_0)$. Действие группы $\pi_1(B, y_0)$ на $\pi_1(F, x_0)$ может быть описано в терминах групп кос. Для этого слой F отождествим с диском D^2, из которого выброшено множество $K = \{x_1, \ldots, x_d\}$ различных точек. Каждый элемент группы $\pi_1(B, y_0)$ определяет биекцию множества K, а следовательно и элемент группы $\mathrm{Br}_d = \mathrm{Br}(D, K)$ кос из d нитей. Таким образом определен гомоморфизм $\pi_1(B, y_0) \to \mathrm{Br}_d$, носящий название гомоморфизма брэйд-монодромии. Этот гомоморфизм позволяет выписать соотношения между образующими группы $\pi_1(E, (x_0, y_0))$.

Переход к рассмотрению амебы $\mathcal{A}_C$ кривой C, то есть образа кривой под действием отображения логарифмической проекции $(x,y) \mapsto (\ln|x|, \ln|y|)$, ставит вопрос о том, каким образом знание амебы может помочь в вычислении фундаментальной группы $\pi_1(\mathbb{C}^2 \setminus C)$. Обозначим через E_ν связную компоненту $\mathbb{R}^2 \setminus \mathcal{A}_C$ порядка $\nu = (\nu_x, \nu_y)$.

Определение и свойства порядка ν можно найти в статье Форсберга-Пассаре-Циха [**3**]. Для нас существенно, что целые числа ν_x и ν_y выражают коэффициенты зацепления петель $\Gamma_x^\nu = \{x = e^{u+it}, y = e^v\}$ и $\Gamma_y^\nu = \{x = e^u, y = e^{v+it}\}$ с кривой C, причем ν_x и ν_y не зависят от выбора точки $(u, v) \in E_\nu$. Пусть $(x_0, y_0) = (e^u, e^v)$, где $(u, v) \in E_\nu$. Тогда классы петель $[\Gamma_x^\nu]$, $[\Gamma_y^\nu] \in \pi_1(\mathbb{C}^2 \setminus C, (x_0, y_0))$ коммутируют. Возникает вопрос, можно ли описать фундаментальную группу дополнения к плоской кривой C, используя в качестве образующих только классы петель, получающихся из петель Γ_x^ν, Γ_y^ν, где ν пробегает порядки всех связных компонент дополнения к амебе кривой?

Чтобы ответить на этот вопрос, мы должны вернуться к описанию группы $\pi_1(\mathbb{C}^2 \setminus C)$ с использованием брэйд-монодромии. Так как старшая степень, с которой переменная x входит в полином f, равна d, то найдется хотя бы одна связная компонента $E_{\widetilde{\nu}}$ из $\mathbb{R}^2 \setminus \mathcal{A}_C$ порядка $\widetilde{\nu} = (d, \nu_y)$. Выберем $(u, v) \in E_{\widetilde{\nu}}$ так, чтобы $v \neq \ln|y_i|$ для всех y_i из дискриминанта Δ полинома f. Тогда для всех $t \in [0, 2\pi]$ диск $\{\ln|x| \leqslant u,\ y = e^{v+it}\}$ содержит ровно d корней полинома $f(x, e^{v+it})$. Следовательно определена коса из d нитей, которую можно сопоставить петле $\Gamma_y^{\widetilde{\nu}} = \{x = e^u, y = e^{v+it}\}$. Отсюда видно, что петли Γ_x^ν, Γ_y^ν естественным образом связаны с брэйд-монодромией кривой, то есть могут претендовать не только на роль образующих группы $\pi_1(\mathbb{C}^2 \setminus C)$, но и давать соотношения между образующими.

В качестве простого примера кривой, фундаментальная группа дополнения которой определяется амебой, рассмотрим дискриминант $\Delta[2, 3]$ кубического уравнения $z^3 + z^2 + xz + y = 0$. Он задается полиномом $27y^2 + 4x^3 + 4y - 18xy - x^2$ и представляет собой каспидальную кривую с невырожденной амебой. Фундаментальная группа $\pi_1(\mathbb{C}^2 \setminus \Delta[2, 3])$ описывается с помощью петель $\Gamma_x^{(3,0)}$ и $\Gamma_y^{(0,2)}$. Брэйд-монодромия дает изоморфизм $\pi_1(\mathbb{C}^2 \setminus \Delta[2, 3]) \cong \mathrm{Br}_3$.

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