Mass Transportation and Disintegration Maps

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1. L. Granieri, F. Maddalena, A Metric Approach to Elastic Reformations
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Problem: Detect isometric shapes

\[ X \quad Y \]
We model (material) shapes as Radon probability measures on compact subsets $X, Y \subset \mathbb{R}^N$. and study a variational model to the aim of quantifying how a target shape $\nu$ on $Y$ differs from an isometric copy of $\mu$ on $X$. 
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$$|u(x) - u(y)| = |x - y|, \quad \forall x, y \in X.$$
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$$\frac{1}{L}|x - y| \leq |u(x) - u(y)| \leq L|x - y|, \quad \forall x, y \in X.$$
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Therefore, the two shapes $X, Y$ could be considered close to be \textit{isometric} as the bi-Lipschitz constant $L$ is close to one, so assuming the bi-Lipschitz constant as a quantifier of the closeness to the isometry.
This global approach has some disadvantages. For instance, the shapes below

looks very close to be isometric but the bi-Lipschitz constant is quite large and far from $L = 1$, whatever the size of the bending part.
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Actually, under some regularity assumptions, by Liouville
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characterizes the isometric maps.
Hence, a reasonable way to quantify how two shapes are
isometric is that of measuring how $\nabla u$ is close to be an
orthogonal matrix.
A Variational Approach

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This approach is pursued in

G. Wolansky, Incompressible, Quasi-Isometric Deformations of 2-Dimensional Domains, SIAM J. Imaging Sciences, 2, No. 4 (2009), 1031-1048

where the admissible maps are incompressible diffeomorphisms, i.e. $u$ such that $\det (\nabla u) = 1$. 
In order to characterize the isometries, a polyconvex function $W$ having minimal value at orthogonal matrices is selected.
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$$\min \left\{ \int_{\Omega_1} W(\nabla u) \, dx \mid u(\Omega_1) = \Omega_2, \, u \in D \right\},$$

where $D$ denotes the set of incompressible diffeomorphisms.
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Figure: An isometric fractured reformation.
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The mass conservation property (1) is a generalized version of incompressibility and it can be always satisfied (provided \( \mu \) has no atom) by some measurable map \( u \).
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The mass conservation property (1) is a generalized version of incompressibility and it can be always satisfied (provided \( \mu \) has no atom) by some measurable map \( u \). Actually, condition (1) is equivalent to the following change of variable formula

\[ \int_X f(u(x)) \, d\mu = \int_Y f(y) \, d\nu, \]  

(2)

for every continuous function \( f : Y \to \mathbb{R} \).
Measurable is too weak

Figure: A piece-wise isometric map for the circle into a square.
The Lipschitz constant can be localized by

\[ e_u(x_0) := \text{Lip}(u)(x_0) = \limsup_{x \to x_0} \frac{|u(x) - u(x_0)|}{|x - x_0|}. \] (3)
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A metric Formulation

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Observe that Wolansky’s approach cannot be pursued in a metric framework. Indeed the mapping \(A \mapsto \varphi(\|A\|)\) is polyconvex only if \(\varphi\) is a positive convex and strictly increasing function, therefore the minimal value cannot be reached at orthogonal matrices \(A\), since they have \(\|A\| = 1\).
Similarly, we introduce the pointwise contraction energy of $u$ at $x_0$ defined by

$$c_u(x_0) := \limsup_{x \to x_0} \frac{|x - x_0|}{|u(x) - u(x_0)|}.$$  \hfill (4)
Similarly, we introduce the pointwise contraction energy of $u$ at $x_0$ defined by

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The pointwise reformation energy of $u$ at $x_0$ is defined by

$$r_u(x_0) = e_u(x_0) + c_u(x_0).$$ \hfill (5)
Definition (Reformation maps)

We shall call reformation map any map \( u : X \to Y \) such that the following conditions hold true:

\[
u \# \mu = \nu, \quad (6)\]

\[
\forall x \in X \ \exists H, K, r > 0 \text{ s.t. } e_u(y) \leq K, \ c_u(y) \leq H \quad (7)
\]

\[
\forall y \in \overline{B}(x, r) \cap X.
\]

We shall denote by \( \text{Ref}(\mu; \nu) \) the set of reformation maps between \( \mu \) and \( \nu \).
By the bounds (7), any $u \in \text{Ref}(\mu;\nu)$ is continuous and, by Stepanov Theorem, for $X = \overline{\Omega}$, it is a.e. differentiable in $\Omega$. 
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In a mechanical perspective, the constraints stated in (7) could be considered as a bound on the maximum expansion or contraction experienced by the material $\Omega$. In this setting, the assumption that of the constants $H, K$ do not depend on the map $u$ in (7) corresponds to a constitututive property of the material under consideration.
We point out that some bounds as in (7) are in some sense necessary to control the geometry of the reformations. For instance, in the case of $\nu = \delta_{y_0}$ we have $e_u = 0, \ c_u = +\infty$ for any map $u$ satisfying (6).
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On the other hand, mapping a bar into a bended one (see Fig. 16) by two piecewise isometries $u_1, u_2$, we have $e_u(x_0) = +\infty$ at the discontinuity point.
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Therefore, roughly speaking, the bound $c_u \leq H$ means no collapsing, while $e_u \leq K$ means no fractures.
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We have the following inversion results

**Theorem (Small reformations are invertible)**

Let \( u \in \text{Ref}(\mu; \nu) \) be such that \( e_u < \sqrt{2} \). Then \( u \) is globally invertible.
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**Theorem (J. Gevirtz, Metric conditions that imply local invertibility, Communications in Pure and Applied Mathematics 23 (1969), 243-264.)**

Let $u \in \text{Ref}(\mu; \nu)$ be an open map such that $HK < 2$. Then $u|_\Omega$ is locally invertible.
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*Theorem (Small reformations are invertible)*

Let \( u \in \operatorname{Ref}(\mu; \nu) \) be such that \( e_u < \sqrt[4]{2} \). Then \( u \) is globally invertible.

*Theorem (J. Gevirtz, Metric conditions that imply local invertibility, Communications in Pure and Applied Mathematics 23 (1969), 243-264.)*

Let \( u \in \operatorname{Ref}(\mu; \nu) \) be an open map such that \( HK < 2 \). Then \( u|_{\Omega} \) is locally invertible.

Key tools: area formula and degree theory for maps in \( \mathbb{R}^N \).
We define the total reformation energy $R(u)$ of a reformation map $u$ of $\mu$ into $\nu$ as follows

$$R(u) := \int_{\chi} r_u(x) \, d\mu.$$
Since $c_u(x) \geq \frac{1}{e_u(x)}$ we get $R(u) \geq 2$. 
Since $c_u(x) \geq \frac{1}{e_{u(x)}}$ we get $R(u) \geq 2$.
Actually, this definition is motivated by the trivial fact that the real function $f(x) = x + 1/x$ reaches its minimum value at $f(1) = 2$. 
Since \( c_u(x) \geq \frac{1}{e_{u(x)}} \) we get \( \mathcal{R}(u) \geq 2 \).
Actually, this definition is motivated by the trivial fact that the real function \( f(x) = x + 1/x \) reaches its minimum value at \( f(1) = 2 \).
Moreover, we have that \( r_u(x) \) reaches its minimum value if \( u : X \rightarrow Y \) is an isometric mapping. Therefore \( \mathcal{R}(u) \) can be viewed as a measure detecting how \( u \) is far from being an isometric map.
Moreover, conversely assume $r_u(x_0) = 2$, then

$$2 = e_u(x_0) + c_u(x_0) \geq e_u(x_0) + \frac{1}{e_u(x_0)} \geq 2,$$
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$$e_u(x_0) + \frac{1}{e_u(x_0)} = 2 \Rightarrow (e_u(x_0) - 1)^2 = 0 \Rightarrow e_u(x_0) = c_u(x_0) = 1.$$
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Taking $x = x_0 + \delta v$ and sending $\delta \to 0$, we get

$$c_u(x_0) = e_u(x_0) = 1 \Rightarrow \frac{|\nabla u(x_0) \cdot v|}{|v|} = 1 \Rightarrow \nabla u(x_0) \in O(N).$$
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Therefore, a Rigidity theorem (in Sobolev spaces) could be applied
We define the elastic reformation energy between $\mu$ and $\nu$ as

$$\mathcal{E}(\mu, \nu) := \inf \left\{ \mathcal{R}(u) \mid u \in \text{Ref}(\mu; \nu) \right\}.$$
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$$E(\mu, \nu) := \inf \{ R(u) \mid u \in \text{Ref}(\mu; \nu) \}.$$ 

We expect to characterize isometric maps as those having the smallest reformation energy.
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$$E(\mu, \nu) := \inf \{ R(u) \mid u \in \text{Ref}(\mu; \nu) \}.$$
Theorem
Let \( \mu \in \mathcal{P}(\Omega) \) and \( \nu \in \mathcal{P}(Y) \) so that \( \mu = \mathcal{L}^N \ll \Omega \), \( \nu = \mathcal{L}^N \ll Y \). Then the variational problem

\[
\min \{ \mathcal{R}(u) \mid u \in \text{Ref}(\mu; \nu), \ e_u < \sqrt[2]{2} \} \tag{8}
\]

admits solutions whenever \( \{ u \in \text{Ref}(\mu; \nu), \ e_u < \sqrt[2]{2} \} \neq \emptyset \).
Sketch of the proof

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Let \((u_n)_{n \in \mathbb{N}}\) be a minimizing sequence. Given \(x_0 \in \Omega\), let \(K, H, r > 0\) as provided by definition of reformation maps. It follows that the sequence \((u_n)_{n \in \mathbb{N}}\) is locally equi-Lipschitz on \(B(x_0, r)\). Therefore, the sequence \(u_n\) is pointwise equicontinuous on \(\Omega\).
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$$\liminf_{n \to +\infty} \left( \int_X \text{Lip}(u_n)(x) \, d\mu + \int_Y \text{Lip}(u_n^{-1})(y) \, d\nu \right) = \liminf_{n \to +\infty} \mathcal{R}(u_n).$$
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\]

We also obtain existence for the variational problem over the set \( \{ u \in \text{Ref}(\mu; \nu), u \text{ incompressible} \} \) or the set \( \{ u \in \text{Ref}(\mu; \nu), u \text{ open s.t. } HK < 2 \} \).
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\liminf_{n \to +\infty} \left( \int_X \text{Lip}(u_n)(x) \, d\mu + \int_Y \text{Lip}(u_n^{-1})(y) \, d\nu \right) = \liminf_{n \to +\infty} \mathcal{R}(u_n).
\]

We also obtain existence for the variational problem over the set \( \{ u \in \text{Ref}(\mu; \nu), u \text{ incompressible} \} \) or the set \( \{ u \in \text{Ref}(\mu; \nu), u \text{ open s.t. } HK < 2 \} \).

This proof could be considered also in a metric framework.
We then characterize isometric measures by the following

Theorem
Let $\mu \in \mathcal{P}(\Omega)$ and $\nu \in \mathcal{P}(Y)$, so that $\mu = \mathcal{L}^N \upharpoonright \Omega$, $\nu = \mathcal{L}^N \upharpoonright Y$, for a given bounded set $Y$. Then, $\mathcal{E}(\mu, \nu) = 2$ if and only if there exists an isometry $u$ such that $u \# \mu = \nu$. 
Generalized Reformations

The notion of reformation map corresponds to the notion of the so-called transport map, i.e. $u : X \rightarrow Y$ such that $u\#\mu = \nu$. 
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Disintegrations

The key tool is the following

**Theorem (Disintegration theorem)**

Let $\gamma \in \mathcal{P}(X \times Y)$ be given and let $\pi^1 : X \times Y \to X$ be the first projection map of $X \times Y$, we set $\mu = (\pi^1)_\# \gamma$. Then for $\mu$ -- a.e. $x \in X$ there exists $\nu_x \in \mathcal{P}(Y)$ such that

(i) the map $x \mapsto \nu_x$ is Borel,

(ii) $\forall \varphi \in C_b(X \times Y) : \int_{X \times Y} \varphi(x, y) d\gamma = \int_X \left( \int_Y \varphi(x, y) d\nu_x(y) \right) d\mu(x)$.

Moreover the measures $\nu_x$ are uniquely determined up to a negligible set with respect to $\mu$. 

As usual we will write $\gamma = \nu_x \otimes \mu$, assuming that $\nu_x$ satisfy the condition (i) and (ii) of Disintegration Theorem.
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\]

We endow \( \mathcal{P}(Y) \) with the Wasserstein metric.

**Definition**

Let \( \mu, \nu \in \mathcal{P}(X) \), the 1-Wasserstein distance between \( \mu \) and \( \nu \) is defined by

\[
W(\mu, \nu) = \inf_{\gamma \in \Pi(\mu, \nu)} \int_X d(x, y) \ d\gamma(x, y). \quad (9)
\]
Finding reformation plans

In the following examples we show that it is possible to compare shapes with regular disintegration maps despite no regular transport map does exist.

Figure: A disconnected target reformation
By results of Dacorogna-Moser, we find two diffeomorphisms $u_1 : X \rightarrow A$, $u_2 : X \rightarrow B$ so that $|\det(\nabla u_1)| = \mathcal{L}^N(A)$, $|\det(\nabla u_2)| = \mathcal{L}^N(B)$. 
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The function $f(x) = \nu_x$ is, at least locally, bi-Lipschitz.

$$W(\nu_x, \nu_{x_0}) = \mathcal{L}^N(A)|u_1(x) - u_1(x_0)| + \mathcal{L}^N(B)|u_2(x) - u_2(x_0)|.$$
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Since \( u_1, u_2 \) are diffeomorphisms, we find constants \( K_{1,2}, H_{1,2}, H, K \) such that

\[ \frac{1}{H_1} |x - x_0| \leq \frac{\mathcal{L}^N(A)}{H_1} |x - x_0| + \frac{\mathcal{L}^N(B)}{H_2} |x - x_0| \leq \]

\[ \mathcal{L}^N(A)|u_1(x) - u_1(x_0)| + \mathcal{L}^N(B)|u_2(x) - u_2(x_0)| = W(\nu_x, \nu_{x_0}) \leq \]
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Since $u_1, u_2$ are diffeomorphisms, we find constants $K_{1,2}, H_{1,2}, H, K$ such that

$$\frac{1}{H}|x - x_0| \leq \frac{\mathcal{L}^N(A)}{H_1}|x - x_0| + \frac{\mathcal{L}^N(B)}{H_2}|x - x_0| \leq K|x - x_0|.$$
Generalized Reformations

It makes sense to compare shapes through Disintegration maps. We define
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Namely, for any reformation plan \( \gamma = \nu_x \otimes \mu \) of \( \mu \) into \( \nu \) we define the pointwise expansion energy

\[ e_\gamma(x_0) := \limsup_{x \to x_0} \frac{W(\nu_x, \nu_{x_0})}{|x - x_0|}, \quad (11) \]
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\[ c_\gamma(x_0) = \limsup_{x \to x_0} \frac{|x - x_0|}{W(\nu_x, \nu_{x_0})}. \quad (12) \]
Since $\mathcal{W}(\delta_x, \delta_y) = d(x, y)$, for $\gamma = (I \times u)\#\mu$ we have

$$e_\gamma(x) = e_u(x), \quad c_\gamma(x) = c_u(x).$$
Since $W(\delta_x, \delta_y) = d(x, y)$, for $\gamma = (I \times u) \# \mu$ we have
\[
  e_\gamma(x) = e_u(x), \quad c_\gamma(x) = c_u(x).
\]

We define the reformation energy of $\gamma$ as follows
\[
  R(\gamma) = \int_X (e_\gamma + c_\gamma) \, d\mu. \tag{13}
\]
Working on a metric space setting some restriction arise.
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We prove the following

**Theorem**

Let $\gamma = f(x) \otimes \mu \in \text{GRef}(\mu; \nu)$ be such that $\mathcal{R}(\gamma) = 2$, $\mu$ absolutely continuous with respect to the Lebesgue measure. Then there exists an open dense subset of $X$ on which the disintegration map $f$ is a local isometry (with respect to the Wasserstein distance).
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The open dense subset is obtained by a Baire Category argument.
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The invertibility restriction can be avoid by taking small reformations.
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**Definition**
We define the set $\text{GRef}(\mu; \nu) \subset \Pi(\mu, \nu)$ as the subset of reformation plans $\gamma$ of $\mu$ into $\nu$ satisfying

$$\forall x_0 \in X : \exists r > 0, H, K \text{ s.t. } e_\gamma(x) \leq K, \ c_\gamma(x) \leq H$$

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(14)
The invertibility restriction can be avoided by taking *small reformatations.*

**Definition**

We define the set $G_{\text{Ref}}(\mu; \nu) \subset \Pi(\mu, \nu)$ as the subset of reformation plans $\gamma$ of $\mu$ into $\nu$ satisfying

$$\forall x_0 \in X : \exists r > 0, H, K \text{ s.t. } e_{\gamma}(x) \leq K, c_{\gamma}(x) \leq H$$

for every $x \in X \cap \overline{B}(x_0, r)$.

**Definition**

Let us define the set of small reformation plans between $\mu$ and $\nu$ as follows

$$G_{\text{Ref}}_0(\mu, \nu) = \{ \gamma \in \Pi(\mu, \nu) \mid e_{\gamma} \leq K, c_{\gamma} \leq H, HK < \sqrt{2} \}.$$
By the metric area formula, small reformations yields globally invertible disintegration maps.
By the metric area formula, small reformations yields globally invertible disintegration maps. Let us introduce the notation

$$\mathcal{E}_G(\mu, \nu) = \inf\{ R(\gamma) \mid \gamma \in \text{GRef}_0(\mu; \nu) \}. \quad (16)$$

We prove the following
By the metric area formula, small reformations yields globally invertible disintegration maps. Let us introduce the notation

\[ E_G(\mu, \nu) = \inf \{ R(\gamma) \mid \gamma \in G\text{Re}f_0(\mu; \nu) \}. \]  

We prove the following

**Theorem**

If \( E_G(\mu, \nu) = 2 \), with \( \mu \) absolutely continuous with respect to the Lebesgue measure, then the infimum is attained at a local isometric reformation plan.
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A natural question concerns the validity of an existence result as done for transport maps. However, we observe that the approach pursued in the proof of such result involve the push-forward of the transport map. Therefore, for generalized reformations, the push-forward of the disintegrations maps is involved. Disintegration maps produce naturally a measure $f_\#\mu$ over the space $(\mathcal{P}(Y), W)$. By the following lemma we see that this point of view is equivalent to fix the second marginal of transport plans correspondent to transport maps.
Lemma

Let \( u, v : X \to Y \) be two given Borel maps, \( \mu \in \mathcal{P}(X) \) and let \( f, g : X \to \mathcal{P}(Y) \) defined by \( f(x) = \delta_{u(x)} \), \( g(x) = \delta_{v(x)} \). Then

\[
u \ll \mu \iff f \ll \mu = g \ll \mu.
\]

(17)
Lemma
Let $u, v : X \to Y$ be two given Borel maps, $\mu \in \mathcal{P}(X)$ and let $f, g : X \to \mathcal{P}(Y)$ defined by $f(x) = \delta_{u(x)}$, $g(x) = \delta_{v(x)}$. Then

$$u \# \mu = v \# \mu \iff f \# \mu = g \# \mu.$$  \hfill(17)

Corollary
Let $u, v : X \to Y$ be two given Borel maps, $\mu \in \mathcal{P}(X)$, let $f, g : X \to \mathcal{P}(Y)$ defined by $f(x) = \delta_{u(x)}$, $g(x) = \delta_{v(x)}$ and let $\gamma = f(x) \otimes \mu$, $\eta = g(x) \otimes \mu$. Then

$$\pi_2 \# \gamma = \pi_2 \# \eta \iff f \# \mu = g \# \mu.$$  \hfill(18)
part of the proof of the above Lemma works for general transport plans $\gamma = f(x) \otimes \mu$, $\eta = g(x) \otimes \mu$, yielding the following
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**Lemma**

Let $\mu \in \mathcal{P}(X)$, $f, g : X \to \mathcal{P}(Y)$, $\gamma = f(x) \otimes \mu$, $\eta = g(x) \otimes \mu$ be given. Then the following implication holds true

$$f_\# \mu = g_\# \mu \Rightarrow \pi^2_\# \gamma = \pi^2_\# \eta.$$  \hfill (19)
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f_\#\mu = g_\#\mu \Rightarrow \pi_2^\#\gamma = \pi_2^\#\eta. \tag{19}
\]

Therefore, also for transport plans, the second marginal can be fixed by fixing the push forward of disintegration maps.
part of the proof of the above Lemma works for general transport plans \( \gamma = f(x) \otimes \mu, \eta = g(x) \otimes \mu \), yielding the following

**Lemma**

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**Lemma**

Let \( \mu \in \mathcal{P}(X), f, g : X \rightarrow \mathcal{P}(Y), \gamma = f(x) \otimes \mu, \eta = g(x) \otimes \mu \) be given. Then the following implication holds true

\[
 f_\# \mu = g_\# \mu \implies \pi_2^\# \gamma = \pi_2^\# \eta. \tag{19}
\]

Therefore, also for transport plans, the second marginal can be fixed by fixing the push forward of disintegration maps. In general the converse of (19) is not true as for \( f : X \rightarrow \mathcal{P}(Y) \) defined by \( f(x) = \nu \) and \( \gamma = f(x) \otimes \mu \). Let \( \eta = g(x) \otimes \mu \) where \( g(x) = \delta_{u(x)} \) for a given transport map \( u : X \rightarrow Y \) with \( u_\# \mu = \nu \)
This discussions allow to distinguish transport plans through the push forward of disintegration maps. We introduce the following notion of transport class.
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**Definition (Transport classes)**

Let $\gamma, \eta \in \Pi(\mu, \nu)$ with $\gamma = f(x) \otimes \mu$, $\eta = g(x) \otimes \mu$ be given. We shall say that $\gamma$ and $\eta$ are equivalent (by disintegration), in symbols $\gamma \approx \eta$, if $f_\# \mu = g_\# \mu$.

For any $\eta \in \Pi(\mu, \nu)$ with $\eta = g(x) \otimes \mu$, we shall call transport class any equivalence class of a transport plan $\eta$ and it will be denoted by $[\eta]$, i.e.

$$[\eta] = \{ \gamma = f(x) \otimes \mu \mid f_\# \mu = g_\# \mu \}. \quad (20)$$
This discussion allows to distinguish transport plans through the push forward of disintegration maps. We introduce the following notion of transport class.

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$$[\eta] = \{ \gamma = f(x) \otimes \mu \mid f_\# \mu = g_\# \mu \}.$$  \hspace{1cm} (20)

It follows that all transport plans induced by transport maps belong to the same transport class.
To better explain the notion of transport class, consider the case of a discrete first marginal $\mu = \sum_i \alpha_i \delta_{x_i}$. 
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$$f_#\mu = \sum_i \alpha_i \delta_{f(x_i)}.$$ 

Therefore, transport classes are fixed by the range of $f$. 
Consider the discrete marginals

\[ \mu = \frac{1}{3} \delta_{x_1} + \frac{1}{3} \delta_{x_2} + \frac{1}{3} \delta_{x_3}, \quad \nu = \frac{1}{6} \delta_{y_1} + \frac{5}{6} \delta_{y_2}. \]
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The above transport plans belong to the same transport class since

\[
f(x_1) = 3(a \delta_{y_1} + b \delta_{y_2}), \quad f(x_2) = \delta_{y_2}, \quad f(x_3) = \delta_{y_2}, \quad a = b = \frac{1}{6}.
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\[ g(x_1) = \delta_{y_2}, \quad g(x_2) = 3(a \delta_{y_1} + b \delta_{y_2}), \quad g(x_3) = \delta_{y_2}, \quad a = b = \frac{1}{6}. \]
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g(x_1) = \delta_{y_2}, \quad g(x_2) = 3(a\delta_{y_1} + b\delta_{y_2}), \quad g(x_3) = \delta_{y_2}, \quad a = b = \frac{1}{6}.
\]

Hence, all the transport plans with one mass split belong to the same transport class.
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\[ h(x_1) = 3(a' \delta y_1 + b' \delta y_2) \]
\[ h(x_2) = 3(c' \delta y_1 + d' \delta y_2) \]
\[ h(x_3) = \delta y_2, \]
\[ a' = \frac{3}{30}, \quad b' = \frac{7}{30}, \quad c' = \frac{2}{30}, \quad d' = \frac{8}{30}. \]
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Maintaining fixed the number of splitting, the transport class may be changed by modifying the amount of traveling masses.

\[ k(x_1) = 3(a''\delta y_1 + b''\delta y_2), \quad k(x_2) = 3(c''\delta y_1 + d''\delta y_2), \quad k(x_3) = \delta y_2, \]

\[ a'' = \frac{1}{30}, \quad b' = \frac{9}{30}, \quad c' = \frac{4}{30}, \quad d' = \frac{6}{30}. \]
Therefore, fixing a transport class means to consider a constrained transport problem, with respect to splitting masses or traveling ones.
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Since the transport maps are dense in $\Pi(\mu, \nu)$ we can state the following

**Proposition**

Let $u : X \to Y$, be a transport map, i.e. such that $u \# \mu = \nu$, with $\mu$ non-atomic, and let $\eta = (I \times u) \# \mu = \delta_{u(x)} \otimes \mu$. If $\gamma \in [\eta]$ then there exists a transport map $v : X \to Y$ such that $\gamma = \delta_{v(x)} \otimes \mu$, i.e. the transport plan $\gamma$ is concentrated on the graph of $v$. 
Monge transport problem can be reformulated as follows

\[
\text{Minimize } \left\{ \int_X c(x, u(x)) \, d\mu : u_\# \mu = \nu \right\} = \\
\text{Minimize } \left\{ \int_{X \times Y} c(x, y) \, d\gamma : \gamma \in [\delta_\nu \otimes \mu] \right\},
\]

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for a transport map \( \nu \). Therefore, Monge problem correspond to minimization of the functional \( \int c \, d\gamma \) in a fixed transport class of \( \Pi(\mu, \nu) \), while the Kantorovich one corresponds to the minimization on the whole \( \Pi(\mu, \nu) \).
Monge-Kantorovich problems over transport classes

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\[
\int_Y \varphi(y) \, d\nu = \int_X \left( \int_Y \varphi(y) \, df(x) \right) \, d\mu = \int_X I_\varphi(f(x)) \, d\mu =
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$$\int_{\mathcal{P}(Y)} I_\varphi(\lambda) \, d\Lambda(\lambda) = \int_{\mathcal{P}(Y)} \left( \int_Y \varphi(y) \, d\lambda \right) \, d\Lambda.$$
Therefore, the measure $\Lambda$ has to satisfy the constraint

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Every probability measure $\Lambda$ over $(\mathcal{P}(Y), W)$ satisfying (21) define a transport class $[\eta] = \{f \otimes \mu : f_{#}\mu = \Lambda\}$. 
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Therefore, the measure \( \Lambda \) has to satisfy the constraint

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Every probability measure \( \Lambda \) over \((\mathcal{P}(Y), W)\) satisfying \((21)\) define a transport class \([\eta] = \{ f \otimes \mu : f#\mu = \Lambda \}\). In this perspective, transport plan in the transport class \( \Lambda \) can be seen as transport map between \( \mu \) and \( \Lambda \). It is then natural to consider the Monge-Kantorovich problem in the class \( \Lambda \) defined as follows

\[
MK_\Lambda(c, \mu, \nu) := \inf_{\gamma} \left\{ \int_{X \times Y} c(x, y) \, d\gamma : \gamma = f \otimes \mu, f#\mu = \Lambda \right\} \tag{22}
\]
The notion of transport class leads naturally to consider an abstract Monge problem between the space $X$ and $\mathcal{P}(Y)$. Consider the following transport cost
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$$\forall (x, \lambda) \in X \times \mathcal{P}(Y) : \tilde{c}(x, \lambda) = \int_Y c(x, y) d\lambda.$$  \hspace{1cm} (23)

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We have the following

**Proposition**

*For every transport class $\Lambda$ we have*

$$M(\tilde{c}, \mu, \Lambda) = MK_\Lambda(c, \mu, \nu).$$
Proof.

It suffices to observe that for any disintegration map $f : X \to \mathcal{P}(Y)$ such that $f_\#\mu = \Lambda$ it results
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Observe that by the above proof it follows that $f$ is a solution of $M(\tilde{c}, \mu, \Lambda)$ if and only if $f \otimes \mu$ is a solution of $MK_\Lambda(c, \mu, \nu)$. 
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\]

Observe that by the above proof it follows that \( f \) is a solution of \( M(\tilde{c}, \mu, \Lambda) \) if and only if \( f \otimes \mu \) is a solution of \( MK_\Lambda(c, \mu, \nu) \). Therefore, every existence result for the Monge problem \( M(\tilde{c}, \mu, \Lambda) \) in the abstract setting corresponds to an existence result for the Monge-Kantorovich problem in the transport class \( \Lambda \).
Coming back to the reformation problem we have the following
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**Theorem**

*(Existence of optimal reformation plans)* Let $\eta \in \text{GRef}_0(\mu; \nu)$ be given. Then, for every $p > 1$ the variational problem

$$
\min_{\text{GRef}_0(\mu; \nu)} \left\{ R^p(\gamma) := \int_{X} (c^p_{\gamma} + e^p_{\gamma}) \, d\mu \mid \gamma \in [\eta] \right\}
$$

admits solutions.
Sketch of the proof

Consider

\[ \int_X c_\gamma(x) d\mu = \int_X \text{Lip}^p(f^{-1})(f(x)) \ d\mu \]  

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Since \(X\) satisfies the doubling condition and the Poincaré inequality, we can apply the theory of Sobolev spaces over the metric space \((\mathcal{P}(Y), W, f_\#\mu)\).
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Since \( X \) satisfies the doubling condition and the Poincaré inequality, we can apply the theory of Sobolev spaces over the metric space \((\mathcal{P}(Y), W, f_\#\mu)\). Moreover, for \( p > 1 \) the pointwise Lipschitz constant \( \text{Lip}(g) \) is the minimal generalized upper gradient of the locally Lipschitz map \( g \) and coincides with the Cheeger \( p \)-energy which is lower semicontinuous with respect to \( L^p \) convergence.
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Thanks for the Attention