Resonance phenomenon for the Galerkin-truncated Burgers and Euler equations

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- The Tyger Phenomenon: 2D Euler Equation
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Equilibrium Statistical Mechanics and Turbulence

- Equilibrium statistical mechanics is concerned with conservative Hamiltonian dynamics, Gibbs states, ...
- Turbulence is about dissipative out-of-equilibrium systems.
- In 1952 Hopf and Lee apply equilibrium statistical mechanics to the 3D Euler equation and obtain the equipartition energy spectrum which is very different from the Kolmogorov spectrum.
- In 1967 Kraichnan uses equilibrium statistical mechanics as one of the tools to predict the existence of an inverse energy cascade in 2D turbulence.
In 1989 Kraichnan remarks the truncated Euler system can imitate NS fluid: the high-wavenumber degrees of freedom act like a thermal sink into which the energy of low-wave-number modes excited above equilibrium is dissipated. In the limit where the sink wavenumbers are very large compared with the anomalously excited wavenumbers, this dynamical damping acts precisely like a molecular viscosity.

In 2005 Cichowlas, Bonaiti, Debbasch, and Brachet discovered long-lasting, partially thermalized, transients similar to high-Reynolds number flow.
The Galerkin-truncated 1D Burgers equation

- The (untruncated) inviscid Burgers equation, written in conservation form, is

\[
\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0; \quad u(x, 0) = u_0(x).
\]

- Let \( K_G \) be a positive integer, here called the Galerkin truncation wavenumber, such that the action of the projector \( P_{K_G} \):

\[
P_{K_G} u(x) = \sum_{|k| \leq K_G} e^{ikx} \hat{u}_k.
\]

- The associated Galerkin-truncated (inviscid) Burgers equation

\[
\partial_t \nu + P_{K_G} \partial_x (\nu^2/2) = 0; \quad \nu_0 = P_{K_G} u_0.
\]
Time Evolution of the Truncated Equation

\[ v(x) \]

\[ x \]

\[ t = 1.00 \]
Tygers in the Galerkin-truncated 1D Burgers equation

Growth of a tyger in the solution of the inviscid Burgers equation with initial condition \( v_0(x) = \sin(x - \pi/2) \). Galerkin truncation at \( K_G = 700 \). Number of collocation points \( N = 16,384 \). Observe that the bulge appears far from the place of birth of the shock.
Tygers only at regions of positive strain

\[ u_0(x) = \sin(x) + \sin(2x + 0.9) + \sin(3x) \]

Three-mode initial condition. Tygers appear at the points having the same velocity as the shock and positive strain.
From tygers to thermalization

![Graphs showing time evolution of v(x), u(x)](image-url)
From tygers to thermalization

\[ u(x) \]

\[ \tilde{u}(x) \]

\[ t = 1.07 \]

\[ t = 1.09 \]

\[ t = 1.11 \]

\[ t = 1.13 \]

\[ t = 1.15 \]

\[ t = 1.17 \]

\[ t = 1.19 \]

\[ t = 1.50 \]
Phenomenological Explanation : 1D Burgers

- A localized strong nonlinearity, such as is present at a preshock or a shock, acts as a source of a truncation wave.
- Away from the source this truncation wave is mostly a plane wave with wavenumber close to \( K_G \).
- The radiation of truncation waves begins only at or close to the time of formation of a preshock.
- Resonant interactions are confined to particles such that 
  \[ \tau \Delta v \equiv \tau |v - v_s| \lesssim \lambda_G. \]
- If \( \tau \) is small the region of resonance will be confined to a small neighborhood of widths \( \sim K_G^{-1/3} \) around the point of resonance.
- In a region of negative strain a wave of wavenumber close to \( K_G \) will be squeezed and thus disappearing beyond the truncation horizon which acts as a kind of black hole.
Truncated 2D Euler

- Numerical integration of the truncated 2D incompressible Euler equation with random initial conditions and resolutions between $512^2$ and $8192^2$.

- Although for the untruncated solution real singularities are ruled out at any finite time, there is strong enhancement of spatial derivatives of the vorticity.

- The highest values of the Laplacian is found in the straight cigar-like structure.
2D Euler

A 2D tyger: before \((t = 0.66)\), early \((t = 0.71)\) and later \((t = 0.75)\). Figures, moderately zoomed, centered on the main cigar. Contours of the Laplacian of vorticity in red, ranging from \(-200\) to \(200\) by increments of 25, streamlines in gray, ranging from \(-1.6\) to \(1.6\) by increments of 2 and positive strain eigendirections in pink segments.
Left: zoomed version of contours of the Laplacian of vorticity at \( t = 0.71 \).
Right: plot of the Laplacian of vorticity along the horizontal segment near \( x_2 = 3 \), shown in the left panel.
Contours of the modulus of the vorticity Fourier coefficients at various times. Negative $k_1$ values not shown because of Hermitian symmetry. Contour values are $10^{-1}$, $10^{-2}$, $\ldots$, $10^{-15}$ from inner to outer (green, blue and pink highlight the values $10^{-5}$, $10^{-10}$, and $10^{-15}$, respectively). Galerkin truncation effects are visible above the rounding level already at $t = 0.49$ and become more and more invasive.
Contours of tri-Laplacian of the vorticity showing a tyger already at $t = 0.49$. 

2D Euler: Magnification of tyger effects
2D Euler: How similar is it to the Burgers equation?

- Most of these tygers appear at places which had no preexisting small-scale activity.
- The streamlines indicate that tyger activity appears at places where the velocity is roughly parallel to the central cigar.
- Considering the cigar as a one-dimensional straight object, the truncation waves generated by the cigar will have crests parallel to the cigar and those fluid particles which move parallel to the crest keep a constant phase and thus have resonant interactions with the truncation waves.
- If we now consider the one-parameter family of straight lines perpendicular to a given cigar, each such line will have some number (possibly zero) of resonance points; altogether they form the tygers.
- There are points where this kind of resonance condition holds but no tyger is seen; this can be interpreted in terms of strain.
width $\propto K_G^{-1/3}$ (using phase mixing arguments)

amplitude $\propto K_G^{-2/3}$ (using energy conservation arguments)
Scaling properties of the early tygers

- **Scaling of the tyger widths**:  
  - By the time $t_\star$, truncation is significant only for a lapse of time $O(K_G^{-2/3})$.  
  - The phase mixing argument tells us that the coherent build up of a tyger will affect only those locations whose velocity differs from that at resonance by an amount 
    \[ \Delta v \lesssim \frac{2\pi}{K_G^{-2/3}K_G} \propto K_G^{-1/3}. \]  
  - Since at such times, the velocity $v$ of the truncated solution is expected to stay close to the velocity $u$ of the untruncated solution and the latter varies linearly with $x$ near the resonance point, the width of the $t_\star$ tyger is itself proportional to $K_G^{-1/3}$.  

- **Scaling of the tyger amplitudes**:  
  - The Galerkin-truncated Burgers equation conserves energy.  
  - The apparent energy loss due to truncation 
    \[ \sim \int_0^{\lambda_G} x^{2/3} dx \sim K_G^{-5/3}. \]  
  - Conservation demands that this energy-loss is transferred to the tygers which gives the tyger-amplitude scaling as 
    \[ \propto K_G^{-2/3}. \]  
  - The above argument is appealing but not rigorous.
Weak solutions?

Plots of solution of the Galerkin-truncated Burgers equation, with $K_G = 5,461$ (green) and $K_G = 21,845$ (black), low-pass filtered at wavenumber $K = 100$, at various times. Initial condition $v_0(x) = \sin(x) + \sin(2x - 0.741)$. The untruncated solution is shown in red.
Define discrepancy $\tilde{u} \equiv v - u$ to obtain

$$\partial_t \tilde{u} + P_{KG} \partial_x \left( u \tilde{u} + \frac{\tilde{u}^2}{2} \right) = \left( I - P_{KG} \right) \partial_x \frac{u^2}{2}, \quad \tilde{u}(0) = 0.$$ 

Decompose $u = u^< + u^>$, where $u^< \equiv P_{KG} u$ and $u^> \equiv (I - P_{KG}) u$.

Similarly the perturbation $u' \equiv P_{KG} \tilde{u}$.

Finally we obtain:

$$\partial_t u' + P_{KG} \partial_x \left( u^< u' + \frac{(u')^2}{2} \right) = P_{KG} \partial_x \left( u^< u^> + \frac{(u^>)^2}{2} \right).$$
The beating input

\[ \text{Im} \hat{f} \]

\[ f \]

\[ k \]

\[ x \]
Birth of tygers: Approximations

\[
\partial_t u' + P_{K_G} \partial_x \left( u^< u' + \frac{(u')^2}{2} \right) = P_{K_G} \partial_x \left( u^< u^> + \frac{(u^>)^2}{2} \right).
\]

**Strategy:**
1. The term \((u')^2\) is discarded;
2. The perturbation \(u'\) is set to zero at time \(t_G\);
3. The untruncated solution is frozen to its \(t_\star\) value.

**With the three approximations the temporal dynamics of the perturbation near \(t_\star\) is**

\[
\frac{d}{d\tau} \hat{u}_k' = \sum_{k' = -K_G}^{K_G} A_{kk'} \hat{u}_{k'}' + \hat{f}_k, \quad \hat{u}_k'(0) = 0,
\]

\[
A_{kk'} \equiv -i k \hat{u}^<_{k_\star, k-k'},
\]

\[
\hat{f}_k \equiv i k \sum_{p+q=k} \left( \hat{u}^<_{k_\star p} \hat{u}^>_{k_\star q} + \frac{1}{2} \hat{u}^>_{k_\star p} \hat{u}^>_{k_\star q} \right).
\]
Are the approximations justified?

Log-linear plot of the compensated amplitude of the tyger $K_G^{2/3} a(K_G)$, calculated (i) from $\tilde{u}$, (ii) from the linearised approximation for $u'$, and (iii) from the freezing plus reinitialization approximation, all versus $K_G$. 
The boundary layer in Fourier space near $K_G$. Shown are the imaginary parts of $\hat{u}'(t_*)$ for three values of $K_G$. The origin is at the preshock. The even-odd oscillations indicate that most of the activity is at the tyger, a distance $\pi$ away.
The envelopes of the various boundary layers shown in earlier (with preshock contributions subtracted out), collapsed into a single curve after rescaling. Red circles: $K_G = 20,000$, blue circles: $K_G = 15,000$, red squares: $K_G = 10,000$, blue squares: $K_G = 5,000$. The thick black line is the exponential fit.
Tygers can be reduced to a problem in linear algebra.

We can solve

$$\frac{d}{d\tau} \hat{u}'_k = \sum_{k'=\pm K_G} K G \hat{u}'_{k'} + \hat{f}_k, \quad \hat{u}'_k(0) = 0$$

at time $\tau_* = t_* - t_G$:

$$u'(\tau_*) = A^{-1} \left( e^{\tau_* A} - I \right) f = \left( \sum_{n=0}^{\infty} A^n \frac{\tau_*^{n+1}}{(n+1)!} \right) f.$$

From this it becomes clear that much will be controlled by the spectral properties of the operator $A$.

We are thus led to consider the associated eigenvalue/eigenvector equation $A\psi = \lambda\psi$. 
Conclusions and Perspective

- Tygers provide a clue as to the onset of thermalization.
- We do not have a complete understanding of the phenomenon.
- Tygers do not modify shock dynamics but modify the flow elsewhere because the tygers induce Reynolds stresses on scales much larger than the Galerkin wavelength; hence the weak limit of the Galerkin-truncated solution as $K_G \to \infty$ is NOT the inviscid limit of the untruncated solution.
- There is good evidence that the key phenomena associated to tygers are also present in the two-dimensional incompressible Euler equation (and also perhaps in three dimensions).
- It is clear that complex-space singularities approaching the real domain within one Galerkin wavelength are the triggering factor in both the 2D Euler and the 1D Burgers case.
- Can we “purge tygers away” and thereby obtain a subgrid-scale method which describes the inviscid-limit solution right down to the Galerkin wavelength?
Purging?

\[ u(x) \]