Chromatic Homotopy Theory
Problem set 1a. Preliminaries on Spectra

Reminder

Recall we defined the category of **spectra** Sp to be the stabilization of the category of spaces $\mathcal{S}$, i.e.

$$\mathrm{Sp} := \varprojlim \left(\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right)$$

We then proved the following list of statements about Sp

- Sp is a stable category, i.e. the square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

  is a pushout diagram if and only if it is a pullback diagram. In particular the suspension Σ and loop Ω functors are mutual inverse of each other. It follows the homotopy category of Sp is a triangulated category with the shift functor [1] being Σ and the class of distinguished triangles $X \to Y \to Z$ consisting of pairs of maps such that

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z \end{array}$$

  is a (co-)fiber diagram.
- There is a canonical projection functor $\Omega^\infty \colon \mathrm{Sp} \to \mathcal{S}_*$, sometimes referred as the null space of a spectrum. If E^* is an extraordinary cohomology theory there exists a unique up to equivalence spectrum E such that E^n is represented by $\Omega^\infty \Sigma^n E \simeq B^n \Omega^\infty E$ in $\mathcal{S}_*$. In particular for any abelian group A there exists the **Eilenberg-MacLane spectrum** HA representing the usual cohomology with coefficients in A (and hence $\Omega^\infty \Sigma^n HA \simeq K(A, n)$). The **generalized Eilenberg-MacLane functor** is then defined as a unique continuous extension of H to the derived category of abelian groups $D(\mathrm{Ab})$.
- The functor Ω^∞ admits a left adjoint $\Sigma^\infty \colon \mathcal{S}_* \to \mathrm{Sp}$. We showed that for a pointed space X there exists a canonical equivalence $\Omega^\infty \Sigma^\infty X \simeq \varinjlim \Omega^n \Sigma^n X$. As a corollary for finite spaces X, Y the natural map

$$\varinjlim \mathrm{Hom}_{\mathcal{S}_*}(\Sigma^n X, \Sigma^n Y) \to \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X, \Sigma^\infty Y)$$

  is an equivalence. The spectra of the form Σ^∞ are called **suspension spectra**. We denote by $\mathbb{S}$ the suspension spectrum of a 0-dimensional sphere $\mathbb{S}^0$.
- For $X, Y \in \mathrm{Sp}$ we denote $[X, Y] := \pi_0 \mathrm{Hom}_{\mathrm{Sp}}(X, Y)$ and $\pi_n(X) := [\mathbb{S}^n, X]$. There is a stable Whitehead's lemma asserting that the map of spectra $f \colon X \to Y$ is an equivalence if and only if the induced map $\pi_n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.
- The category of spectra is generated under co-limits by $\mathbb{S}$. In particular any spectrum X admits a cell filtration $X_\bullet$ such that $\mathrm{gr}_s X_\bullet := X_s / X_{s-1}$ is equivalent to the direct sum of $\mathbb{S}^s$. A spectrum X is called **finite** if it lies in a smallest stable subcategory of Sp generated by the sphere spectrum, or equivalently if it admits a cell decomposition with only finitely many cells.
- The category of spectra admits a symmetric monoidal structure $\wedge \colon \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ called the **smash product** such that $\wedge$ commutes with all colimits in each argument and Σ^∞ is symmetric monoidal with respect to the smash product on $\mathcal{S}_*$ and Sp. There is also the **mapping spectrum functor** $\mathrm{Map}_{\mathrm{Sp}}(-,-)$ determined by the following property

$$\mathrm{Hom}_{\mathrm{Sp}}(X \wedge Y, Z) \simeq \mathrm{Hom}_{\mathrm{Sp}}(X, \mathrm{Map}_{\mathrm{Sp}}(Y, Z))$$

- For spectra E and X we will denote

$$E_n(X) := \pi_n(X \wedge E) \qquad E^n(X) := \pi_{-n}\operatorname{Map}_{\mathrm{Sp}}(X, E) \simeq [X, \Sigma^n E]$$

  and will call $E_*(X)$ and $E^*(X)$ E**-homology** and E**-cohomology of** X respectively. In the case $X = \mathbb{S}$ we will sometimes write just $E_n := E_n(\mathbb{S}) \simeq \pi_n(E)$ and $E^n := E^n(\mathbb{S}) \simeq \pi_{-n}(E)$.

1 First properties and examples

Problem 1.

a) Prove that $\pi_0(\mathbb{S}) \simeq \mathbb{Z}$.

b) Prove that $\pi_1(\mathbb{S}) \simeq \mathbb{Z}/2\langle\eta\rangle$, where η is the stabilization of the Hopf fibration map $\mathbb{S}^3 \to \mathbb{S}^2$. (*Hint: one way to proceed is as follows. Let the map* $\alpha\colon \mathbb{S}^3 \to K(\mathbb{Z}, 3)$ *classify generator in* $H^3(\mathbb{S}^3, \mathbb{Z})$ *and denote by* $\mathbb{S}^3_{\geq 4}$ *the fiber of* α*. Using Hochschild-Serre spectral sequence for an induced fiber sequence* $\mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2) \to \mathbb{S}^3_{\geq 4} \to \mathbb{S}^3$ *compute that* $H^4(\mathbb{S}^3_{\geq 4}, \mathbb{Z}) \simeq \mathbb{Z}/2$*. Then use Hurewicz isomorphism and Freudenthal suspension theorems.*)

c) Let X be a finite spectrum. Prove that $\pi_i(X)$ are finitely generated abelian groups for all $i \in \mathbb{Z}$.

Problem 2. Let X be a space. Prove that

$$\Sigma^\infty X_+ \wedge H\mathbb{Z} \simeq C_*(X, \mathbb{Z}) \qquad \operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ X, H\mathbb{Z}) \simeq C^*(X, \mathbb{Z})$$

where $C_*(X, \mathbb{Z})$ and $C^*(X, \mathbb{Z})$ are chain and cochain complexes of X respectively considered as a spectrum via the generalized Eilenberg-MacLane spectrum functor.

Problem 3. Recall that a non-empty category I is called **filtered** if every finite diagram in I has a cone.

a) Let $\{X_i\}_{i\in I}$ be a filtered diagram of spectra. Prove that for all $n \in \mathbb{Z}$ the canonical map

$$\varinjlim \pi_n(X_i) \to \pi_n(\varinjlim X_i)$$

   is an isomorphism.

b) Deduce that

$$H\mathbb{Q} \simeq \varinjlim(\mathbb{S} \xrightarrow{\cdot 2!} \mathbb{S} \xrightarrow{\cdot 3!} \mathbb{S} \xrightarrow{\cdot 4!} \dots)$$

   Hint: You may use that by the rational homotopy theory

$$\pi_i(\mathbb{S}^{2n+1}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \begin{cases} \mathbb{Q}, & i = 2n+1 \\ 0, & i \neq 2n+1 \end{cases}$$

c) Deduce that for any spectrum X the canonical map $\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_*(X, \mathbb{Q})$ is an isomorphism.

d) Prove that $H\mathbb{Q} \wedge H\mathbb{Q} \simeq H\mathbb{Q}$.

Problem 4. Let $A_\bullet$

$$\dots \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1 \xrightarrow{f_0} A_0$$

be a tower of abelian groups. Consider the map $\alpha\colon \prod_i A_i \to \prod_i A_i$ defined by $\alpha(\{a_i\}) := \{a_i - f_i(a_{i+1})\}$. Set $\varprojlim^1 A_\bullet := \operatorname{coker}(\alpha)$.

a) Let $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ be a short exact sequence of towers of abelian group. Prove that there exists a long exact $\varprojlim$-$\varprojlim^1$ sequence

$$\begin{aligned} 0 \to \varprojlim A_\bullet \to \varprojlim B_\bullet \to \varprojlim C_\bullet \to \\ \to \varprojlim{}^1 A_\bullet \to \varprojlim{}^1 B_\bullet \to \varprojlim{}^1 C_\bullet \to 0 \end{aligned}$$

b) Prove that if all maps f_i are surjective, then $\varprojlim^1 A_\bullet$ vanish.

c) (Milnor exact sequence) Let $\{X_i\}_{i\in I}$ be a tower of spectra. Prove that for each $n \in \mathbb{Z}$ there exists a short exact sequence

$$0 \to \varprojlim^1 \pi_{n+1}(X_i) \to \pi_n(\varprojlim X_i) \to \varprojlim \pi_n(X_i) \to 0$$

Problem 5.

a) Let X, Y be a pair of connective spectra. Prove that $\pi_0(X \wedge Y) \simeq \pi_0(X) \otimes_{\mathbb{Z}} \pi_0(Y)$. (*Hint: prove that for a connective spectrum Z homotopy group $\pi_0(Z)$ depends only on 1-skeleton*).

b) (Stable Hurewicz isomorphism) Deduce that for a spectrum X, and n minimal such that $\pi_n(X) \not\simeq 0$ the Hurewicz map $X \simeq X \wedge \mathbb{S} \to X \wedge H\mathbb{Z}$ induces an isomorphism $\pi_n(X) \simeq H_n(X, \mathbb{Z})$. By taking $X := \Sigma^\infty Y$ of a pointed simply connected space Y deduce the usual Hurewicz theorem.

c) Prove that the map of eventually connective spectra $X \to Y$ is an equivalence if and only if it induces an isomorphism in integral homology.

Problem 6. (Moore spectra)

a) Let A be an abelian group. Prove there exists a unique (up to equivalence) connective spectrum $\mathbb{S}A$ such that

$$H_n(\mathbb{S}A, \mathbb{Z}) \simeq \begin{cases} A, & n = 0 \\ 0, & n > 0 \end{cases}$$

This $\mathbb{S}A$ is called the **Moore spectrum of** A.

b) (π_*-universal coefficient formula) Let A be an abelian group and X be a spectrum. For each $i \in \mathbb{Z}$ prove that homotopy group $\pi_i(X \wedge \mathbb{S}A)$ fits into a short exact sequence

$$0 \to \pi_i(X) \otimes_{\mathbb{Z}} A \to \pi_i(X \wedge \mathbb{S}A) \to \mathrm{Tor}_1^{\mathbb{Z}}(\pi_{i-1}(X), A) \to 0$$

c) Compute $\mathbb{S}\mathbb{Z}$, $\mathbb{S}\mathbb{Z}[\frac{1}{p}]$ and $\mathbb{S}\mathbb{Q}$.

Problem 7.

a) Let $\mathrm{Sp}_{\mathbb{Q}}$ denote the full subcategory of Sp on spectra X such that $\pi_i(X)$ is a $\mathbb{Q}$-vector space for all $i \in \mathbb{Z}$. Prove that the generalized Eilenberg-MacLane functor induces an equivalence

$$D(\mathrm{Vect}_{\mathbb{Q}}) \xrightarrow[\sim]{H} \mathrm{Sp}_{\mathbb{Q}}$$

where $D(\mathrm{Vect}_{\mathbb{Q}})$ denotes the derived category of the category of $\mathbb{Q}$-vector spaces.

b) Deduce that for any spectrum X there is a splitting

$$X \wedge H\mathbb{Q} \simeq \bigoplus_{n\in\mathbb{Z}} \Sigma^n H(\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q})$$

Problem 8. (Spainer-Whitehead category) The **Spainer-Whitehead category** $\mathcal{SW}$ is defined as follows

- An object of $\mathcal{SW}$ is a pair (X, n), where X is a pointed space and $n \in \mathbb{Z}$.
- For $(X, n), (Y, m) \in \mathcal{SW}$ the set of morphisms is defined to be

$$\mathrm{Hom}_{\mathcal{SW}}((X, n), (Y, m)) := \operatorname*{colim}_{i\to\infty} [\Sigma^{n+i} X, \Sigma^{m+i} Y]$$

In this problem You will study the relation between $\mathcal{SW}$ and Sp.

a) Prove that for a finite pointed space X and any pointed space Y the natural map

$$\varinjlim \mathrm{Hom}_{\mathcal{S}_*}(\Sigma^n X, \Sigma^n Y) \to \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X, \Sigma^\infty Y)$$

is an equivalence.

b) Let X be a finite spectrum. Prove that there exists a finite pointed space Y such that $\Sigma^n X \simeq \Sigma^\infty Y$ for some $n \in \mathbb{Z}_{\geq 0}$. Deduce that the homotopy category of finite spectra is equivalent to the full subcategory of $\mathcal{SW}$ on objects (X, n) such that X is a finite space.

c) Prove that the functor $\mathcal{S}_* \to \mathcal{SW}, X \mapsto (X, 0)$ does not commute with infinite coproducts. Deduce that for a pair of pointed spaces X, Y the natural map

$$\mathrm{Hom}_{\mathcal{SW}}((X,0),(Y,0)) \to [\Sigma^\infty X, \Sigma^\infty Y]$$

is not an isomorphism in general.

Problem 9.

a) Let X be a finite spectrum such that $X \wedge H\mathbb{Q} \simeq 0$. Prove that there exists an integer $n \in \mathbb{Z}$ such that $\pi_*(X)$ is n-torsion. *(Hint: first prove the statement for* $\mathrm{Map}_{\mathrm{Sp}}(X,X)$*).*

b) Prove that $\pi_*(\mathbb{S}/p)$ is at most p^2-torsion.

2 t-structure on the category of spectra and Postnikov towers

Recall we proved that the category of spectra admits a canonical t-structure such that $\mathrm{Sp}_{\geq n}$ and $\mathrm{Sp}_{\leq n}$ are full subcategories of n-**connective** and n-**coconnective** (or n-**truncated**) spectra respectively, i.e.

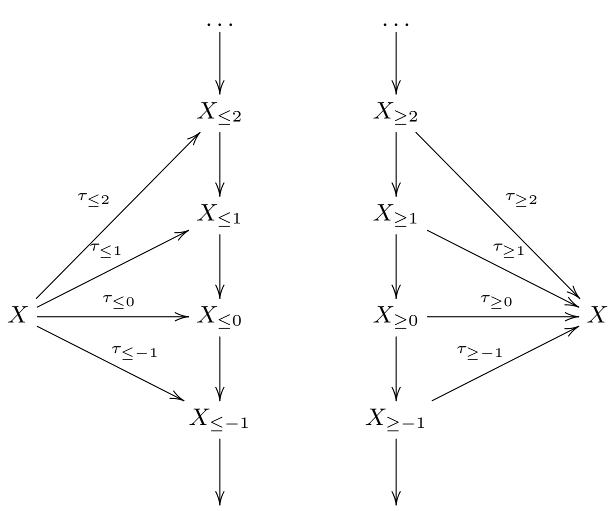
$$\mathrm{Sp}_{\geq n} = \{X \in \mathrm{Sp} \mid \pi_i(X) \simeq 0 \text{ for } i < n\} \qquad \mathrm{Sp}_{\leq n} = \{X \in \mathrm{Sp} \mid \pi_i(X) \simeq 0 \text{ for } i > n\}$$

and $[\mathrm{Sp}_{\geq n}, \mathrm{Sp}_{<n}] \simeq 0$. The inclusion of n-truncated spectra admits a left adjoint $\tau_{\leq n}$ and the inclusion of n-connective spectra admits a right adjoint $\tau_{\geq 0}$. In particular for each spectrum X there is a functorial fiber sequence

$$X_{>n} \to X \to X_{\leq n}$$

where $X_{>n} := \tau_{\geq n}(X) \in \mathrm{Sp}_{>n}$ is an n-connected cover of X and $X_{\leq n} := \tau_{\leq n}(X)$ is its n-th truncation. We proved that $\mathrm{Sp}^{\heartsuit} := \mathrm{Sp}_{\leq 0} \bigcap \mathrm{Sp}_{\geq 0} \simeq \mathrm{Ab}$ and the natural map $D(\mathrm{Ab}) \to \mathrm{Sp}$ is given by the generalized Eilenberg-MacLane functor. We also define the full subcategory of **eventually connective spectra** $\mathrm{Sp}_{>-\infty}$ to be the union $\bigcup_{n \in \mathbb{Z}} \mathrm{Sp}_{\geq n}$.

The natural towers of truncation and connective covers associated with the t-structure on Sp



are called **Postnikov** and **Whitehead towers** respectively. As a consequence of the Whitehead's lemma we obtain

Corollary 2.1. *Let X be a spectrum. Then Postnikov tower converges and Whitehead filtration is exhaustive, i.e. the natural maps*

$$X \xrightarrow{\sim} \lim_{n\to\infty} X_{\le n} \qquad \operatorname*{colim}_{n\to -\infty} X_{\ge n} \xrightarrow{\sim} X$$

are equivalencies.

Definition 2.2 (k-invariants)**.** Let X be a spectrum. Then there is a natural fiber sequence

$$X_{\le n} \to X_{\le n-1} \xrightarrow{k_n} \Sigma^{n+1} H\pi_n(X)$$

showing that $X_{\le n}$ can be restored from $X_{\le n-1}$ and the so-called k-**invariant**, a class $\widetilde{k_n} \in H^{n+1}(X_{\le n-1}, \pi_n(X))$ classifying a map k_n. Note that since Postnikov tower converges, the sequence of k-invariants completely determine eventually connective spectrum X.

Problem 10. Let A, B be pair of abelian groups. Prove that for $i \in \{0, 1\}$

$$[HA, \Sigma^i HB] \simeq \operatorname{Ext}^i_{\mathrm{Ab}}(A, B)$$

Problem 11. Let $ku := KU_{\ge 0}$ be a connective cover of the complex K-theory spectrum KU. Let us denote the Bott element (generator of $\pi_2(ku) \simeq \mathbb{Z}$) by β.

a) Prove that $KU_{\ge 2n} \simeq \Sigma^{2n} ku$ and that under this identification the maps in the Whitehead tower

$$\Sigma^{2(n+1)} ku \simeq KU_{\ge 2(n+1)} \to KU_{\ge 2n} \simeq \Sigma^{2n} ku$$

are given by multiplication by β.

b) Prove that the first nontrivial k-invariant for ku, $k_2 \colon H\mathbb{Z} \to \Sigma^3 H\mathbb{Z}$ is given by the third integral Steenrod square $\mathrm{Sq}^3_{\mathbb{Z}}$, which is defined to be the following composite

$$H\mathbb{Z} \to H\mathbb{Z}/2 \xrightarrow{\mathrm{Sq}^2} \Sigma^2 H\mathbb{Z}/2 \xrightarrow{\beta_2} \Sigma^3 H\mathbb{Z}$$

Problem 12. (Atiyah-Hirzebruch spectral sequence)

a) Let E be a spectrum. Prove there exists a functorial in $X \in \mathrm{Sp}$ cohomological spectral sequence

$$E_2^{p,q} := H^p(X, \pi_{-q}(E)) \;\Rightarrow\; E^{p+q}(X), \qquad |d_r^{*,*}| = (r, -(r-1))$$

such that

- All pages $(E_r^{*,*}, d_r^{*,*})$ are functors from Sp to the category of graded complexes of $\mathcal{A}_E^*$-modules, where $\mathcal{A}_E^* := \pi_{-*} \operatorname{Map}_{\mathrm{Sp}}(E, E)$ is the E-Steenrod algebra. In particular for all r, p the differential $d_r^{p,*} \colon E_r^{p,*} \to E_r^{p+r,*-(r-1)}$ is a natural transformation of functors from Sp to $\mathrm{Mod}_{\mathcal{A}_E^*}$.
- The action of $\mathcal{A}_E^*$ on $E_{r+1}^{*,*}$ is induced from the action on $E_r^{*,*}$. The action on $E_2^{*,*} \simeq H^*(X, \pi_{-*}(E))$ is induced from the action of $\mathcal{A}_E^*$ on $\pi_{-*}(E)$.

(*Hint: consider filtration on* $\operatorname{Map}_{\mathrm{Sp}}(X, E)$ *induced by the Postnikov filtration on* E.)

b) Prove that all differentials $d_r^{p,q}$ are torsion, i.e. $nd_r^{p,q} = 0$ for some $n \in \mathbb{Z}$.

c) Let $E = KU_{(p)}$ be a p-local complex K-theory spectrum. Prove that in the corresponding Atiyah-Hirzebruch spectral sequence all differentials $d_r^{*,*}$ vanish for $r < p$ and that $pd_p^{*,*} = 0$. (*Hint: use that the Adams operations* ψ_k *act on* $(E_r^{*,*}, d_r^{*,*})$).

d) Let $E = KU$ a complex K-theory spectrum. Prove that in the corresponding Atiyah-Hirzebruch spectral sequence there is an isomorphism of graded algebras

$$E_3^{*,*} \simeq E_2^{*,*} \simeq H^*(X, \mathbb{Z})[\beta^{\pm 1}], \qquad |\beta| = (0, -2), |x| = (p, 0) \text{ for } x \in H^p(X, \mathbb{Z})$$

and differential d_3 is determined by $d_3(\beta) = 0$, $d_3(x) = \mathrm{Sq}^3_{\mathbb{Z}}(x)\beta$ for $x \in H^*(X, \mathbb{Z})$.

3 Dualities in Sp

Definition 3.1. Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a symmetric monoidal category. Object X of $\mathcal{C}$ is called

- **dualizable** if there exists an object $X^\vee$ and a pair of morphisms $\mathbb{I} \xrightarrow{\text{coev}} X \otimes X^\vee$, $X^\vee \otimes X \xrightarrow{\text{ev}} \mathbb{I}$ such that the composites

$$X \simeq \mathbb{I} \otimes X \xrightarrow{\text{coev} \otimes 1_X} X \otimes X^\vee \otimes X \xrightarrow{1_X \otimes \text{ev}} X \otimes \mathbb{I} \simeq X$$

$$X^\vee \simeq X^\vee \otimes \mathbb{I} \xrightarrow{1_{X^\vee} \otimes \text{coev}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\text{ev} \otimes 1_{X^\vee}} \mathbb{I} \otimes X \simeq X^\vee$$

  are identity morphisms.
- **invertible** if there exists an object $X^{-1} \in \mathcal{C}$ such that $X \otimes X^{-1} \simeq \mathbb{I}$.

If $X^\vee$ or X^{-1} exist they unique up to equivalence. In the case $\mathcal{C} = \text{Sp}$ the spectrum $X^\vee$ is called the **Spainer-Whithead dual of** X.

Finally, if $f \colon X \to X$ is an endomorphism of a dualizable object of $\mathcal{C}$ the **trace** $\text{tr}(f) \in \text{End}_{\mathcal{C}}(\mathbb{I})$ **of** f is defined as the following composite

$$\mathbb{I} \xrightarrow{\text{coev}} X \otimes X^\vee \xrightarrow{f \otimes 1_{X^\vee}} X \otimes X^\vee \simeq X^\vee \otimes X \xrightarrow{\text{ev}} \mathbb{I}$$

Problem 13.

a) Let $\mathcal{C}$ be a symmetric monoidal category. Prove that the following conditions are equivalent

- The monoidal unit $\mathbb{I}_{\mathcal{C}}$ is compact.
- Every dualizable object of $\mathcal{C}$ is compact.

b) Let X be a spectrum. Prove that the following conditions are equivalent

- X is compact.
- X is a retract of a finite spectrum.
- X is dualizable.

c) Let X be a spectrum. The functor $- \wedge X$ being a left adjoint of $\text{Map}_{\text{Sp}}(X, -)$ commutes with all colimits but not with limits in general: prove that the functor $- \wedge X$ commutes with limits if and only if X is dualizable.

Problem 14. Prove that $X \in \text{Sp}$ is invertible if and only if $X \simeq \mathbb{S}^n$ for some $n \in \mathbb{Z}$.

Problem 15. Let X be a finite CW-complex. Prove that

a) The suspension spectrum $\Sigma^\infty_+ X$ is dualizble.

b) For an abelian group A there exists a canonical equivalence

$$(\Sigma^\infty_+ X)^\vee \wedge HA \simeq C^*(X, A)$$

where $C^*(X, A)$ denotes a cochain complex of X with coefficients in A considered as a spectrum via the generalized Eilenberg-MacLane spectrum functor.

c) For an endomorphism $f \colon X \to X$ define **the Lefschetz number of** f to be

$$L(f) := \sum_i (-1)^i \, \text{tr}\, H_i(f)$$

Prove that

$$L(f) = \text{tr}_{\text{Sp}} \left(\Sigma^\infty_+ X \xrightarrow{\Sigma^\infty_+(f)} \Sigma^\infty_+ X \right) \quad \text{in particular} \quad \chi(X) = \text{tr}_{\text{Sp}}(1_{\Sigma^\infty_+ X})$$

Problem 16. Consider the functor

$$I^*\colon \mathrm{Sp} \to \mathrm{Ab}^\bullet \qquad X \mapsto \mathrm{Hom}^*_{\mathrm{Ab}}(\pi_*(X), \mathbb{Q}/\mathbb{Z})$$

a) Prove that I^* is a generalized cohomology theory.

   By Brown representability I^* is represented by some $I \in \mathrm{Sp}$ which is called **Brown-Comenetz spectrum**.

b) For a spectrum E define **Brown-Comenetz dual** IE **of** E to be $\mathrm{Map}_{\mathrm{Sp}}(E, I)$. Prove that there exists a functorial in $E, X \in \mathrm{Sp}$ isomorphism

$$(IE)^*(X) \simeq \mathrm{Hom}^*(E_*(X), \mathbb{Q}/\mathbb{Z})$$

c) Prove that the canonical map $X \to I^2(X) := I(I(X))$ is an equivalence if and only if all $\pi_i(X)$ are finite.

d) Compute I^2HA for finitely generated abelian group A and $I^2\mathbb{S}$.