The Kervaire Invariant One Problem, Talk 5
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The main aim of this lecture is to prove the Browder theorem. The main reference is W. Browder's paper [Bro69].

1 Introduction

At the previous lecture we have defined the Kervaire invariant of a framed manifold of dimension $2q$. Recall that this invariant is a bordism invariant, so it defines a homomorphism:

$$c\colon \Omega_{2q}^{fr} \to \mathbb{Z}/2.$$

Here Ω_{2q}^{fr} is the group of framed cobordisms of dimension $2q$ and by the Pontryagin-Thom theorem this group is equal to $\pi_{2q}^s(S^0)$.

The Kervaire Invariant One Problem asks for what q the homomorphism c is surjective. The Browder theorem is a milestone in the solution of this problem. In order to formulate theorem we need to introduce some notation.

Notation. Denote by $\mathcal{A}_2$ the Steenrod algebra mod 2. Recall that there exists the Adams spectral sequence:

$$E^2_{s,t} = \mathrm{Ext}^{\mathcal{A}_2}_{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow {}_2\pi^s_{t-s}(S^0),$$

which converges to the 2-primary component of stable homotopy groups $\pi_*^s(S^0)$. Let us to describe two first rows at the second page of this spectral sequence.

Recall that $\mathcal{A}_2$ is a Hopf algebra, so its dual is also a Hopf algebra. Denote by $\mathrm{Prim}(\mathcal{A}_2^*)$ the vector space of primitive elements of $\mathcal{A}_2^*$. Now it is not hard to see (for instance, the proof can be found in [Swi75, Proposition 19.17]) that

$$\mathrm{Ext}^{\mathcal{A}_2}_{1,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Prim}(\mathcal{A}_2^*),$$

The vector space $\mathrm{Prim}(\mathcal{A}_2^*)$ has the basis $\langle h_0, h_1, \ldots, h_i, \ldots\rangle$, such that for any $i, j \in \mathbb{N}$

$$h_i(\mathrm{Sq}^{2^j}) = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

Moreover, any h_i is evaluated by zero on decomposable elements in $\mathcal{A}_2$. All in all,

$$\mathrm{Ext}^{\mathcal{A}_2}_{1,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\langle h_0, \ldots, h_i, \ldots \mid h_i \in \mathrm{Ext}^{\mathcal{A}_2}_{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)\rangle$$

There exists the Yoneda multiplication on Ext-groups and the group $\mathrm{Ext}^{\mathcal{A}_2}_{2,*}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by products. More precisely [Ada60],

$$\mathrm{Ext}^{\mathcal{A}_2}_{2,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\langle h_i h_j, \mid i \geq j,\, i \neq j+1\rangle$$

Finally, we can formulate the Browder theorem. Recall that a class $\alpha \in E^p_{*,*}$ at some finite page is called a permanent cycle if α persists to E^∞.

Theorem (Browder)**.** *1) There is no a framed closed manifold of dimension $2q$ with the Kervaire invariant one, if $q+1$ is not a power of two.*

2) If $q = 2^i - 1$, then a such manifold (i.e framed, closed and dimension $2q$) exists if and only if the class h_i^2 is a permanent cycle in the Adams spectral sequence.

Remark. Form of this theorem is very close to the Adams observation on the Hopf invariant problem. He obtained that there exists an element $\alpha \in \pi_{2n-1}(S^n)$ with the Hopf invariant one if and only if $n = 2^i$ and h_i is a permanent cycle.

2 Plan of the proof

Let M be a closed smooth manifold of dimension $2q$. Denote by ν the stable normal bundle of M and denote by $T(\nu)$ the Thom spectrum of ν.

In Lecture 4 we defined the Browder spectrum $MO\langle v_{q+1}\rangle$ and by any mock $MO\langle v_{q+1}\rangle$-orientation $\eta\colon T(\nu) \to MO\langle v_{q+1}\rangle$ we constructed the quadratic refinement of the intersection form restricted to $(\ker(D\eta)^*)^q$. We also proved that if η comes from a framing of M, then we have the equality $(\ker(D\eta)^*)^q = H^q(M, \mathbb{Z}/2)$. Denote by $\operatorname{Im}\Omega_{2q}^{fr}$ the image of Ω_{2q}^{fr} under natural map

$$\varepsilon\colon \Omega_{2q}^{fr} = \pi_{2q}(S) \to \pi_{2q}(MO\langle v_{q+1}\rangle) = MO_{2q}\langle v_{q+1}\rangle.$$

Define the map $\tilde{c}\colon \operatorname{Im}\Omega_{2q}^{fr} \to \mathbb{Z}/2$ by the rule: $\tilde{c}(M,\eta) = c(M,\eta,H^q(M,\mathbb{Z}/2))$. Then the Kervaire invariant factors as

$$\begin{tikzcd} \Omega_{2q}^{fr} \arrow[r, "c"] \arrow[d, "\varepsilon"] & \mathbb{Z}/2 \\ \operatorname{Im}\Omega_{2q}^{fr} \arrow[ur, "\tilde{c}"'] & \end{tikzcd}$$

Here $\tilde{c}$ is the Kervaire invariant of a v_{q+1}-oriented manifold.

The Browder's idea was to bound the order of $\operatorname{Im}\Omega_{2q}^{fr}$. In order to do this, he first considered the "generalized Whitehead tower" of $MO\langle v_{q+1}\rangle$.

Theorem 1. *There exists the diagram*

$$\begin{tikzcd} MO\langle v_{q+1}\rangle \arrow[d, "G_0"] & MO\langle v_{q+1}\rangle^{(1)} \arrow[l, "F_0"'] \arrow[d, "G_1"] & MO\langle v_{q+1}\rangle^{(2)} \arrow[l, "F_1"'] \arrow[d, "G_2"] & MO\langle v_{q+1}\rangle^{(3)} \arrow[l, "F_2"'] \\ MO & \Sigma^{-1}MO \wedge K_{q+1} & \Sigma^{2q}\, H\mathbb{Z}/2 & \end{tikzcd}$$

Here

1) the map $G_0\colon MO\langle v_{q+1}\rangle \to MO$ is the natural map of "forgetting of v_{q+1}-orientation";

2) sequences

$$MO\langle v_{q+1}\rangle^{(1)} \xrightarrow{F_0} MO\langle v_{q+1}\rangle \xrightarrow{G_0} MO,$$

$$MO\langle v_{q+1}\rangle^{(2)} \xrightarrow{F_1} MO\langle v_{q+1}\rangle^{(1)} \xrightarrow{G_1} \Sigma^{-1}MO \wedge K_{q+1},$$

$$MO\langle v_{q+1}\rangle^{(3)} \xrightarrow{F_2} MO\langle v_{q+1}\rangle^{(2)} \xrightarrow{G_2} \Sigma^{2q}\, H\mathbb{Z}/2$$

are fiber sequences;

3) the spectrum $MO\langle v_{q+1}\rangle^{(3)}$ is $2q$-connected.

Notice that with such diagram we can associate two maps:

$$k_1 \colon MO \xrightarrow{\delta} \Sigma^1 MO\langle v_{q+1}\rangle^{(1)} \xrightarrow{\Sigma G_1} MO \wedge K_{q+1},$$

and

$$k_2 \colon MO \wedge K_{q+1} \xrightarrow{\Sigma\delta} MO\langle v_{q+1}\rangle^{(2)} \xrightarrow{\Sigma^2 G_2} \Sigma^{2q+2}\, H\mathbb{Z}/2\,.$$

Moreover, the composition $k_2 \circ k_1$ is trivial, so it defines the secondary cohomological operation

$$\Phi \colon S_{MO} \to T_{\Sigma^{2q+1}\, H\mathbb{Z}/2}.$$

Now consider the morphism $f \colon S^{2q} \to S^0$. We are interested in the map $\varepsilon \circ f$. Since any framed manifold is a non-oriented boundary, the composition $G_0 \circ (\varepsilon \circ f)$ is trivial, so there exists a map $\tilde{\varepsilon} \colon \mathrm{Cone}(f) \to MO$ such that the following diagram is commutative:

$$\begin{tikzcd}
S^{2q} \arrow[r, "f"] & S^0 \arrow[r, "\varepsilon"] \arrow[d, "d"] & MO\langle v_{q+1}\rangle \arrow[r, "G_0"] & MO \\
& \mathrm{Cone}(f) \arrow[urr, dashed, "\tilde{\varepsilon}"'] & &
\end{tikzcd}$$

The map $\tilde{\varepsilon}$ is non-trivial because the composition $\tilde{\varepsilon} \circ d = G_0 \circ \varepsilon$ is non-trivial.

Proposition 1. *The composition $\varepsilon \circ f$ is non-trivial if and only if the following dichotomy is satisfied*

1) the composition $k_1 \circ \tilde{\varepsilon}$ is non-trivial;

2) or $\Phi(\tilde{\varepsilon})$ is defined (i.e the composition $k_1 \circ \tilde{\varepsilon}$ is trivial) and $\Phi(\tilde{\varepsilon})$ is non-trivial.

Proof. Exercise. Hint: suppose that $\overline{\varepsilon \circ f}$ is a lifting of $\varepsilon \circ f$ along F_0. Consider the diagram:

$$\begin{tikzcd}
S^{2q} \arrow[r, "f"] \arrow[d, "\overline{\varepsilon \circ f}"] & S^0 \arrow[r, "d"] \arrow[d, "\varepsilon"] & \mathrm{Cone}(f) \arrow[d, "\bar{\varepsilon}"] \\
MO\langle v_{q+1}\rangle^{(1)} \arrow[r, "F_0"] \arrow[d, "G_1"] & MO\langle v_{q+1}\rangle \arrow[r, "G_0"] \arrow[d] & MO \arrow[d, "k_1"] \\
\Sigma^{-1} MO \wedge K_{q+1} \arrow[r] \arrow[d, "k_2"] & * \arrow[r] \arrow[d] & MO \wedge K_{q+1} \arrow[d, "k_2"] \\
\Sigma^{2q+1}\, H\mathbb{Z}/2 \arrow[r] & * \arrow[r] & \Sigma^{2q+2}\, H\mathbb{Z}/2\,.
\end{tikzcd}$$

□

Proposition 2. *Spectra MO and $MO \wedge K_{q+1}$ are sums of Eilenberg-Maclane spectra.*

Proof. The spectrum MO is an Eilenberg-Maclane spectrum by the Thom theorem [Swi75, Theorem 20.8]. Since the smash product of any spectrum with an Eilenberg-Maclane spectrum is a sum of Eilenberg-Maclane spectra, the spectrum $MO \wedge K_{q+1}$ is also an Eilenberg-Maclane spectrum. □

Now notice that the map k_1 acts between Eilenberg-Maclane spectra, therefore k_1 is a sum of Steenrod squares. But any Steenrod square acts by zero on $H^*(\mathrm{Cone}(f), \mathbb{Z}/2)$. So the composition $k_1 \circ \varepsilon$ is trivial and we can only study the secondary cohomological operation Φ.

Since all spectra MO, $MO \wedge K_{q+1}$ and $\Sigma^{2q+2} H\mathbb{Z}/2$ are sums of Eilenberg-Maclane spectra, the secondary cohomological operation Φ is the sum of the secondary cohomological operations Φ_j, such that each Φ_j based on the relation of the form:

$$\sum_i \mathrm{Sq}^{a_i}\mathrm{Sq}^{b_i} = 0.$$

By Proposition 1, we are interested in conditions when such operation can detect an element in $\pi^s_{2q}(S^0)$.

Theorem (Adams, [Ada60])**.** *The secondary cohomology operation based on the relation* $\sum \mathrm{Sq}^{a_i}\mathrm{Sq}^{b_i} = 0$ *detects an element in* $\pi^s_*(S^0)$ *if and only if there exists a permanent cycle* $h_j h_k \in E^2_{2,*}$ *in the Adams spectral sequence, such that the experesion*

$$h_j h_k\Big(\sum_i \mathrm{Sq}^{a_i}\mathrm{Sq}^{b_i}\Big) := \sum_i h_j(\mathrm{Sq}^{a_i})h_k(\mathrm{Sq}^{b_i})$$

is equal to one.

After the computation of maps k_1 and k_2 and applying the Adams theorem we can prove the following statement.

Theorem 2. *1) If* $q+1$ *is not a power of two, then* $\mathrm{Im}\,\Omega^{fr}_{2q} = 0$*;*

2) If $q+1 = 2^i$ *and* h_i^2 *is not a permanent cycle, then* $\mathrm{Im}\,\Omega^{fr}_{2q} = 0$*;*

3) If $q+1 = 2^i$ *and* h_i^2 *is a permanent cycle, then* $\mathrm{Im}\,\Omega^{fr}_{2q} = \mathbb{Z}/2$*.*

Moreover, in the last case the generator of $\mathrm{Im}\,\Omega^{fr}_{2q}$ *is cobordant to* $S^q \times S^q$ *with the* v_{q+1}*-orientation* η *such that* $\tilde{c}(S^q \times S^q, \eta) = 1$ *(see subsection 2.3 in Lecture 4).*

Evidently, Theorem 2 proves the Browder theorem.

3 Proof of Theorem 1

Let us construct the map $\Sigma G_1 \colon MO/MO\langle v_{q+1}\rangle \to MO \wedge K_{q+1}$.

Lemma 1. *There exists a map* $h \colon MO/MO\langle v_{q+1}\rangle \to MO \wedge K_{q+1}$ *such that* h^* *induces an isomorphism on* $H^i(-, \mathbb{Z}/2)$ *for* $i \le 2q+1$ *and* $(\ker h^*)^{2q+2}$ *generated by one element* β*.*

Proof. Denote by B the space BO, by E the space $BO\langle v_{q+1}\rangle$, and denote by $\pi \colon E \to B$ the embedding of the fiber. Consider the homotopy colimit $\widehat{E}$ of the diagram:

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "\pi"] & B \arrow[d, "\sigma"] \\ B \arrow[r, "\sigma"] & \widehat{E}. \end{tikzcd}$$

There exists the natural map $\hat{\pi} \colon \widehat{E} \to B$ such that the following diagram is commutative:

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[d, "\pi"] & B \arrow[d, "\sigma"] \arrow[ddr, "\mathrm{id}"] & \\ B \arrow[r, "\sigma"] \arrow[drr, "\mathrm{id}"'] & \widehat{E} \arrow[dr, dashed, "\hat{\pi}"] & \\ & & B \end{tikzcd}$$

Then the map σ is a homotopy section $\hat{\pi}$ and $\mathrm{Fib}(\hat{\pi}) \cong \Sigma \,\mathrm{Fib}(\pi) \cong \Sigma K_q$. Denote by γ the universal vector bundle over $B = BO$. Set $\bar{\gamma} := \pi^*(\gamma)$ and $\hat{\gamma} := \hat{\pi}^*(\gamma)$. The following diagram of Thom spectra is commutative and cocartesian:

$$\begin{tikzcd} T(\bar{\gamma}) \arrow[r, "T\pi"] \arrow[d, "T\pi"'] & T(\gamma) \arrow[d, "T\sigma"] \\ T(\gamma) \arrow[r, "T\sigma"] & T(\hat{\gamma}). \end{tikzcd}$$

Recall that by definition $T(\gamma) = MO$ and $T(\bar{\gamma}) = MO\langle v_{q+1}\rangle$. So we have the homotopy equivalence

$$MO/MO\langle v_{q+1}\rangle \cong \mathrm{Cone}(T\sigma) = T(\hat{\gamma})/T(\gamma).$$

It means that it is enough to construct a map h from $T(\hat{\gamma})/T(\gamma)$ to $MO \wedge K_{q+1}$.

Denote by $j\colon \Sigma K_q \to \widehat{E}$ the embedding of the fiber of tha map $\hat{\pi}$. Since the map $\hat{\pi}\colon \widehat{E} \to B$ has the homotopy section $\sigma\colon B \to \widehat{E}$, there exists an element $g \in H^{q+1}(\widehat{E}, \mathbb{Z}/2) = [\widehat{E}, K_{q+1}]$ such that $j^*(g) = \Sigma(\iota_q)$ and $\hat{\pi}^*(g) = 0$. Consider the map $H\colon \widehat{E} \to K_{q+1} \times B$ which is defined by the rule $H(x) = (g(x), \hat{\pi}(x))$. Then there exists the commutative diagram

$$\begin{tikzcd} \Sigma K_q \arrow[r, "j"] \arrow[d, "R"'] & \widehat{E} \arrow[r, bend left, "\hat{\pi}"] \arrow[d, "H"'] & B \arrow[l, bend left, "\sigma"] \arrow[d, "\mathrm{id}"] \\ K_{q+1} \arrow[r, "i_1"] & K_{q+1} \times B \arrow[r, bend left, "p_2"] & B. \arrow[l, bend left, "i_2"] \end{tikzcd}$$

Here rows are fiber sequences, $p_2\colon K_{q+1} \times B \to B$ is the projection map, i_1 and i_2 are standard embeddings, and the map $R\colon \Sigma K_q \to K_{q+1}$ represents $\Sigma(\iota_q) \in H^{q+1}(\Sigma K_q, \mathbb{Z}/2)$. Well-known that the map R induces an isomorphism on $H^i(-, \mathbb{Z}/2)$ for $i \le 2q+1$ and $(\ker R^*)^{2q+2}$ is generated by $\mathrm{Sq}^{q+1}(\iota_{q+1})$. By Serre spectral sequences for maps $\hat{\pi}$ and i_2 the map H induces an isomorphism on $H^i(-, \mathbb{Z}/2)$ for $i \le 2q+1$ and $(\ker H^*)^{2q+2}$ is generated by the cohomological class α such that $i_1^*(\alpha) = \mathrm{Sq}^{q+1}(\iota_{q+1})$.

Now consider the diagram of Thom spectra:

$$\begin{tikzcd} \Sigma^\infty(\Sigma K_q)_+ \arrow[r, "\bar{j}"] \arrow[d, "R"'] & T(\hat{\gamma}) \arrow[r, bend left, "T\hat{\pi}"] \arrow[d, "H"'] & T(\gamma) \arrow[l, bend left, "T\sigma"] \arrow[d, "\mathrm{id}"] \\ \Sigma^\infty(K_{q+1})_+ \arrow[r, "Ti_1"] & \Sigma^\infty(K_{q+1})_+ \wedge T(\gamma) \arrow[r, bend left, "Tp_2"] & T(\gamma). \arrow[l, bend left, "Ti_2"] \end{tikzcd}$$

Now rows is not fiber sequence any more, but the diagram is still commutative. So there exists the map

$$h\colon T(\hat{\gamma})/T(\gamma) = \mathrm{Cone}(T\sigma) \to \mathrm{Cone}(Ti_2) = (\Sigma^\infty(K_{q+1})_+ \wedge T(\gamma))/T(\gamma).$$

There exists an isomorphism

$$(\Sigma^\infty(K_{q+1})_+ \wedge T(\gamma))/T(\gamma) \cong \Sigma^\infty K_{q+1} \wedge T(\gamma) = MO \wedge K_{q+1}.$$

So we constructed the map $h\colon T(\hat{\gamma})/T(\gamma) \to \Sigma^\infty K_{q+1} \wedge T(\gamma)$. By the Thom isomorphism h^* is an isomorphism on $H^i(-, \mathbb{Z}/2)$ for $i \le 2q+1$ and $(\ker h^*)^{2q+2}$ is generated by the cohomological class β such that $T(i_1)^*(\beta) = \mathrm{Sq}^{q+1}(\iota_{q+1})$. $\square$

Lemma 2. *1) The map h induces an isomorphism on $\pi_i(-)$ for $i \le 2q$.*

2) There exists the exact sequence

$$0 \to \mathbb{Z}/2 \to \pi_{2q+1}(MO/MO\langle v_{q+1}\rangle) \xrightarrow{h_*} \pi_{2q+1}(MO \wedge K_{q+1}) \to 0.$$

3) Moreover, the kernel of h_* is generated by the image of $\pi_{2q+1}(\Sigma^\infty \Sigma K_q)$ under the map

$$\bar{j}_*\colon \pi_{2q+1}(\Sigma^\infty \Sigma K_q) \to \pi_{2q+1}(T(\hat{\gamma})) \to \pi_{2q+1}(T(\hat{\gamma})/T(\gamma)).$$

Proof. Exercise. Hint: use Lemma 1 and the fact that $MO \wedge K_{q+1}$ is a sum of Eilenberg-Maclane spectra. □

These two lemmas prove Theorem 1. Indeed, the map $G_1 \colon MO\langle v_{q+1}\rangle^{(1)} \to \Sigma^{-1}MO \wedge K_{q+1}$ is the map $\Sigma^{-1}h$ and the map $G_2 \colon MO\langle v_{q+1}\rangle^{(2)} \to \Sigma^{2q} H\mathbb{Z}/2$ is such that $\Sigma^2 G_2$ represents an element $\beta \in (\ker h^*)^{2q+2}$. By Lemma 2 the fiber of G_2 is $2q$-connected.

4 Proof of Theorem 2

By Theorem 1 we have the relation $MO \xrightarrow{k_1} MO \wedge K_{q+1} \xrightarrow{k_2} \Sigma^{2q+2} H\mathbb{Z}/2$. Let us compute $k_2^*(\iota_{2q+2})$.

Lemma 3. *We have the equality in* $H^{2q+2}(MO \wedge K_{q+1}, \mathbb{Z}/2)$*:*

$$k_2^*(\iota_{2q+2}) = \mathrm{Sq}^{q+1}(U \cup \iota_{q+1}) + v_{q+1} \cup U \cup \iota_{q+1} + \sum_{\substack{0 \le j \le q \\ i+j=q+1}} (w_i \cup U) \cup \mathrm{Sq}^j \iota_{q+1}.$$

Here $U \in H^0(MO, \mathbb{Z}/2)$ *is the Thom class,* $v_{q+1} \in H^{q+1}(BO, \mathbb{Z}/2)$ *is the universal* $(q+1)$*-th Wu class,* $w_j \in H^j(BO, \mathbb{Z}/2)$ *is the universal* j*-th Stiefel-Whitney class and* $\iota_{q+1} \in H^{q+1}(K_{q+1}, \mathbb{Z}/2)$*.*

Proof. Denote by X the expression

$$\mathrm{Sq}^{q+1}(U \cup \iota_{q+1}) + v_{q+1} \cup U \cup \iota_{q+1} + \sum_{\substack{0 \le j \le q \\ i+j=q+1}} (w_i \cup U) \cup \mathrm{Sq}^j \iota_{q+1}.$$

By Lemma 1 the element $k_2^*(\iota_{2q+2})$ is a unique non-trivial element in $(\ker h^*)^{2q+2}$. So it is enough to prove two facts on X: $X \neq 0$ and $h^*(X) = 0$.

The element X is not equal to zero, because

$$\bar{j}'^*(X) = \mathrm{Sq}^{q+1}(\iota_{q+1}) \in H^{2q+2}(K_{q+1}, \mathbb{Z}/2).$$

The map $\bar{j}' \colon \Sigma^\infty K_{q+1} \to MO \wedge K_{q+1}$ is the smash of the unit map $S \to MO$ with $\Sigma^\infty K_{q+1}$.

Let us compute $h^*(X)$. First,

$$\begin{aligned} h^*(\mathrm{Sq}^{q+1}(U \cup \iota_{q+1})) &= \mathrm{Sq}^{q+1}(U \cup g) \\ &= \sum_{i+j=q+1} \mathrm{Sq}^i U \cup \mathrm{Sq}^j g \\ &= U \cup g^2 + \sum_{\substack{0 \le j \le q \\ i+j=q+1}} \mathrm{Sq}^i U \cup \mathrm{Sq}^j g \\ &= U \cup g^2 + \sum_{\substack{0 \le j \le q \\ i+j=q+1}} w_i \cup U \cup \mathrm{Sq}^j g. \end{aligned}$$

Here $g \in H^*(MO/MO\langle v_{q+1}\rangle, \mathbb{Z}/2)$, so $g^2 = \delta^*(g) \cup g$. But $\delta^*(g)$ is a non-zero element in the kernel of the map

$$\pi^* \colon H^{q+1}(MO, \mathbb{Z}/2) \to H^{q+1}(MO\langle v_{q+1}\rangle, \mathbb{Z}/2),$$

so $\delta^*(g) = v_{q+1} \cup U$. Hence,

$$h^*(\mathrm{Sq}^{q+1}(U \cup \iota_{q+1})) = v_{q+1} \cup U \cup g + \sum_{\substack{0 \le j \le q \\ i+j=q+1}} w_i \cup U \cup \mathrm{Sq}^j g.$$

But

$$h^*(v_{q+1} \cup U \cup \iota_{q+1}) = v_{q+1} \cup U \cup g,$$

$$h^*\Bigg(\sum_{\substack{0 \le j \le q \\ i+j=q+1}} (w_i \cup U) \cup \mathrm{Sq}^j \iota_{q+1} \Bigg) = \sum_{\substack{0 \le j \le q \\ i+j=q+1}} w_i \cup U \cup \mathrm{Sq}^j g$$

So $h^*(X) = 0$. □

Now we can prove Theorem 2. The relation

$$MO \xrightarrow{k_1} MO \wedge K_{q+1} \xrightarrow{k_2} \Sigma^{2q+2} H\mathbb{Z}/2$$

defines the sum of secondary cohomological operations Φ_j, where any Φ_j is based on a relation of the form

$$\sum a_i b_i = 0.$$

Here a_i, b_i is some Steenrod squares. By proposition 1 we want to know when Φ_j can detect elements in π^s_{2q}. By the Adams theorem it is enough to evaluate $h_j h_k \in \mathrm{Ext}^{\mathcal{A}_2}_{2,*}(\mathbb{F}_2, \mathbb{F}_2)$ on the relation $\sum a_i b_i = 0$.

Notice that $\dim(a_i b_i) = 2q+2$. Since $MO \wedge K_{q+1} = \Sigma^{q+1} H\mathbb{Z}/2 \vee \Sigma^b A$, where A is an Eilenberg-Maclane spectrum and $b > q+1$, we have $\dim(b_i) \geq q+1$. So by dimensional reason if $j > k$, then

$$h_j h_k \Big(\sum a_i b_i \Big) = 0$$

So we can consider only elements $h_k^2 \in \mathrm{Ext}^{\mathcal{A}_2}_{2,2^{k+1}}(\mathbb{F}_2, \mathbb{F}_2)$. But if $q+1$ is not a power of two, then $h_k^2(\sum a_i b_i) = 0$ for any k. So in this case Φ does not detect any element in π^s_{2q}, and $\mathrm{Im}\, \Omega^{fr}_{2q} = 0$. It proves the first statement of Theorem 2.

Now suppose that $q = 2^k - 1$. Then the Steenrod square Sq^{q+1} is indecomposable, so

$$\chi(\mathrm{Sq}^{q+1}) = \mathrm{Sq}^{q+1} + D,$$

where D is the sum of decomposable elements in $\mathcal{A}_2$. It means that the Wu class $v_{q+1} = w_{q+1} + e$, where the element e belongs to the subalgebra of $H^*(BO, \mathbb{Z}/2)$ generated by w_i, $i < q+1$.

Hence,

$$k_2^*(\iota_{2q+2}) = \mathrm{Sq}^{q+1}(U \cup \iota_{q+1}) + \sum_{0 \leq i \leq q} U \cup x_i \cup \zeta_i.$$

Here $\zeta_i \in H^{q+i+1}(K_{q+1}, \mathbb{Z}/2)$ and $x_i \in H^{q+1-i}(BO, \mathbb{Z}/2)$.

Now $k_1^*(U \cup \iota_{q+1}) = \chi(\mathrm{Sq}^{q+1}) U$ in $H^{q+1}(MO, \mathbb{Z}/2)$. So in the relation, which defines the secondary cohomological operation Φ,

$$\sum a_i b_i = 0$$

we have $b_1 = \chi(\mathrm{Sq}^{q+1})$, Since $q = 2^k - 1$, $h_k(\chi(\mathrm{Sq}^{q+1})) = 1$. Also, we have $a_1 = \mathrm{Sq}^{q+1}$, so $h_k^2(a_1 b_1) = 1$.

Now show that $h_k^2(k_1^*(\sum U \cup x_i \cup \zeta_i)) = 0$. Consider the decomposition

$$\sum_{0 \leq i \leq q} U \cup x_i \cup \zeta_i = \sum_{i \geq 2} a_i c_i,$$

where $a_i \in \mathcal{A}_2$ and c_i are generators of $H^*(MO \wedge K_{q+1}, \mathbb{Z}/2)$ over the Steenrod algebra $\mathcal{A}_2$. Notice that $\dim(c_i) \geq q+1$.

Suppose that $\dim(c_2) = q+1$ and $\dim(c_i) > q+1$ for $i > 2$. Then $c_2 = U \cup \iota_{q+1}$ and

$$h_k^2 \Big(\sum a_i k_1^*(c_i) \Big) = h_k(a_2) h_k(k_1^* c_2) = h_k(a_2).$$

Suppose that $h_k(a_2) = 1$, then $a_2 = \mathrm{Sq}^{q+1} + D$, where D is a sum of decomposable elements. Then

$$\bar{j}'^* \Big(\sum a_i c_i \Big) = \mathrm{Sq}^{q+1} \iota_{q+1} + D'(\iota_{q+1}) \neq 0.$$

Here D' is another sum of decomposable elements. On the other hand,

$$\bar{j}'^* \left(\sum_{0 \leq i \leq q} U \cup x_i \cup \zeta_i \right) = 0,$$

because $\dim(x_i) > 0$ and $\bar{j}'^*(x_i) = 0$. It means that $h_k(a_2) = 0$ and

$$h_k^2(k_1^* k_2^*(\iota_{2q+2})) = 1.$$

By the Adams theorem Φ detects an element in π^s_{2q} if and only if h_k^2 is a permanent cycle. So we proved the second and the third statements of Theorem 2.

By Lemma 2 we have the inclusion $\operatorname{Im}\Omega^{fr}_{2q} \subset \operatorname{Im}(j_*)$, where $j\colon \Sigma^\infty K_q \to MO\langle v_{q+1}\rangle$ is the map which comes from the fiber sequence $K_q \to BO\langle v_{q+1}\rangle \to BO$. It is an exercise to show that $\operatorname{Im}(j_*)$ is generated by cobordism class of $(S^q \times S^q, \eta)$ from subsection 2.3 in Lecture 4. But the Kervaire invariant of $(S^q \times S^q, \eta)$ is equal to one, so we proved the Browder theorem.

References

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