The Kervaire Invariant One Problem, Talk 0 (Introduction)
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This is an introductory lecture which should (very roughly) explain what we will study during the semester.

1 Formulation

Recall that a nondegenerate quadratic form $q\colon V \to k$ over a field k of characteristic 2 can always be decomposed into a sum of nondegenerate binary forms (quadratic form of rank 2) $q \simeq q_1 \oplus q_2 \ldots \oplus q_n$. As any other nondegenerate binary form, in some basis the form q_i looks like

$$q_i(x, y) = a_i x^2 + xy + b_i y^2, \quad a_i, b_i \in k$$

If we change the basis, the product $a_i b_i$ will differ by some element of the form $\wp(x) := x^2 + x, x \in k$. Hence we define

Definition 1.1. The *Arf invariant* of the quadratic form q is

$$\mathrm{Arf}(q) := \sum_i a_i b_i \in k/\wp(k)$$

Arf showed that this is indeed an invariant of q. Moreover, he proved the following

Theorem 1.2 (Arf)**.** *Over a perfect field of characteristic 2 the rank and the Arf invariant determine q up to isomorphism.*

Example 1.3. The Arf invariant of the quadratic form q over the field with two elements $\mathbb{F}_2$ is just 0 or 1. It is easy to see, that $\mathrm{Arf}(q)$ in this case is equal to the value which is assumed most often by q.

Now recall that over the field of characteristic not equal to 2 theories of quadratic forms and symmetric bilinear forms are equivalent by the pair of mutually inverse maps

$$b \mapsto q_b(x) := \frac{1}{2} b(x, x) \qquad q \mapsto b_q(x, y) := q(x + y) - q(x) - q(y)$$

Over the field of characteristic 2 it is not quite so, instead we define

Definition 1.4. The *quadratic refinement* of a bilinear form b is a quadratic form q such that

$$b(x, y) = q(x + y) - q(x) - q(y)$$

To see why we are interested in the definition above recall first the following

Definition 1.5. A *framing* of a smooth manifold M is a choice of trivialization of the stable tangent bundle of M.

Now given a smooth manifold M of dimension $4n + 2$ we have a nondegenerate bilinear Poincare pairing

$$\cap \colon H_{2n+1}(X, \mathbb{F}_2) \otimes H_{2n+1}(X, \mathbb{F}_2) \to \mathbb{F}_2.$$

The framing gives us a quadratic refinement of $\cap$ which we will denote by q_M. This motivates the following

Definition 1.6. The *Kervaire invariant* of a smooth framed manifold M of dimension $4n+2$ is the Arf invariant of q_M.

Example 1.7. The bilinear form above for the 2-dimensional torus $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$ has the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The framing induced by the standard embedding $\mathbb{T}^2 \hookrightarrow \mathbb{R}^3$ gives the refinement $q_{\mathbb{T}^2} = xy$. Hence the torus with the standard framing has zero Kervaire invariant.

From the manifold point of view the Kervaire invariant is interesting because of the following result of the surgery theory

Theorem 1.8. *The manifold is framed cobordant to the homotopy sphere if and only if its Kervaire invariant vanishes.*

We now wish to explain how stable homotopy theory enters the picture. Let $\mathcal{A} := [H\mathbb{F}_2, H\mathbb{F}_2]_*$ be the mod 2 Steenrod algebra, where $H\mathbb{F}_2$ is the mod 2 Eilenberg-Maclane spectrum. Then there is an Adams spectral sequence with the second page

$$E_2^{p,q} = \mathrm{Ext}_{\mathcal{A}}^{p,q}(\mathbb{F}_2, \mathbb{F}_2)$$

which converges to $\pi_{q-p}(\mathbb{S})$ modulo odd torsion.

To start a calculation it is convenient to use the resolution of $\mathbb{F}_2$ as an $\mathcal{A}$-module which begins as

$$\ldots \to \bigoplus_j \mathcal{A}\langle \mathrm{Sq}^{2^j} \rangle \to \mathcal{A} \xrightarrow{\varepsilon} \mathbb{F}_2 \to 0$$

(the kernel of ε above is generated by Sq^{2^j} as they form a basis of $\mathcal{A}$ as an $\mathbb{F}_2$-algebra).

This defines a set of distinguished elements $h_j \in \mathrm{Ext}_{\mathcal{A}}^{1,2^j}(\mathbb{F}_2, \mathbb{F}_2)$. Now there is a classical

Theorem 1.9 (Browder). *Manifolds with nonzero Kervaire invariant may exist only in dimensions $2^{j+1}-2$. Such a manifold exists in dimension $2^{j+1}-2$ if and only the class h_j^2 is a permanent cycle.*

We will denote the classes in $\pi_{2^{j+1}-2}(\mathbb{S})$ coming from h_j^2 by θ_j and will call them the *Arf-Kervaire elements.*

There are examples of manifolds with nonzero Kervaire invariant of dimensions 2, 6, 14, 30, 62. In the previous century there were many unsuccessful attempts to construct such a manifolds in higher dimensions. The question remained open until recently, when in 2010 the following was proved

Theorem 1.10 (Hill, Hopkins, Ravenel). *For $j \geq 7$ the Kervaire elements θ_j are all equal to zero.*

So now only the dimension 126 is left unknown.

2 Idea of the proof

Very schematically the proof of theorem 1.10 goes as follows. One constructs a ring spectrum Ω with the following properties:

- *Detection theorem.* The map induced by the unit $\mathbb{S} \to \Omega$ is injective on the Arf-Kervaire elements θ_j.
- *Periodicity theorem.* The homotopy groups $\pi_*\Omega$ are 256-periodic.
- *Gap theorem.* We have $\pi_k\Omega \simeq 0$ for $-4 \leq k \leq -1$.

From this and the result of Browder we immediately get the proof of theorem 1.10. Below we will spell out some of the details. But first we need a brief tour into the world of

Equivariant homotopy theory

Let G be a discreet group. Define then a category $\mathbb{B}G$ with only one object $*$ and $\mathrm{Hom}_{\mathbb{B}G}(*,*) := G$. For a category C and an object $X \in C$ one can define an action of G on X to be a functor $a\colon \mathbb{B}G \to C$, which send $*$ to X. For example, if C is the category of sets, we have the canonical equivalence

$$\mathrm{Fun}(\mathbb{B}G, \mathcal{S}\mathrm{et}) \simeq G - \mathcal{S}\mathrm{et}$$

where the category on the right is the category of sets with the action of G.

Analogously for a topological group G we can construct an $(\infty,1)$-category $\mathbb{B}G$ and for any $(\infty,1)$-category $\mathcal{C}$ the category of G-object in $\mathcal{C}$ is defined to be

$$\mathcal{C}^{hG} := \mathrm{Fun}(\mathbb{B}G, \mathcal{C})$$

Note that the construction of the category $\mathcal{C}^{hG}$ above depends only in the homotopy type of G and in fact we may assume G to be any $\mathbb{E}_1$-space (an associative up to a coherent choice of homotopies monoid in the category of spaces). That is, we see, that the definition above is not appropriate unless we are interested only on the homotopy type of G and do not care about other structure (such as being smooth) G can have.

In order to avoid this problem, for a Lie group G we define instead the *G-equivariant homotopy category* $\mathcal{T}\mathrm{p}^G$ as the localization of the (strict) category of G-topological spaces by the collection of weak G-equivariant homotopy equivalences, which are defined to be the maps $f\colon X \to Y$ such that the induced map f^H between H-fixed subspaces

$$f^H\colon X^H \to Y^H$$

is an (ordinary) weak homotopy equivalence for any closed subgroup H of G. There is always forgetful functor $\mathcal{T}\mathrm{p}^G \to \mathcal{T}\mathrm{p}^{hG}$, but for nontrivial G it is very far from being an equivalence.

One important classes of G-spaces is the class of spaces of the form $\Sigma^n G/H$, where H is a closed subgroup of G. The reason for this is that they play the role of cells in the equivariant world. This remark allows us to move towards the stable setting: namely, recall that classically one obtains the category of spectra by inverting the suspension functor on the category of spaces. Since the suspension functor is given by the smash product with the sphere, in the equivariant setting one could define the category of G-spectra as the universal category in which the smash product with the orbit spheres $\Sigma^{\infty+n}_{+} G/H$ are equivalences. We will call the resulting category Sp^{nG}, where n stand for "naive".

The reason for the name "naive" is that in fact the category Sp^{nG} does not behave well enough. One of the main reasons is that we do not have a reasonable duality theory in this setting: for example, one would like at least all the orbit spheres to be dualizable. But in our setting this is clearly false: if $\Sigma^\infty_+ G/H \in \mathrm{Sp}^{nG}$ would be dualizable, then its dual would have to be $\mathrm{Map}_{\mathrm{Sp}^{nG}}(\Sigma^\infty_+ G/H, \mathbb{S})$. But as G acts trivially on the right, by the standard adjunction we deduce

$$\mathrm{Map}_{\mathrm{Sp}^{nG}}(\Sigma^\infty_+ G/H, \mathbb{S}) \simeq \mathrm{Map}_{\mathrm{Sp}^{nG}}((\Sigma^\infty_+ G/H)/G, \mathbb{S}) \simeq \mathbb{S}$$

which is nonsense. So we do not have a well-behaved duality even for zero dimensional cells.

In order to fix the problem, notice that for any orthogonal representation V of G we may consider the one point compactification $\mathbb{S}^V$ of V which inherits a G-action and will be called a *V-representation sphere*. Now to obtain the category of *genuine G-spectra* Sp^G one needs not only to invert all the G-spaces of the form $\Sigma^n_+ G/H$ but also all the representation spheres $\mathbb{S}^V$, where V ranges over all the orthogonal representations of G, .

The category Sp^G has many good properties. In particular, there is a forgetful functor $h\colon \mathrm{Sp}^G \to \mathrm{Sp}^{hG}$ which allows to produce the following

Definition 2.1. For a genuine G-spectrum $X \in \mathrm{Sp}^G$ define a *homotopy fixed points spectrum* $X^{hG} \in \mathrm{Sp}$ to be the limit of the diagram $\mathbb{B}G \to \mathrm{Sp}$ which corresponds to $h(X)$.

In fact, we can do even better:

Definition 2.2. For a genuine G-spectrum $X \in \mathrm{Sp}^G$ define an *honest* or a *categorical fixed point spectrum* functor $X \mapsto X^G$ as the right adjoint to the obvious inclusion $\mathrm{Sp} \to \mathrm{Sp}^G$.

In order to move further, notice that for a subgroup $H \hookrightarrow G$ we have a restriction functor

$$\mathrm{Res}^G_H \colon \mathrm{Sp}^G \to \mathrm{Sp}^H .$$

It admits both left and right adjoints, the induction $G \wedge_H -$ and coinduction $\mathrm{Map}_H(G, -)$. Moreover, for a finite group G and a genuine G-spectrum $X \in \mathrm{Sp}^G$ one has a canonical equivalence

$$G \wedge_H X \xrightarrow{\sim} \mathrm{Map}_H(G, X)$$

which can be considered as an equivariant version of the statement that in spectra finite coproducts are canonically equivalent to finite products.

Analogously, for a forgetful functor $\mathrm{CRing}^G \to \mathrm{CRing}^H$ one has the left adjoint *norm functor* Nm^G_H with plenty of plausible properties. We will need it to construct Ω, to which we turn now.

The spectrum Ω

We are now ready to define the spectrum Ω.

First take the complex cobordism spectrum $MU \in \mathrm{Sp}$. The cyclic group C_2 acts on it by the "conjugation of complex structure" and one can refine this action to the structure of a genuine C_2-spectrum $MU_{\mathbb{R}} \in \mathrm{Sp}^{C_2}$. We then take the norm $\mathrm{Nm}^{C_{2^n}}_{C_2}(MU_{\mathbb{R}})$ and will call this spectrum $MU_{(2^n)} \in \mathrm{Sp}^{C_{2^n}}$. The C_8-equivariant spectrum $\tilde{\Omega}$ is defined to be the localization $MU_{(8)}[D^{-1}]$ for an appropriate choice of D. Finally we define Ω to be the categorical fixed points spectrum

$$\Omega := \tilde{\Omega}^{C_8}$$

The underlying spectrum of $\tilde{\Omega}$ is just a localization of $MU \wedge MU \wedge MU \wedge MU$, but the structure of the C_8-fixed points spectrum is not obvious. The main tool here is the concept of the

The slice filtration. Recall that classically the Postnikov tower of a space X is the sequence of spaces $\tau_{\leq n}X$ and maps $X \to \tau_{\leq n}X$ such that

$$X \xrightarrow{\sim} \varprojlim \tau_{\leq n}X$$

and

$$\pi_i(\tau_{\leq n}X) = \begin{cases} 0, \ i > n \\ \pi_i(X), \ i \leq n \end{cases}$$

and the maps $X \to \tau_{\leq n}X$ induce an equivalence on π_i for $i \leq n$.

By the general theory of higher categories, one has an analogue of the Postnikov tower for any object in the category Sp^G. However, it turns out that it is not such a useful tool for the analysis of the spectrum $\tilde{\Omega}$ (for example, the associated graded factors are *not* the Eilenberg-Mac Lane spectra but something much harder), so we have to invent something new.

In the category of non-equivariant spectra Sp one can define the n-th Postnikov truncation functor $\tau_{\leq n}$ as the Bousfield localization with respect to the subcategory generated under colimits and extensions by the sphere spectra $\mathbb{S}^m, m > n$. In a similar manner, the Postnikov tower in the category Sp^G corresponds to the Bousfield localisation with respect to subcategories generated by $\Sigma_+^{\infty+m}(G/H)$. But as we have discussed before we also have another important kind of spheres, namely, the representation spheres. This motivates the following

Definition 2.3. Define the the *n-th slice truncation functor* $\tau_{\leq n}$ as the Bousfield localization functor with respect to the subcategory generated by genuine G-spectra of the form

$$\Sigma^\infty(G/H_+ \wedge \mathbb{S}^{k\rho_H - \varepsilon}),$$

where ρ_H is the regular representation of H, $k \cdot |H| - \varepsilon \geq n$ and $\varepsilon = 0, 1$.

The advantage of the slice filtration is that one can very explicitly compute the associated graded factors of $\widetilde{\Omega}$:

Theorem 2.4 (Slice theorem)**.** *For any $n \geq 1$ the associated graded spectrum of the slice filtration on $MU_{(2^n)}$ is equivalent to $MU_{(2^n)} \wedge \mathbb{H}\underline{\mathbb{Z}}$.*

This gives just enough tools to make computations needed to prove Gap theorem doable.

The Periodicity and Detection theorems are proved for *homotopy* fixed points of $\widetilde{\Omega}$. Both theorems heavily relies on the accurate choose of Δ. The proof of the Periodicity theorem is mostly computations with the slice spectral sequence (which becomes accessible after inverting Δ). The proof of Detection theorem uses a bit of chromatic theory.

Periodicity theorem is very flexible and by choosing different Δ one can get smaller period. The period 256 coming from the proof of Detection theorem.

Finally to tie results about homotopy and honest fixed points together one uses the following comparison theorem

Theorem 2.5 (Homotopy fixed points theorem)**.** *The canonical map $\widetilde{\Omega}^{C_{2^n}} \to \widetilde{\Omega}^{hC_{2^n}}$ is an equivalence.*