The Kervaire Invariant One Problem, Talk 10
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1 Slice tower

As any presentable category the category of genuine spectra Sp^G admits the theory of Postnikov towers. Moreover we have seen that this towers always converge in Sp^G. In this section for finite G we will introduce another (slice) filtration of the category of G-spectra more tightly connected with representation theory of G. One advantage of slice tower is that it is better suited for some important examples of equivariant spectra like K-theory or cobordisms.

Let us first give a variation of how one can define the Postnikov tower. For an integer n consider the full subcategory $\mathrm{Sp}^{G,\geq n+1}$ of $n+1$-connected G-spectra. This category is presentable, hence by adjoint functor theorem the inclusion $i_{n+1}\colon \mathrm{Sp}^{G,\geq n+1} \hookrightarrow \mathrm{Sp}^G$ admits the right adjoint $\widetilde{\tau}_{\geq n+1}$. Denote the composite $i_{n+1} \circ \widetilde{\tau}_{\geq n+1}$ just by $\tau_{\geq n+1}$. The counit of adjunction induces the natural transformation $\tau_{\geq n+1} \to 1_{\mathrm{Sp}^G}$ and we define $\tau_{\leq n}$ to be the cofiber of this map. It is easy to see that $\tau_{\leq n}$ coincide with the usual Postnikov truncation functors.

Now note that the category $\mathrm{Sp}^{G,\geq n}$ may be defined as a smallest subcategory of Sp^G closed under colimits and extensions and containing cells $G/H \wedge \mathbb{S}^m$ of dimension m greater than n. We obtain the slice filtration by replacing ordinary cells with

Definition 1.1. For $m \in \mathbb{Z}$ the *slice sphere* $\widehat{S}(m, K)$ is $G \wedge_K \mathbb{S}^{m\rho_K}$, where ρ_K is the regular representation of K. The dimension of $\widehat{\mathbb{S}}(m, K)$ is defined to be $m \cdot |K|$ (it is the dimension of the underlying manifold of $\widehat{\mathbb{S}}$).

For an integer n let us define $\mathrm{Sp}^G_{>^s n}$ to be minimal full subcategory of Sp^G closed under colimits and extensions and containing all slice spheres $\widehat{\mathbb{S}}$ of dimension grater then n.

Lemma 1.2. *The category $\mathrm{Sp}^G_{>^s n}$ is generated under colimits by the small set of objects.*

It follows that $\mathrm{Sp}^G_{>^s n}$ is presentable. So we can define $\tau_{>^s n}$ and $\tau_{\leq^s n}$ as for Postnikov towers. For a G-spectrum X we will sometime denote $\tau_{\leq^s X}$ just by $X_{\leq^s n}$ and analogously for $\tau_{>^s n}X$. We will say that a G-spectrum X is *n-slice* if $X \in \mathrm{Sp}^G_{\geq^s n} \cap \mathrm{Sp}^G_{\leq^s n}$.

We will also not the following proposition, which readily follows from definition

Proposition 1.3. *The functor $\tau_{\leq^s n}$ is the left adjoint to the inclusion of the full subcategory of slice n-truncated spectra (i.e. such G-spectra X, that $\mathrm{Hom}_{\mathrm{Sp}^G}(\widehat{\mathbb{S}}, X) \simeq 0$ for all slice spheres $\widehat{\mathbb{S}}$ of dimension larger then n).*

The slice filtration behave well with respect to the restriction and induction functors.

Proposition 1.4. *Let X be a G-spectrum and Y be an H-spectrum for some subgroup $H \overset{i}{\hookrightarrow} G$. Then*

1. *(a) If $Y \in \mathrm{Sp}^H_{>^s n}$ then $G \wedge_H Y \in \mathrm{Sp}^G_{>^s n}$.*
   (b) If $Y \in \mathrm{Sp}^H_{\leq^s n}$ then $G \wedge_H Y \in \mathrm{Sp}^G_{\leq^s n}$.
2. *(a) If $X \in \mathrm{Sp}^G_{>^s n}$ then then $i^*X \in \mathrm{Sp}^H_{>^s n}$.*
   (b) If $X \in \mathrm{Sp}^G_{\leq^s n}$ then $i^*X \in \mathrm{Sp}^H_{\leq^s n}$.

Proof. 1. (a) The induction functor is exact and commutes with colimits, hence it is enough to prove the statement for the slice spheres $Y = G \wedge_K \mathbb{S}^{m\rho_k}$. But this immediately follows from the functoriality of the induction functors with respect to group homomorphisms.

(b) By the Wirtmuller isomorphism $i_* \simeq i_!$. Hence for a G-slice sphere $\widehat{\mathbb{S}}$ of dimension grater then n we have

$$\mathrm{Hom}_{\mathrm{Sp}^G}(\widehat{\mathbb{S}}, G \wedge_H Y) \simeq \mathrm{Hom}_{\mathrm{Sp}^H}(i^*\widehat{\mathbb{S}}, Y) \simeq 0$$

where we have used that $i^*\widehat{\mathbb{S}}$ is a sum of H-slice spheres of the same dimension as dimension of $\widehat{\mathbb{S}}$.

2. (a) This is immediate from definitions, because the restriction of a slice sphere is a sum of the slice spheres of the same dimension.

(b) For a slice H-sphere $\widehat{\mathbb{S}}$ of dimension grater then n we have

$$\mathrm{Hom}_{\mathrm{Sp}^H}(\widehat{\mathbb{S}}, i^*X) \simeq \mathrm{Hom}_{\mathrm{Sp}^G}(G \wedge_H \widehat{\mathbb{S}}, X) \simeq 0$$

where we have used that $G \wedge_H \widehat{\mathbb{S}}$ is in $\mathrm{Sp}^G_{>^s n}$ by the previous part.

□

The following proposition will tell us something about the relation between slice and ordinary connectivity.

Proposition 1.5. *For $n \geq 0$ all spectra $\Sigma^{\infty+n}_G G/H_+$ are in $\mathrm{Sp}^G_{\geq^s n}$.*

Proof. By the previous proposition it is enough to treat the case $H = G$. We have the cofiber sequence of G-spaces

$$\mathbb{S}(n\rho_G - n) \to \mathbb{S}^0 \to \mathbb{S}^{n\rho_G - n}$$

By smashing it with $\mathbb{S}^n$ we obtain the cofiber sequence of G-spectra

$$\Sigma^{\infty+n}_G \mathbb{S}(n\rho_G - n) \to \mathbb{S}^n \to \mathbb{S}^{n\rho_G}$$

The right hand side term of this sequence is by definition in $\mathrm{Sp}^G_{\geq^s n}$ and the spectrum on the left consists of induced G-cells, hence is in $\mathrm{Sp}^G_{\geq^s n}$ by inductive hypothesis. □

We can now give an example

Example 1.6. For $n = 0$ by the previous proposition $\Sigma_G^\infty G/H_+$ are in $\mathrm{Sp}^G_{\geq^s 0}$, hence $\mathrm{Sp}^G_{\geq 0} \subseteq \mathrm{Sp}^G_{\geq^s 0}$. On the other hand, the only generators of $\mathrm{Sp}^G_{\geq^s 0}$ are $\Sigma^\infty G/H_+ \in \mathrm{Sp}^G_{\geq 0}$, hence $\mathrm{Sp}^G_{\geq^s 0} \subseteq \mathrm{Sp}^G_{\geq 0}$. It follows $\mathrm{Sp}^G_{\geq^s 0} = \mathrm{Sp}^G_{\geq 0}$ and $\tau_{\leq^s -1} \simeq \tau_{\leq -1}$.

The analogues statement holds for $n = 1$. Indeed, by the previous proposition all $\Sigma_G^{\infty+1} G/H_+$ are in $\mathrm{Sp}^G_{\geq^s 1}$. On the other hand by dimension reasons again, the only generator of $\mathrm{Sp}^G_{\geq^s 1}$ is $G \wedge \mathbb{S}^1$. The rest is the same as in the previous paragraph.

It follows that for a G-spectrum X the fiber $X_{^s 0}$ of the canonical map $X_{\leq^s 0} \to X_{\leq^s -1}$ (the zero slice of X) is equivalent to $H\underline{\pi_0}(X)$.

Here is a useful criterion, which helps to describe slice tower in some cases

Proposition 1.7 (Slice recognition)**.** *Let X be a G-spectrum.*

1. *Let $X'_{>^s n} \to X \to X'_{\leq^s n}$ be a fiber sequence of G-spectra, such that $X'_{>^s n} \in \mathrm{Sp}^G_{>^s n}$ and $X'_{\leq^s n} \in \mathrm{Sp}^G_{\leq^s n}$. Then the canonical maps $X'_{>^s n} \to X_{>^s n}$ and $X_{\leq^s n} \to X'_{\leq^s n}$ are equivalences.*

2. *Let $\tau_n \colon X \to X'_{\leq^s n}$ be a tower. Then τ is a slice tower if and only if $X'_{\leq^s n} \in \mathrm{Sp}^G_{\leq^s n}$ and the fiber of $X \to X'_{\leq^s n}$ is in $\mathrm{Sp}^G_{>^s n}$.*

Proof.
1. Let $Y \in \mathrm{Sp}^G_{\leq^s n}$. Then we have the fiber sequence

$$\mathrm{Hom}_{\mathrm{Sp}^G}(X'_{\leq^s n}, Y) \to \mathrm{Hom}_{\mathrm{Sp}^G}(X, Y) \to \mathrm{Hom}_{\mathrm{Sp}^G}(X'_{>^s n}, Y) \simeq 0$$

hence by Yoneda lemma the canonical map $X_{\leq^s n} \to X'_{\leq^s n}$ is an equivalence.

2. Follows immediately from the previous part.

□

Corollary 1.8. *Let $i \colon H \hookrightarrow G$ be a subgroup of G.*

1. *For X an H-spectrum the induction of the slice tower $X \to X_{\leq^s n}$ is the slice tower for the induction $G \wedge_H X$.*

2. *For Y a G-spectrum the restriction of the slice tower $Y \to Y_{\leq^s n}$ is the slice tower of the restriction i^*Y.*

Proof. Both statements immediately follow from proposition 1.4 and the proposition above.
□

Remark 1.9. In particular taking $H = \{1_G\}$ in the second part of the corollary above, we see that the slice tower of a G-spectrum Y is an equivariant refinement of the Postnikov tower of Y^u.

In ordinary world smashing with $\mathbb{S}^1$ just shifts the Postnikov filtration by one. We have an analogues statement for the slice filtration

Proposition 1.10. *For any integer n the smashing with $\mathbb{S}^{\rho_G}$ induces an equivalence $\mathrm{Sp}^G_{>^s n} \to \mathrm{Sp}^G_{>^s n+|G|}$ with inverse induced by the smashing with $\mathbb{S}^{-\rho_G}$.*

Proof. It is enough to prove that $\mathbb{S}^{\rho_G} \wedge \mathrm{Sp}^G_{>^s n} \subseteq \mathrm{Sp}^G_{>^s n+|G|}$ and $\mathbb{S}^{-\rho_G} \wedge \mathrm{Sp}^G_{>^s n} \subseteq \mathrm{Sp}^G_{>^s n-|G|}$. We will do it by induction on $|G|$.

For $K = G$ we have

$$\widehat{\mathbb{S}}(m, G) \wedge \mathbb{S}^{\pm\rho_G} = \mathbb{S}^{m\rho_G} \wedge \mathbb{S}^{\pm\rho_G} \simeq \mathbb{S}^{(m\pm1)\rho_G} \in \mathrm{Sp}^G_{\dim \widehat{\mathbb{S}}(m,G)\pm|G|}$$

For $K \neq G$ by projection formula

$$\widehat{\mathbb{S}}(m, K) \wedge \mathbb{S}^{\pm\rho_G} = (G \wedge_K \mathbb{S}^{m\rho_K}) \wedge \mathbb{S}^{\pm\rho_G} \simeq G \wedge_K (\mathbb{S}^{m\rho_K} \wedge i^*_K \mathbb{S}^{\pm\rho_G})$$

By the second part of the proposition 1.4 $i^*_K \mathbb{S}^{\pm\rho_G}$ is in $\mathrm{Sp}^K_{\geq^s \pm|G|}$. By induction $\mathbb{S}^{\rho_K} \wedge i^*_K \mathbb{S}^{\pm\rho_G}$ is in $\mathrm{Sp}^K_{\geq^s \dim \widehat{\mathbb{S}}(m,K)\pm|G|}$. Hence $G \wedge_K (\mathbb{S}^{m\rho_K} \wedge i^*_K \mathbb{S}^{\pm\rho_G}) \in \mathrm{Sp}^G_{\dim \widehat{\mathbb{S}}(m,K)\pm|G|}$ by the first part of the proposition 1.4.

We conclude, because slice spheres of dimension grater or equal to n generate $\mathrm{Sp}^G_{\geq^s n}$ under colimits and extensions and the functor $\mathbb{S}^{\pm\rho_G} \wedge -$ commutes with colimits and exact. □

Corollary 1.11. *Let X be a G-spectrum. If $\tau_{\leq^s \bullet} \colon X \to X_{\leq^s \bullet}$ is the slice tower of X, then $\mathbb{S}^{\rho_G} \wedge \tau_{\leq^s \bullet}$ is the slice tower of $\mathbb{S}^{\rho_G} \wedge X$. In particular*

$$\mathbb{S}^{\rho_G} \wedge X_{>^s n} \simeq (\mathbb{S}^{\rho_G} \wedge X)_{>^s n} \quad \mathbb{S}^{\rho_G} \wedge X_{\leq^s n} \simeq (\mathbb{S}^{\rho_G} \wedge X)_{\leq^s n} \quad \mathbb{S}^{\rho_G} \wedge X_{^s n} \simeq (\mathbb{S}^{\rho_G} \wedge X)_{^s n}$$

1.1 Multiplicative properties of the slice towers

Definition 1.12. For an integer $n \geq 0$ define $\mathrm{Sp}^G_{[0;n]^s}$ as the intersection $\mathrm{Sp}^G_{\geq^s 0} \cap \mathrm{Sp}^G_{\leq^s n}$.

Proposition 1.13.

1. *The category $\mathrm{Sp}^G_{[0;n]^s}$ admit a (essentially) unique symmetric monoidal structure, such that $\tau_{\leq^s n} \colon \mathrm{Sp}^G_{\geq^s 0} \to \mathrm{Sp}^G_{[0;n]^s}$ promotes to a monoidal functor.*

2. *Let $\mathrm{E_k Alg}_{[0;n]^s}(\mathrm{Sp}^G)$ denote the full subcategory of $\mathrm{E_k Alg}(\mathrm{Sp}^G)$ consisting of algebras A, such that the underlying spectrum of A is in $\mathrm{Sp}^G_{[0;n]^s}$. Then there is a canonical equivalence $\mathrm{E_k Alg}(\mathrm{Sp}^G_{[0;n]^s}) \simeq \mathrm{E_k Alg}_{[0;n]^s}(\mathrm{Sp}^G)$.*

Proof. 1. Everything is in Lurie (somewhere, you just need to look hard enough). In particular this statement is [Lur13, Proposition 2.2.1.9], but to verify the assumptions we need to prove the following fact: let $f \colon X \to Y$ be a map of connective G-spectra and assume that the induced map $X_{\leq^s n} \to Y_{\leq^s n}$ is an equivalence, then for any connective G-spectrum V the map $(X \wedge V)_{\leq^s n} \to (Y \wedge V)_{\leq^s n}$ is an equivalence. Equivalently, the functor $\widetilde{\tau}_{\leq^s n}(V \wedge -)$ maps n-equivalences to equivalences. Every such functor is a colimit of $\widetilde{\tau}_{\leq^s n}(G/H_+ \wedge -)$, hence it is enough to prove the statement for $V = G/H_+$. For $H = G$ the statement is tautological and for $H \neq G$ it follows by induction using projection formula $G/H_+ \wedge X \simeq G \wedge_H (i^*_H X)$ and corollary 1.8.

2. We have the canonical functor
$$\varphi\colon \mathrm{E_kAlg}_{[0;n]^s}(\mathrm{Sp}^G) \hookrightarrow \mathrm{E_kAlg}(\mathrm{Sp}^G) \xrightarrow{\tau_{\leq^s n}} \mathrm{E_kAlg}(\mathrm{Sp}^G_{[0;n]^s})$$
The right adjoint $i_{\leq^s n}$ to the truncation is right lax-monoidal (as every right adjoint of a monoidal functor), hence it induces a functor
$$\widetilde{\psi}\colon \mathrm{E_kAlg}(\mathrm{Sp}^G_{[0;n]^s}) \to \mathrm{E_kAlg}(\mathrm{Sp}^G)$$
which factors through $\mathrm{E_kAlg}_{[0;n]^s}(\mathrm{Sp}^G)$. We will denote the factorization by ψ.

Both composites $\psi \circ \varphi$ and $\varphi \circ \psi$ are equivalent to the identity functors on the level of underlying spectra. We conclude, because the forgetful functor from algebras to spectra is conservative.

□

Remark 1.14. It follows that there is a commutative diagram
$$\begin{tikzcd}
\mathrm{E_kAlg}(\mathrm{Sp}^G_{\geq^s 0}) \arrow[r, "\tau_{\leq^s n}"] \arrow[d] & \mathrm{E_kAlg}_{[0;n]^s}(\mathrm{Sp}^G) \arrow[d] \\
\mathrm{Sp}^G_{\geq^s 0} \arrow[r, "\tau_{\leq^s n}"] & \mathrm{Sp}^G_{[0;n]^s}
\end{tikzcd}$$

In particular let A be a connective E_k-algebra in Sp^G. Then $A_{\leq^s n}$ admits a canonical structure of E_k-algebra, such that the truncation map $A \to A_{\leq^s n}$ is a morphism of E_k-algebras.

2 Slice spectral sequence

Also we already studied some properties of the slice tower, we haven't proved its convergence yet. We will handle it in this section and establishing some important results interesting in their own right along the way.

Lemma 2.1. *For $m \geq 0$ the slice sphere $\widehat{\mathbb{S}}(m, K)$ may be decomposed as a colimit of $\Sigma^{\infty+k} G/H_+$ where $m \leq k \leq m|K|$ and $H \leq K$. For $m \leq 0$ there is a similar decomposition with $m|K| \leq k \leq m$.*

Proof. If $\widehat{\mathbb{S}}$ is induced, it admits the induced cell decomposition with desired properties. For $m > 1$ cell decomposition of $\mathbb{S}^{\rho_G}$ induces cell decomposition of $\mathbb{S}^{m\rho_G} = (\mathbb{S}^{\rho_G})^m$ with cells in desired range. The equivariant Spainer-Whithead duality gives desired decomposition of $\mathbb{S}^{-m\rho_G}$.

So it is enough to treat the case $\mathbb{S}^{\rho_G}$. It is enough to prove that the space $\mathbb{S}^{\rho_G - 1}$ admits an equivariant cell decomposition with cells of dimension $0, \ldots, |G|-1$. Now note that $\mathbb{S}^{\rho_G-1} \simeq \Sigma\mathbb{S}(\rho_G - 1)$. Consider $\rho_G - 1$ as a real vector space. The boundary of the standard $|G| - 1$-dimensional simplex is equivariantly equivalent to $\mathbb{S}(\rho_G - 1)$. Barycentric subdivision of standard cell decomposition of this simplex gives the desired decomposition of $\mathbb{S}(\rho_G - 1)$. □

Corollary 2.2. *For $n \geq 0$ we have that $\mathrm{Sp}^G_{\geq^s n} \subseteq \mathrm{Sp}^G_{\geq \lceil |G|/n \rceil}$. If $n \leq 0$ then $\mathrm{Sp}^G_{\geq^s n} \subseteq \mathrm{Sp}^G_{\geq n}$.*

Corollary 2.3. *For a G-spectrum X the canonical map $X \to X_{\leq^s n}$ induces an isomorphism on $\underline{\pi}_k$ for*

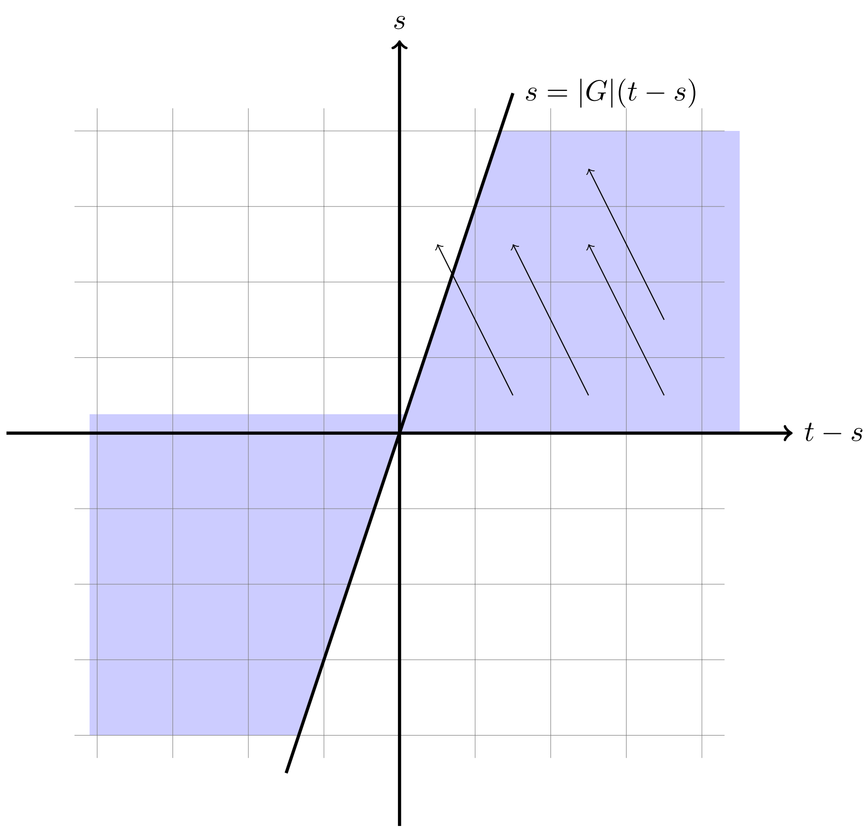
$$\begin{cases} k < \lceil |G|/n \rceil & n \geq 0 \\ k \leq n & n \leq 0 \end{cases}$$

Corollary 2.4. *Slice towers converge, in the sense that for any G-spectrum X the canonical maps $X \to \varprojlim X_{\leq^s n}$ and $\varinjlim X_{>^s n} \to 0$ are equivalences.*

Corollary 2.5. *If X is in n-slice then $\underline{\pi}_k(X) \simeq 0$ unless*

$$\begin{cases} \lceil n/|G| \rceil \leq k \leq n & n \geq 0 \\ n \leq k < \lceil (n+1)/|G| \rceil & n \leq 0 \end{cases}$$

Corollary 2.6. *For the E_2-page of the slice spectral sequence of a G-spectrum X ($E_2^{s,t} = \pi_{t-s} X_{^s t} \Rightarrow \pi_{t-s} X$) the non-vanishing region is marked blue on the diagram below*



2.1 Pure G-spectra and Gap Theorem

In this section we will introduce a family of G-spectra, for which one can guarantee the vanishing of some homotopy groups (General Gap Theorem). Later we will prove, that our spectrum Ω from the introduction is (almost) of this kind, which will allow us to deduce $\pi_{-2}(\Omega) \simeq 0$, needed for the solution to the Kervaire Invariant Problem.

Proposition 2.7. *For a slice sphere $\widehat{\mathbb{S}}$ of dimension d the spectrum $\widehat{\mathbb{S}} \wedge H\underline{\mathbb{Z}}$ is a d-slice.*

Proof. By projection formula $(G \wedge_K \mathbb{S}^{m\rho_K}) \wedge H\underline{\mathbb{Z}} \simeq G \wedge_K (\mathbb{S}^{m\rho_K} \wedge H\underline{\mathbb{Z}})$, hence by proposition 1.4 we may assume $K = G$.

We know $H\underline{\mathbb{Z}}$ is a zero slice by example 1.6. It is left to note that the smash of a slice $H\underline{\mathbb{Z}}$ with $\mathbb{S}^{m\rho_G}$ is a slice of dimension $\dim \mathbb{S}^{m\rho_G}$ by corollary 1.11. □

Definition 2.8. The spectrum is called *pure* if all of its slice layers are sums of $\widehat{\mathbb{S}} \wedge H\underline{\mathbb{Z}}$ where $\widehat{\mathbb{S}}$ is an isotrophic slice sphere.

I learned the following argument from Akhil Mathew's talk [HR16, Talk 16]. For readers convenience I will reproduce it here

Theorem 2.9 (General Gap theorem)**.** *Let X be a pure isotrophic G-spectrum for G a non-trivial 2-group. Then $\pi_k(X) \simeq 0$ for $k = -2, -1$.*

Proof. It is enough to treat the case $X = \widehat{\mathbb{S}} \wedge H\mathbb{Z}$, result for general X follows by convergence of the slice tower. Also by adjunction of restriction and co-induction and projection formula

$$\pi_*^G((G \wedge_K \mathbb{S}^{m\rho_K}) \wedge H\underline{\mathbb{Z}}) \simeq \pi_*^G(G \wedge_K (\mathbb{S}^{m\rho_k} \wedge H\underline{\mathbb{Z}})) \simeq \pi_*^K(\mathbb{S}^{m\rho_K} \wedge H\underline{\mathbb{Z}})$$

Hence we may assume $K = G$.

Let $\widehat{\mathbb{S}}(m, G)$ be a slice sphere $S^{m\rho_G}$. First assume $m \geq 0$. Then $\pi_{<0}(\mathbb{S}^{m\rho_G} \wedge H\underline{\mathbb{Z}})$ vanish, because the spectrum $\mathbb{S}^{m\rho_G} \wedge H\underline{\mathbb{Z}}$ is connective.

For $m = -l < 0$ we want to prove that

$$\pi_{-i}(\mathbb{S}^{-l\rho_G} \wedge H\underline{\mathbb{Z}}) \simeq \mathrm{Hom}_{\mathrm{Sp}^G}(\mathbb{S}^{l\rho_G - i}, H\underline{\mathbb{Z}}) \simeq H^i(\mathbb{S}^{l\rho_G}/G, \mathbb{Z})$$

vanish. For $l \geq 2$ the space $\mathbb{S}^{l\rho_G}$ is equivalent to $\Sigma^2\mathbb{S}^{l\rho_G-2}$, hence it is at least 2-connected. Taking coinvariants $\Sigma^2\mathbb{S}^{l\rho_G-2}/G \simeq \Sigma^2(S^{l\rho_G-2}/G)$ preserves connectedness in this case (because the factor $S^{l\rho_G-2}/G$ of a connected space $S^{l\rho_G-2}/G$ is connected), hence $H^{1,2}(\mathbb{S}^{l\rho_G}/G, \mathbb{Z}) \simeq 0$. The case $l = 1, k = -1$ can be treated similarly.

So it is left to prove that $H^2(\mathbb{S}^{\rho_G}/G, \mathbb{Z}) \simeq 0$. Again $\mathbb{S}^{\rho_G} \simeq \Sigma\mathbb{S}^{\rho_G - 1}$. For the latter space we have the cofiber sequence

$$\mathbb{S}(\rho_G - 1)_+ \to \mathbb{S}^0 \to \mathbb{S}^{\rho_G - 1}$$

and hence

$$H^2(\mathbb{S}^{\rho_G}/G, \mathbb{Z}) \simeq \widetilde{H}^0((\mathbb{S}(\rho_G - 1)/G), \mathbb{Z}) \simeq 0$$

(the last isomorphism follows from the fact, that $\mathbb{S}(\rho_G - 1)/G$ is connected). □

Remark 2.10. In fact with more efforts one can prove that π_{-3} also vanishes for pure G-spectra, but we don't need this.

References

[HR16] M. Hill and D. Ravenel, *MIT Talbot 2016: Equivariant stable homotopy theory and the Kervaire invariant* (2016), available at http://math.mit.edu/conferences/talbot/index.php?year=2016&sub=talks.

[Lur13] J. Lurie, *Higher Algebra* (2013), available at http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf.