The Kervaire Invariant One Problem, Talk 6
Independent University of Moscow, Fall semester 2016

1 Unstable G-equivariant homotopy theory

Notation 1.1. For the rest of this section G will be a compact real Lie group.

Denote $\mathrm{T}op^G$ for the topologically enriched category of G-topological spaces, i.e. topological spaces with continuous action of G and continuous G-equivariant maps.

This category has all (co)limits and forgetfull functor to ordinary topological spaces preserves it. For example the product $X \times_G Y$ is $X \times Y$ with diagonal action of G. Also $\mathrm{T}op^G$ has an inner hom G-space $\mathrm{Map}_G(X, Y)$ again with underlining topological space the usual $\mathrm{Map}(X, Y)$ (with compact open topology) and an action of G given by

$$(gf)(x) = gf(g^{-1}x), \quad g \in G, f \in \mathrm{Map}(X, Y)$$

Usual adjunctions hold

$$\mathrm{Map}_G(X \times_G Y, Z) \simeq \mathrm{Map}_G(X, \mathrm{Map}_G(Y, Z))$$

Definition 1.2. Let V be a finite dimensional real representation of G. We denote a one point compactification of V by $\mathbb{S}^V_G$ and will call it a *V-sphere*. It is a a pointed G-equivariant space with ∞ as a base point.

Having defined G-spheres we have a notion of a suspension and loop spaces functors for every G-representation V. Namely for any pointed G-space X we set

$$\Sigma^V_G X = X \wedge_G \mathbb{S}^V_G \qquad \Omega^V_G = \mathrm{Map}_G(\mathbb{S}^V_G, X)$$

Example 1.3. We always have the usual sphere $\mathbb{S}^n$ corresponding to the trivial n-dimensional representation of G. Correspondingly we have a usual suspension functor Σ^n_G.

For any closed subgroup $H \leqslant G$ we have a G-equivariant space G/H, an *H-orbit space*. These play a role of 0-cells in equivariant setting as we will see below. We have G-spheres $\Sigma^n_G(G/H)_+$ and G-cells $\mathbb{D}^n \wedge_G (G/H)+$. Define G-CW complex to be a G-space which can be obtained by usual procedure of gluing of these generalized G-cells along G-spheres.

Definition 1.4. For a pointed G-space X we define

$$\pi^H_n(X) = \mathrm{Map}_G(\mathbb{S}^n, X)^H \simeq \mathrm{Map}_G(\Sigma^n_G(G/H)_+, X) \simeq \mathrm{Map}(\mathbb{S}^n, X^H) = \pi_n(X^H)$$

Definition 1.5. A G-equivariant homotopy between two maps $f, g\colon X \to Y$ is a G-equivariant map

$$h\colon X \times_G I \to Y$$

such that $h_{|X\times\{0\}} = f$ and $h_{|X\times\{1\}} = g$ (where I is an interval $[0;1]$ with the trivial G-action).

Definition 1.6. We say that $f\colon X \to Y$ is a weak G-homotopy equivalence if it induces an (ordinary) weak homotopy equivalence

$$f^H\colon X^H \to Y^H$$

for any closed subgroup $H \leqslant G$ (we denote X^H, Y^H for *ordinary* or *honest* fixed point spaces).

Example 1.7. Of cause any G-equivariant homotopy equivalence is a G-equivariant weak homotopy equivalence, because G-equivariant homotopy equivalence restricts to homotopy equivalence of H-fixed point spaces for any H.

Definition 1.8. We define $\mathcal{T}\mathrm{p}^G$ to be (simplicial) localization of $\mathrm{T}op^G$ with respect to weak G-equivariant homotopy equivalences and will call it the *G-equivariant homotopy type category.*

Here are some examples of objects in $\mathcal{T}\mathrm{p}^G$.

Example 1.9. We have the fully-faithful embedding $\mathcal{T}\mathrm{p} \hookrightarrow \mathcal{T}\mathrm{p}^G$ as topological spaces with trivial action of G.

Example 1.10. Let $\mathbb{E}G$ be a contractible topological space with free G-action. We have a G-equivariant map $p\colon \mathbb{E}G \to *$. For G nontrivial p is *not* a G-equivariant equivalence, because

$$p^G\colon \varnothing \simeq (\mathbb{E}G)^G \to *^G \simeq *$$

is not a weak homotopy equivalence.

We now want to prove some structural results about $\mathcal{T}\mathrm{p}^G$. But first we need few definitions and technical results.

Definition 1.11. Let $\mathcal{O}_G$, the *orbit category of* G, be the full subcategory of $\mathcal{T}\mathrm{p}^G$ on orbit spaces G/H. We will sometime denote the object $G/\{1_G\} \in \mathcal{O}_G$ just by G.

Note that by definition we have

$$\mathrm{Hom}_{\mathcal{O}_G}(G/H_1, G/H_2) \simeq (G/H_2)^{H_1} \quad \mathrm{End}_{\mathcal{O}_G}(G/H) \simeq G/N_G(H) =: W_G H$$

where $N_G(H)$ is the normalizer of H in G.

We will use the following results

Theorem 1.12. *Every G-space is G-weakly equivalent to a retract of G-CW complex.*

Proposition 1.13 (Hovey, proposition 2.4.2)**.** *Every compact topological space is compact object of* $\mathrm{T}op$ *with respect to the diagram of cellular inclusions of CW complexes.*

Lemma 1.14. *The fixed point functor* $-^H \colon \mathcal{T}\mathrm{p}^G \to \mathcal{T}\mathrm{p}$ *preserves colimits.*

Proof. Every colimit is a filtered colimit of finite ones, so it is enough to threat these two cases.

Note that $-^H \simeq \mathrm{Hom}_G(G/H, -)$. Hence we need to prove that G/H are compact objects of $\mathcal{T}\mathrm{p}^G$. It is enough to prove that $\mathrm{Hom}_{\mathrm{T}op^G}(G/H, -)$ preserves filtered diagrams of G-CW-complexes with cellular closed inclusions. The functor $-^{\{e\}} \simeq \mathrm{Hom}_G(G, -)$ preserves all colimits, because it admits right adjoint $G \times_G -$. For closed subgroup H we have that G/H is the colimit of $G \times_G H \rightrightarrows G$ in $\mathrm{T}op^G$. Compact objects are closed under finite colimits, hence it is enough to prove that $G \times_G H$ is compact. Let X_α be a filtered diagram og G-CW-complexes with cellular inclusions in $\mathrm{T}op^G$. Then

$$\mathrm{Hom}_G(G \times_G H, \varinjlim X_\alpha) \simeq \mathrm{Hom}_{\mathrm{T}op}(H, \varinjlim X_\alpha) \simeq \varinjlim \mathrm{Hom}_{\mathrm{T}op}(H, X_\alpha) \simeq \varinjlim \mathrm{Hom}_G(G \times_G H, X_\alpha)$$

where we have used that H is a compact Lie group, hence compact as an object of $\mathrm{T}op$.

For finite colimits it is enough to prove statement for initial object (which is obvious) and pushouts. Let

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

be a pushout diagram. Without loss of generality we can assume that $X \to Y$ and $X \to Z$ are injective. We always have the canonical map

$$\varphi \colon Y^H \coprod_{X^H} Z^H \to W^H$$

and we want to prove that in our case it is an equivalence. It is enough to prove that ψ is bijection. But as G-set

$$Y \coprod_X Z \simeq X \coprod (Y \setminus X) \coprod (Z \setminus X)$$

and fixed points obviously commute with disjoint unions. □

Remark 1.15. The same result doesn't hold in $\mathrm{T}op^G$. For example the pushout of the diagram $* \leftarrow G \to *$ is $*$ and for nontrivial G the pushout of invariants $* \leftarrow \varnothing \to *$ is $* \coprod *$.

Let $\mathcal{F}_- \colon \mathcal{T}\mathrm{p}^G \to \mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{T}\mathrm{p})$ be the restricted Yoeneda functor, i.e. the composite

$$\mathcal{T}\mathrm{p}^G \xrightarrow{Y} \mathrm{Fun}((\mathcal{T}\mathrm{p}^G)^{op}, \mathcal{T}\mathrm{p}) \longrightarrow \mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{T}\mathrm{p})$$

Note that for a G-space X by definition we have

$$\mathcal{F}_X(G/H) = \mathrm{Hom}_{\mathcal{T}\mathrm{p}^G}(G/H, X) \simeq X^H$$

Let us denote the canonical inclusion $\mathcal{O}_G \hookrightarrow \mathcal{T}\mathrm{p}^G$ by i. We then can consider the left Kan extension of i along the Yoneda embedding, which we will denote by $|-|$

$$\begin{array}{ccc} \mathcal{O}_G & \overset{i}{\hookrightarrow} & \mathcal{T}\mathrm{p}^G \\ {\scriptstyle Y}\big\downarrow & \nearrow_{|-|} & \\ \mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{T}\mathrm{p}) & & \end{array}$$

Now we can prove the following result

Theorem 1.16 (Elmendorf)**.** *Functors*

$$|-|\colon \mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{T}\mathrm{p}) \leftrightarrows \mathcal{T}\mathrm{p}^G \colon \mathcal{F}_{-}$$

are mutually inverse equivalences.

Proof. By definition we have the adjunction $|-| \dashv \mathcal{F}_{-}$. By co-Yoneda's lemma every presheaf $\mathcal{X} \in \mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{T}\mathrm{p})$ is a colimit of G/H_α. By Yoneda's lemma for 0-cells G/H_α the unit of adjuntion is identity, hence the unit of adjunction

$$\mathcal{X} \to \mathcal{F}_{|\mathcal{X}|}$$

is a weak homotopy equivalence, because both $|-|$ and $-^H$ (by lemma 1.14) preserve colimits. Hence, $|-|$ is fully-faithful.

On the other hand by theorem 1.12 every object in $\mathcal{T}\mathrm{p}^G$ is equivalent to a colimits of G/H, hence $|-|$ is essentially surjective. ☐

Example 1.17. By Elmendorf's theorem G-space $\mathbb{E}G$ is equivalent to presheaf

$$\mathcal{F}_{\mathbb{E}G}(G/H) \simeq \begin{cases} *, & H = \{e\} \\ \varnothing, & H \neq \{e\} \end{cases}$$

Using Elmendorf's theorem we can give a few interesting examples of G-spaces.

Example 1.18. Let $\mathcal{U}$ be a family of subgroups of G closed under subconjugations. Consider presheaf

$$\mathcal{F}_{\mathcal{U}}(G/H) = \begin{cases} *, & H \in \mathcal{U} \\ \varnothing, & H \notin \mathcal{U} \end{cases}$$

We will denote the corresponding space by $\mathbb{E}_{\mathcal{U}}G$ ($\mathbb{E}G$ is the special case with $\mathcal{U}$ consisting only of trivial subgroup). The space $\mathbb{E}_{\mathcal{U}}$ has the following universal property: for any G-space X there is a map from X to $\mathbb{E}_{\mathcal{U}}$ if and only if $X^H \simeq \varnothing$ for any $H \notin \mathcal{U}$.

Example 1.19. Let $\mathcal{A}$ be a presheaf of abelian group on $\mathcal{O}_G$. For any $n \geq 0$ we then have a G-space $K(\mathcal{A}, n)$ (or $\mathbb{B}^n\mathcal{A}$) such that

$$\underline{\pi_i} K(\mathcal{A}, n) \simeq \begin{cases} \mathcal{A}, & i = n \\ 0, & i \neq n \end{cases}$$

1.1 Aside: Sullivan's conjecture

Construction 1.20. We have the inclusion $\mathbb{B}G^{op} \hookrightarrow \mathcal{O}_G$, $* \mapsto G/\{1_G\}$ which induces a restriction functor $\mathcal{T}\mathrm{p}^G \to \mathcal{T}\mathrm{p}^{\mathbb{B}G}$ (it is the same as to consider G-space X just as a weak G-representation in $\mathcal{T}\mathrm{p}$). For a G-spaces X and a closed subgroup $H \leqslant G$ we define

$$X^{hH} := \lim(\mathbb{B}H \to \mathbb{B}G \to \mathcal{O}_G^{op} \xrightarrow{\mathcal{F}_X} \mathcal{T}\mathrm{p})$$

and call it an *H-homotopy fixed points of X*. Homotopy coinvariants X_{hH} are defined as the colimit of the diagram above.

Note that by definition

$$\mathrm{Map}(X^H, X^{hH}) \simeq \mathrm{Map}(X^H, X)^{hH}$$

The latter space has a point corresponding to $X^H \to X$. Hence the inclusion $X^H \hookrightarrow X$ factors canonically as

$$X^H \to X^{hH} \to X$$

and analogously for coinvariants.

Also the difference between homotopy $-^{hG}$ and honest $-^G$ fixed points in general is quite big, in this section we will formulate one beautiful result, showing that in some special cases this difference vanishes. The exposition here is generally follows Lurie's lecture notes [Lur07].

The key step is the following theorem

Theorem 1.21. *For any finite-dimensional $\mathbb{F}_p$-vector space V there exists* Lannes T-functor $T_V : \mathrm{Mod}_{F_p} \to \mathrm{Mod}_{F_p}$ *such that*

- *T_V is exact.*
- *For any space X there is a canonical equivalence of $\mathbb{F}_p$-modules*

$$T_V C^*(X, \mathbb{F}_p) \simeq C^*(X^{hV}, \mathbb{F}_p)$$

For the proof consult Lurie's lectures or [May, Chapter VIII, 2].

Proposition 1.22. *Let G be a finite p-group. Then the functor*

$$-^{hG} : \mathcal{T}\mathrm{p}_p^{hG} \to \mathcal{T}\mathrm{p}_p$$

preserves finite colimits.

Proof. Every finite p-group is solvable, hence there is a filtration $\{1_G\} = G_0 \subset G_1 \subset \ldots \subset G_n = G$ such that $G_i/G_{i-1} \simeq V_i$ where V_i is a finite dimensional $\mathbb{F}_p$-vector space. We will prove the proposition by induction on i.
Abelian case. For a finite digram $\{X_k\}_{k \in K}$ of p-complete spaces the canonical map

$$\varphi \colon \operatorname*{colim}_K X_k^{hV_i} \to (\operatorname*{colim}_K X_k)^{hV_i}$$

induces an equivalence

$$C^*(\operatorname*{colim}_K X_k^{hV_i}, \mathbb{F}_p) \xrightarrow{\sim} C^*((\operatorname*{colim}_K X_k)^{hV_i}, \mathbb{F}_p)$$

by exactness of T_{V_i} of theorem 1.21. By definition of Bousfield localization φ itself is an equivalence.

Induction step. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fiber sequence of spaces such that functors

$$\operatorname*{colim}_B \colon \operatorname{Fun}(B, \mathcal{T}\mathrm{p}) \to \mathcal{T}\mathrm{p} \qquad \operatorname*{colim}_F \colon \operatorname{Fun}(B, \mathcal{T}\mathrm{p}) \to \mathcal{T}\mathrm{p}$$

preserve finite colimits. We will prove that functor colim_E also preserves finite colimits. To do it, note that

$$\operatorname*{colim}_E \simeq \operatorname*{colim}_B \circ p_!$$

where $p_!$ is the left Kan extension of colim_E along p. Hence by assumptions on B it is enough to prove that $p_!$ preserves finite colimits. For a diagram $\mathcal{F}_\alpha \in \operatorname{Fun}(E, \mathcal{T}\mathrm{p})$ we always have the canonical map

$$\varphi\colon \operatorname*{colim}_{\operatorname{Fun}(B,\mathcal{T}\mathrm{p})} p_!(\mathcal{F}_\alpha) \to p_! \left(\operatorname*{colim}_{\operatorname{Fun}(E,\mathcal{T}\mathrm{p})} \mathcal{F}_\alpha \right)$$

For a finite diagram we want to prove that φ is an equivalence and one can check this fiberwise. But p is a Grothendieck fibration, hence for any $b \in B$ we have

$$p_!(-)_b \simeq \operatorname*{colim}_F i^* -$$

By assumption on F the right hand side functor preserves finite colimits, hence φ_b is an equivalence for all $b \in B$ and therefore so is φ.

The induction step follows by taking $F = G_{i-1}$, $E = G_i$ and $B = G_i/G_{i-1}$. □

Theorem 1.23 (Sullivan conjecture)**.** *Let G be a finite p-group and X a finite G-space. Then the canonical map*

$$(X^G)_p \to (X^{hG})_p \to X_p^{hG}$$

is a homotopy equivalence.

Proof. The completion and honest fixed point functors always preserve colimits and homotopy fixed point functor preserves finite colimits by proposition 1.22. It follows that the full subcategory of G-spaces for which the lemma is true is closed under finite colimits.

So it is enough to prove the theorem for $X = G/H$ for all closed subgroups H of G. For $H = G$ both $(X^G)_p$ and X_p^{hG} are just points. For $H \neq G$ both spaces X^G and $X_p^{hG} \simeq X^{hG}$ are empty. □

References

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