The Kervaire Invariant One Problem, Talk 7
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1 G-spectra

A doubly naive approach to G-equivariant stable category would be to define

Definition 1.1. The category of **doubly naive** or **weak G-spectra** Sp^{hG} is $\mathrm{Fun}(\mathbb{B}G, \mathrm{Sp})$.

We've seen this doesn't work even for G-spaces. Instead a naive approach to the stable G-category would be

Definition 1.2. The category of **naive G-spectra** Sp^{nG} is the stabilization of $\mathcal{S}^G$, which is by Elmendorf's theorem is equivalent to $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Sp})$.

The category Sp^{nG} remembers more about G, but for various reasons may be not satisfying for what one would like too call 'the category of cohomology theories on G-spaces'. We will indicate two points here

- We have
$$\mathrm{Map}_{\mathrm{Sp}^{nG}}(\Sigma^\infty_+ G/H, \mathbb{S}) \simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+(G/H)/G, \mathbb{S}) \simeq \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, \mathbb{S}) \simeq \mathbb{S}$$
because G-acts trivially on $\mathbb{S}$. This shows that for $H \neq G$ the stabilization of zero dimensional cell $\Sigma^\infty_+ G/H$ can not be dualizable in Sp^{nG}. So we do not expect in general a good behavior (such as Poincar duality or pushforwards in cohomology) from a cohomology theory represented by a naive G-spectrum.

- In ordinary homotopy theory we have the Freudenthal suspension theorem, stating that the canonical map, stating that the canonical map $Y \to \Omega\Sigma Y$ is a $2n$-equivalence, if Y is n-connected. In equivariant homotopy theory we have the generalization of this not only for ordinary suspension, but for any representation sphere $\mathbb{S}^V$

  Proposition 1.3. *Let Y be a G-space and $n\colon \mathcal{O}_G \to \mathbb{N}$ a function, such that*

  1. *$n(H) \leq 2\,\mathrm{conn}(Y^H) + 1$ for all subgroups H such that $V^H \neq 0$.*
  2. *$n(H) \leq \mathrm{conn}(Y^K)$ for all pairs of subgroups H, K such that $V^H \neq V^K$.*

  Then the canonical map $Y \to \Omega^V \Sigma^V Y$ induces an isomorphism
$$\pi_k^H(Y) \to \pi_k^H(\Omega^V \Sigma^V Y)$$
  for $k \leq n(H)$. Therefore if $\dim X^H < n(H)$ for all $H \leq G$, then the canonical map
$$[X, Y] \to [\Sigma^V X, \Sigma^V Y]$$
  is bijective.

The second point is very suggestive. Namely, recall that the category of spectra is an universal (in an appropriate sense) category with the functor $\Sigma^\infty\colon \mathcal{S}_+ \to \mathrm{Sp}$ with the property that the suspension functor is invertible in Sp. Now the suspension is given by the smash product with $\mathbb{S}^1$ (which one may consider as a representation sphere for one dimensional trivial representation), so in equivariant world we may want to invert multiplication by representation spheres $\mathbb{S}^V$ for nontrivial representations V as well. Before proceed with the actual construction, we will explain the idea in a simple example.

Example 1.4. Let R be an ordinary commutative ring and x some element of R. Then the localization $R[x^{-1}]$ is isomorphic to the direct limit

$$R[x^{-1}] \simeq \varinjlim(R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \dots)$$

taken in the category of R-modules. One way to deduce R-algebra structure on $R[x^{-1}]$ is to not that

$$R[x^{-1}] \otimes_R R[x^{-1}] \simeq (R[x^{-1}])[x^{-1}] \simeq R[x^{-1}]$$

induces multiplication map $x \otimes y \mapsto xy$.

Now let R be an object in symmetric monoidal category $\mathcal{C}$ and $x\colon R \to R$ some map. Assume we want to define localization $R[x^{-1}]$. Generally it is hard to write down explicit formulas in higher categorical setting, so instead one may prefer to adopt the approach above, and it is a result of Lurie [Lur13, Section 4.8.2], that one can actually do it.

In particular we are interested in the case where $\mathcal{C}$ is the category of presentable stable categories $\mathcal{P}r^{st,L}$. Recall this category has a monoidal structure $\widehat{\otimes}$ (just a completion of the usual Cartesian monoidal structure), such that commutative algebras in $(\mathcal{P}r^{st,L}, \widehat{\otimes})$ are precisely presentably monoidal categories. For a category $\mathcal{D}$ we will use the procedure indicated above to invert the family of objects $X_i \in \mathcal{D}$.

One last note is that one sometimes wants to invert not all representation spheres, but just some family of them, and it is equally easy to develop the theory in this generality, so we will do it.

Let $\mathcal{U}$ be a set of orthogonal finite-dimensional G-representations.

Construction 1.5. Let $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers with usual linear order. Let $I_{\mathcal{U}}$ be the subset of the set $(\mathbb{Z}_{\geq 0})^{\mathcal{U}}$ of all sequences $\{n_V\}_{V \in \mathcal{U}}$, such that only finitely many n_V are nonzero. There is an induced partial order on $I_{\mathcal{U}}$ (lexicographical one), so we can consider $I_{\mathcal{U}}$ as a category. It is easily to see that $I_{\mathcal{U}}$ is filtered.

There is a functor $\mathcal{I}_{\mathcal{U}}\colon I_{\mathcal{U}} \to \mathrm{Mod}_{\mathrm{Sp}^{nG}}$ which sends every object of $I_{\mathcal{U}}$ to the category Sp^{nG} and the morphism $\{n_V\} \to \{m_V\}$ to the smashing functor

$$X \mapsto X \wedge \bigwedge_{V \in \mathcal{U}} \mathbb{S}^{(m_V - n_V)V}$$

Define **$\mathcal{U}$-indexed genuine stable G-category** $\mathrm{Sp}^{\mathcal{U}}$ to be the colimit

$$\mathrm{Sp}^{\mathcal{U}} := \varinjlim \mathcal{I}_{\mathcal{U}}$$

in the category of Sp^{nG}-module categories.

We will introduce some notations right away

Notation 1.6.

- We will denote the canonical functor $\mathrm{Sp}^{nG} \to \mathrm{Sp}^{\mathcal{U}}$ by $\widetilde{\Sigma}^\infty_{\mathcal{U}}$ and its right adjoint (which exists by adjoint functor theorem) by $\widetilde{\Omega}^\infty_{\mathcal{U}}$. By composing $\widetilde{\Sigma}^\infty_{\mathcal{U}}$ and $\widetilde{\Omega}^\infty_{\mathcal{U}}$ with $\Sigma^\infty\colon \mathcal{S}^G \to \mathrm{Sp}^{nG}$ and $\mathcal{S}^G \leftarrow \mathrm{Sp}^{nG}\colon \Omega^\infty$ respectively we obtain the pair of adjoint functors

$$\Sigma^\infty_{\mathcal{U}}\colon \mathcal{S}^G \leftrightarrows \mathrm{Sp}^{\mathcal{U}}\colon \Omega^\infty_{\mathcal{U}}$$

- We will also define the forgetful functor $-^u \colon \mathrm{Sp}^{\mathcal{U}} \to \mathrm{Sp}^{hG}$ by the composition of $\widetilde{\Omega}^\infty_{\mathcal{U}}$ and the forgetful functor $\mathrm{Sp}^{nG} \to \mathrm{Sp}^{hG}$ (which is the evaluation on G).

- We will denote the monoidal unit of $\mathrm{Sp}^{\mathcal{U}}$ by $\mathbb{S}_{\mathcal{U}}$ of just by $\mathbb{S}$ when $\mathcal{U}$ is clear from the context.

- For a pointed G-space X an $\mathcal{U}$-spectrum E we will abbreviate

$$X \wedge E := \Sigma^\infty_{\mathcal{U}} X \wedge E \qquad \mathrm{Map}_{\mathrm{Sp}^{\mathcal{U}}}(X, E) := \mathrm{Map}_{\mathrm{Sp}^{\mathcal{U}}}(\Sigma^\infty_{\mathcal{U}} X, E)$$

More generally we will sometimes abbreviate $\Sigma^\infty_{\mathcal{U}} X$ just by X, when there is no risk of confusion.

Remark 1.7. By adjoint functor theorem the category of presentable stable categories and colimit preserving functors $\mathcal{P}r^{st,L}$ is canonically equivalent to the category $\mathcal{P}r^{st,R}$ of presentable stable categories and functor, preserving limits and filtered colimits for big enough cardinal. The equivalence is identity on objects and send functor to its adjoint. It follows that

$$\mathrm{Sp}^{\mathcal{U}} \simeq \varprojlim \mathcal{I}^R_{\mathcal{U}}$$

where $\mathcal{I}^R_{\mathcal{U}}$ defined similarly to $\mathcal{I}_{\mathcal{U}}$ but with $\Omega^V(-)$ instead of Σ^V-.

From this description we see that an object of $\mathrm{Sp}^{\mathcal{U}}$ is given by the set $\{X_W\}_{W \in \mathcal{U}}$ of naive G-spectra with the coherent set of equivalences $\Omega^V X_{W \oplus V} \simeq X_W$. The disadvantage of this description is that it is not obvious why this limit category admits a good monoidal structure, or why $\widetilde{\Sigma}^\infty_{\mathcal{U}}$ is a monoidal functor.

Remark 1.8. By what we've learned in the talk on presentable and stable categories, we know that for $X, Y \in \mathrm{Sp}^{nG}$ we have

$$\mathrm{Hom}_{\mathrm{Sp}^G}(\widetilde{\Sigma}^\infty_{\mathcal{U}} X, \widetilde{\Sigma}^\infty_{\mathcal{U}} Y) \simeq \varinjlim \mathrm{Hom}_{\mathrm{Sp}^{nG}}(\Sigma^V X, \Sigma^V Y)$$

Remark 1.9. We could also start from the category of G-spaces $\mathcal{S}^G$ to define $\mathrm{Sp}^{\mathcal{U}}$. In this case we should've add the trivial representation $\mathbb{R}$ to $\mathcal{U}$.

Directly from definition we can establish the following basic properties of $\mathrm{Sp}^{\mathcal{U}}$

Proposition 1.10. *1. For any $V \in \mathcal{U}$ the object $\widetilde{\Sigma}^\infty_{\mathcal{U}} \mathbb{S}^V$ is invertible in $\mathrm{Sp}^{\mathcal{U}}$.*

2. $\mathrm{Sp}^{\mathcal{U}}$ is symmetric presentably monoidal stable category and the functor $\widetilde{\Sigma}^\infty_{\mathcal{U}}$ is monoidal.

3. The unit $\widetilde{\Sigma}_{\mathcal{U}} \circ \widetilde{\Omega}^\infty_{\mathcal{U}} \to 1_{\mathrm{Sp}^{\mathcal{U}}}$ and counit $1_{\mathrm{Sp}^{nG}} \to \widetilde{\Omega}^\infty_{\mathcal{U}} \circ \widetilde{\Sigma}_{\mathcal{U}}$ of adjunction $\widetilde{\Sigma}_{\mathcal{U}} \dashv \widetilde{\Omega}^\infty_{\mathcal{U}}$ are equivalences of weak G-spectra.

Proof. 1. On the level of colimit diagram for $\mathrm{Sp}^{\mathcal{U}}$ the smashing with $\widetilde{\Sigma}^\infty_{\mathcal{U}} \mathbb{S}^V$ corresponds to the inclusion of cofinal subcategory $\{n_W\}_{W \in \mathcal{U}}, n_V > 0$ of $I_{\mathcal{U}}$.

2. By [Lur13, proposition 4.8.2.9.] it is enough to prove that the canonical map

$$\mathrm{Sp}^{\mathcal{U}} \simeq \mathrm{Sp}^{\mathcal{U}} \otimes \mathrm{Sp}^{nG} \to \mathrm{Sp}^{\mathcal{U}} \otimes \mathrm{Sp}^{\mathcal{U}}$$

is an equivalence. But

$$\mathrm{Sp}^{\mathcal{U}} \otimes \mathrm{Sp}^{\mathcal{U}} \simeq \mathrm{Sp}^{\mathcal{U}} \otimes \varinjlim \mathrm{Sp}^{nG} \simeq \varinjlim (\mathrm{Sp}^{\mathcal{U}} \otimes \mathrm{Sp}^{nG}) \simeq \varinjlim \mathrm{Sp}^{\mathcal{U}} \simeq \mathrm{Sp}^{\mathcal{U}}$$

where we have used that all functors in the diagram

$$\mathrm{Sp}^{\mathcal{U}} \otimes (\mathrm{Sp}^{nG})_{I_{\mathcal{U}}}$$

are smashings with $\mathbb{S}^V$, hence equivalences by the previous.

3. The composition $\widetilde{\Omega}^\infty_{\mathcal{U}} \circ \widetilde{\Sigma}^\infty_{\mathcal{U}}$ sends a naive G-spectrum X to $\varinjlim \Omega^V \Sigma^V X$. The forgetful functor commutes with (co)limits and for any representation V the following diagrams are commutative

$$\begin{tikzcd} \mathrm{Sp}^{nG} \arrow[r, "\Sigma^V"] \arrow[d, "-^u"'] & \mathrm{Sp}^{nG} \arrow[d, "-^u"] \\ \mathrm{Sp} \arrow[r, "\Sigma^{\dim V}"] & \mathrm{Sp} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sp}^{nG} \arrow[r, "\Omega^V"] \arrow[d, "-^u"'] & \mathrm{Sp}^{nG} \arrow[d, "-^u"] \\ \mathrm{Sp} \arrow[r, "\Omega^{\dim V}"] & \mathrm{Sp} \end{tikzcd}$$

where $-^u$ is the forgetful functor. It follows that

$$(\widetilde{\Omega}^\infty_{\mathcal{U}} \widetilde{\Sigma}^\infty_{\mathcal{U}} X)^u \simeq (\varinjlim \Omega^V \Sigma^V X)^u \simeq \varinjlim \Omega^{\dim V} \Sigma^{\dim V} X^u \simeq X^u$$

Analogues statement for co-unit of adjunction follows from definition of forgetful functor and triangular identities.

□

Definition 1.11. The universe $\mathcal{U}$ consist of all finite-dimensional representations of G is called **complete**. For the complete universe $\mathcal{U}$ we will call $\mathrm{Sp}^{\mathcal{U}}$ the category of **genuine G-spectra** or just **G-spectra** and will denote it by Sp^G.

The rest of this section is devoted to the proof of stable analogue of Elmendorf's theorem. In what follows we will need the following geometric fact

Theorem 1.12. *Any smooth closed G-manifold admits a finite G-cell decomposition.*

Corollary 1.13. *Any closed smooth G-manifold is finite G-space.*

Lemma 1.14. *For any universe $\mathcal{U}$ stabilization of orbits $\Sigma^\infty_{\mathcal{U}+} G/H$ are compact in $\mathrm{Sp}^{\mathcal{U}}$.*

Proof. By corollary 1.13 representation spheres $\mathbb{S}^V$ are compact objects of Sp^{nG}, hence loop spaces functors $\mathrm{Map}_{\mathrm{Sp}^{nG}}(\mathbb{S}^V, -)$ are continuous. From this observation and remark 1.7 we see that the category $\mathrm{Sp}^{\mathcal{U}}$ is equivalent to the limit in the category of **presentable** categories, hence the canonical projection functor $\widetilde{\Omega}^\infty_{\mathcal{U}}$ is continuous.

Now let X_α be a filtered diagram of $\mathcal{U}$-spectra. Then

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sp}^{\mathcal{U}}}(\Sigma^\infty_{\mathcal{U}+} G/H, \varinjlim X_\alpha) \simeq \mathrm{Hom}_{\mathcal{S}^G}(G/H, \varinjlim \Omega^\infty_{\mathcal{U}} X_\alpha) \simeq \\ \simeq \varinjlim \mathrm{Hom}_{\mathcal{S}^G}(G/H, \Omega^\infty_{\mathcal{U}} X_\alpha) \simeq \varinjlim \mathrm{Hom}_{\mathrm{Sp}^{\mathcal{U}}}(\Sigma^\infty_{\mathcal{U}+} G/H, X_\alpha) \end{aligned}$$

where we have used that G/H are compact in $\mathcal{S}^G$. □

Proposition 1.15. *The functor $\widetilde{\Omega}^\infty_{\mathcal{U}}$ is conservative.*

Proof. By remark 1.7 and appendix to the second talk it is enough to prove that the functor $\Omega^V \colon \mathrm{Sp}^{nG} \to \mathrm{Sp}^{nG}$ is conservative. Let $f \colon X \to Y$ be a map of naive G-spectra. Then $\Omega^V(f)$ is an equivalence if and only if the cofiber $\Omega^V(Y)/\Omega^V(X)$ vanishes. Hence it is enough to prove that $\Omega^V Z \neq 0$ for $Z \neq 0$ and any representation V.

Let $Z \neq 0$ be a spectrum with G-action and V a representation of G. First assume that for some proper subgroup $i_H \colon H \hookrightarrow G$ the restriction $i_H^* Z$ is non-zero. Then by induction on dimension and the number of connected components we have

$$i_H^* \Omega^V Z \simeq \Omega^{i_H^* V}(i_H^* Z) \neq 0$$

hence $\Omega^V Z \neq 0$.

Now assume $i_H^* Z \simeq 0$ for all proper subgroups H of G. Then $Z^H \simeq 0$ for all $H < G$ and $Z^G \neq 0$ since $Z \neq 0$. Hence Z is concentrated as a presheaf over G/G and

$$(\Omega^V Z)^G = \mathrm{Map}_{\mathrm{Sp}^{nG}}(\mathbb{S}^V, Z)^G \simeq \mathrm{Hom}_{\mathrm{Sp}}((\mathbb{S}^V)^G, Z^G) \simeq \Omega^{V^G} Z^G \neq 0$$

hence $\Omega^V Z \neq 0$. $\square$

Corollary 1.16 (Stable G-Whitehead lemma)**.** *For any universe $\mathcal{U}$ the morphism $X \to Y$ in $\mathrm{Sp}^{\mathcal{U}}$ is an equivalence if and only if the induced morphism*

$$\pi_*^H(X) \to \pi_*^H(Y)$$

is an isomorphism for any H.

Proof. Let $f\colon X \to Y$ be a morphism inducing isomorphism on all homotopy groups π_*^H. By the previous proposition it is enough to prove that $\widetilde{\Omega}_{\mathcal{U}}^\infty(f)$ is an equivalence. But

$$\pi_i^H(X) = \pi_i \mathrm{Hom}_{\mathrm{Sp}^{\mathcal{U}}}(\Sigma_{\mathcal{U}}^\infty G/H_+, X) \simeq \pi_i \mathrm{Hom}_{\mathrm{Sp}^{nG}}(\Sigma^\infty G/H_+, \widetilde{\Omega}_{\mathcal{U}}^\infty X) = \pi_i^H(\widetilde{\Omega}_{\mathcal{U}}^\infty X)$$

and analogously for Y. Hence $\widetilde{\Omega}_{\mathcal{U}}^\infty(f)$ is an equivalence by the usual Whitehead's lemma. $\square$

Definition 1.17. The **stable orbit category** is the full subcategory of $\mathrm{Sp}^{\mathcal{U}}$ on $\Sigma_{\mathcal{U}}^\infty G/H_+$.

Theorem 1.18 (Stable Elmendorf's theorem)**.** *For any universe $\mathcal{U}$ the restricted Yoneda functor*

$$\mathcal{Y}_-\colon \mathrm{Sp}^{\mathcal{U}} \to \mathrm{Fun}(\mathcal{SO}_{\mathcal{U}}^{op}, \mathrm{Sp})$$

is an equivalence.

Proof. The Yoneda embedding $\mathcal{Y}_-$ commutes with limits and (by lemma 1.14) with filtered colimits, hence it admits a left adjoint $|-|$. By Yoneda lemma the counit of adjunction is an equivalence for representable presheafs. By co-Yoneda lemma every presheaf is a colimit of representables, hence the counit of adjunction $\mathcal{Y}_{|\mathcal{F}|} \to \mathcal{F}$ is equivalence for every presheaf $\mathcal{F} \in \mathrm{Fun}(\mathcal{SO}_{\mathcal{U}}^{op}, \mathrm{Sp})$.

It is only left to prove that $\mathcal{Y}_-$ is essentially surjective. By the fully-faithful part it is enough to prove that the smallest subcategory of $\mathrm{Sp}^{\mathcal{U}}$ closed under colimits and finite limits and containing $\Sigma_{\mathcal{U}}^\infty G/H_+$ is $\mathrm{Sp}^{\mathcal{U}}$. Let X be an object of $\mathrm{Sp}^{\mathcal{U}}$. Consider the simplicial $\mathcal{U}$-spectrum $X_\bullet$ (the bar resolution of X)

$$\ldots \widetilde{\Sigma}_{\mathcal{U}}^\infty \widetilde{\Omega}_{\mathcal{U}}^\infty \widetilde{\Sigma}_{\mathcal{U}}^\infty \widetilde{\Omega}_{\mathcal{U}}^\infty X \rightrightarrows \widetilde{\Sigma}_{\mathcal{U}}^\infty \widetilde{\Omega}_{\mathcal{U}}^\infty X$$

where edge and degeneration maps are induced by unit and co-unit of adjuncton $\widetilde{\Sigma}_{\mathcal{U}}^\infty \dashv \widetilde{\Omega}_{\mathcal{U}}^\infty$. I claim that the canonical map $p\colon |X_\bullet| \to X$ is an equivalence. Indeed, the map

$$|\widetilde{\Omega}_{\mathcal{U}}^\infty X_\bullet| \simeq \widetilde{\Omega}_{\mathcal{U}}^\infty |X_\bullet| \xrightarrow{\widetilde{\Omega}_{\mathcal{U}}^\infty(p)} \widetilde{\Omega}_{\mathcal{U}}^\infty X$$

is an equivalence, because it admits a splitting. But $\Omega_{\mathcal{U}}^\infty$ is conservative by proposition 1.15, hence p is also equivalence.

So every $\mathcal{U}$-spectrum is a colimit of spectra of the form $\widetilde{\Sigma}_{\mathcal{U}}^\infty Y, Y \in \mathrm{Sp}^{nG}$. It is left to note that by co-Yoneda's lemma every Y is a colimit of $\Sigma^\infty G/H_+$. $\square$

Remark 1.19. The presheaf category $\mathrm{Fun}(\mathcal{SO}_{\mathcal{U}}^{op}, \mathrm{Sp})$ is monoidal with pointwise monoidal structure. The equivalence above is **not** monoidal in general. For example we will see later that $\mathcal{Y}_{\mathbb{S}}(*) = \mathbb{S}^G \not\simeq \mathbb{S}$ for nontrivial G.

Remark 1.20. The second part of the above proof shows that Sp^G are monadic over Sp^{nG}. The corresponding monad send a naive G-spectrum X to

$$\varinjlim \Omega^V \Sigma^V X$$

where V ranges over closure of $\mathcal{U}$ under direct sums.

It follows, that one can think about $\mathcal{U}$-spectrum as of naive G-spectrum with some sort of additional data. Moreover one can explicitly describe this additional data (at least when $\mathcal{U}$ is the complete universe). It is encoded by the so called **transfer maps**. We will return to this point in the next talk.

2 Functoriality of equivariant spectra and fixed points functors

In this section we will introduce some functoriality of equivariant spectra and define various notions of fixed points. The material is more or less trivial, but will be actively used later on. For simplicity we will stick to complete universes everywhere.

Let $\varphi\colon G_1 \to G_2$ be a homomorphism of Lie groups. By restriction of action φ induces the restriction functor $\varphi^*\colon \mathrm{T}op^{G_1} \leftarrow \mathrm{T}op^{G_2}$. It is easy to see that φ^* preserves limits, colimits and weak equivariant homotopy equivalences. It follows that φ induces the restriction functor $\varphi^*\colon \mathcal{S}^{G_1} \leftarrow \mathcal{S}^{G_2}$ which has both left adjoint $\varphi_!$ and right adjoint φ_*. E.g. for $i\colon H \to G$ an inclusion we have

$$i_! X \simeq G \times^H X \qquad i_* X \simeq \mathrm{Map}_H(G, X)$$

It is also easy to check that we have analogous adjunctionon on the level of naive spectra and that φ^* is closed monoidal.

For a naive G_2-spectrum X and G_2-representation V we have that

$$\varphi^*(X \wedge \mathbb{S}^V) \simeq \varphi^*(X) \wedge \varphi^*(\mathbb{S}^V) \simeq \varphi^*(X) \wedge \mathbb{S}^{\varphi^* V}$$

Hence we obtain the induces functor

$$\varphi^*\colon\ \mathrm{Sp}^{G_1} \leftarrow \mathrm{Sp}^{G_2}$$

We have the following basic properties

Proposition 2.1. *1. The functor φ^* admits both left $\varphi_!$ and right φ_* adjoints. We will call these functors **induction** and **co-induction** respectively.*

2. The following diagrams are commutative

$$\begin{tikzcd} \mathcal{S}_*^{G_1} \arrow[r, "\varphi_!"] \arrow[d, "\Sigma^\infty_{G_1}"'] & \mathcal{S}_*^{G_2} \arrow[d, "\Sigma^\infty_{G_2}"] \\ \mathrm{Sp}^{G_1} \arrow[r, "\varphi_!"] & \mathrm{Sp}^{G_2} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}_*^{G_1} \arrow[r, "\varphi_*"] & \mathcal{S}_*^{G_2} \\ \mathrm{Sp}^{G_1} \arrow[r, "\varphi_*"] \arrow[u, "\Omega^\infty_{G_1}"] & \mathrm{Sp}^{G_2} \arrow[u, "\Omega^\infty_{G_2}"'] \end{tikzcd}$$

3. For a G_2-spectrum X we have the canonical equivalences

$$\varphi_!\varphi^* X \simeq \Sigma^\infty_{G_2} G_2/\varphi(G_1)_+ \wedge X \qquad \varphi_*\varphi^* X \simeq \mathrm{Map}_{\mathrm{Sp}^{G_2}}(\Sigma^\infty_{G_2} G_2/\varphi(G_1)_+, X)$$

4. (Projection formulas). Let X be a G_1-spectrum, Y be a G_2-spectrum and Z be a dualiazble G_2-spectrum. Then there is a canonical equivalence

$$\varphi_!(X \otimes \varphi^* Y) \simeq \varphi_!(X) \otimes Y \qquad \varphi_*(X \otimes \varphi^* Z) \simeq \varphi_*(X) \otimes Z$$

5. *Let X be a G_1-spectrum, Y be a G_2-spectrum. Then*

$$\mathrm{Map}_{\mathrm{Sp}^{G_2}}(\varphi_! X, Y) \xrightarrow{\sim} \varphi_* \mathrm{Map}_{\mathrm{Sp}^{G_1}}(X, \varphi^* Y)$$

Remark 2.2. For an inclusion of a closed subgroup $i\colon H \hookrightarrow G$ the induction $i_!$ and co-induction i_* functors are usually denoted as $G \wedge_H -$ and $\mathrm{Map}_H(G, -)$. It is natural choice of notation e.g. by the description of corresponding functors on the level of G-spaces and by the second property above.

Fixed point functors. Recall, that in G-spaces we have two kind of fixed points: homotopical and honest. In this section we will describe naturally defined fixed points functors for G-spectra. First we have

Definition 2.3. Let X be a G-spectrum. For a closed subgroup H of G we define **homotopy H-(co)invariant** as

$$X_{hH} := X^u_{hH} \qquad X^{hH} := (X^u)^{hH}$$

It is easy to see that

$$X_{hH} \simeq (\mathbb{E}G_+ \wedge X)^u/H \qquad X^{hH} \simeq \mathrm{Hom}_{\mathrm{Sp}^G}(\mathbb{E}G_+, X)$$

This is the weakest notion of fixed points. On the opposite we have

Definition 2.4. Let X be a G-spectrum and H a normal closed subgroup of G. We define the **honest** or **categorical homotopy H-fixed points** as $X^H := p_*(X)$, where $p\colon G \twoheadrightarrow G/H$ is the quotient map.

For H not necessarily normal, just restrict X to $N_G(H)$-spectrum and then take fixed points in the above sense. So X^H is always a $W_G H$-spectrum. It is easy to see that the underlying spectrum of X^H is $\mathrm{Hom}_{\mathrm{Sp}^G}(G/H_+, X)$.

Honest fixed points is usually what we are after, but they are hard to deal with. For example

$$(X \wedge Y)^G \not\simeq X^G \wedge Y^G \qquad (\Sigma^\infty_G Z)^G \not\simeq \Sigma^\infty Z^G$$

in general (where $X, Y \in \mathrm{Sp}^G$, $Z \in \mathcal{S}^G$). We have the third type of fixed points in stable setting, which fixes this problem

Construction 2.5. The smash products in Sp^{nG} are computed pointwise. In particular for a normal close subgroup H of G and G-representation V the following diagram

$$\begin{array}{ccc} \mathrm{Sp}^{nG} & \xrightarrow{\Sigma^V} & \mathrm{Sp}^{nG} \\ {\scriptstyle (-)^H}\downarrow & & \downarrow{\scriptstyle (-)^H} \\ \mathrm{Sp}^{nG/H} & \xrightarrow{\Sigma^{V^H}} & \mathrm{Sp}^{nG/H} \end{array}$$

is commutative. It follows that there is an induces functor $\Phi^H\colon \mathrm{Sp}^G \to \mathrm{Sp}^{G/H}$.

For H not necessarily normal again first restrict spectrum to $N_G H$ and then apply the construction above. We will call Φ^H the **geometric H-fixed points functor**.

It enjoys the following properties

Proposition 2.6. *1. For a pointed G-space X we have the canonical equivalence*

$$\Phi^H(\Sigma^\infty_G X) \simeq \Sigma^\infty_{W_G H} X^H$$

2. Geometric fixed point functors Φ^H are monoidal.

3. *Let $\mathcal{P}$ be a family consisting of all proper subgroups of G. There is a canonical equivalence*

$$\Phi^G(X) \simeq \left(\widetilde{\mathbb{E}}\mathcal{P} \wedge X\right)^G$$

Proof. 1. This is just the special case of the definition of Φ^H.

2. The restriction to $\mathrm{Sp}^{nN_G H} \leftarrow \mathrm{Sp}^{nG}$ and H-fixed points $-^H\colon \mathrm{Sp}^{nG} \to \mathrm{Sp}$ are monoidal functors (because smash products of naive G-spectra are computed pointwise), hence Φ^H is monoidal as a filtered colimit of monoidal functors.

3. We want to prove that the natural transformation $-^G \to \Phi^G(-)$ induces an equivalence

$$(\widetilde{\mathbb{E}}\mathcal{P} \wedge X)^G \to \Phi^G(\widetilde{\mathbb{E}}\mathcal{P} \wedge X) \simeq \Phi^G(\widetilde{\mathbb{E}}\mathcal{P}) \wedge \Phi^G(X) \simeq \Phi^G(X)$$

All functors in the diagram above preserves colimits and finite limits, hence it is enough to prove the statement for $X = \Sigma^\infty_G G/H_+$. In this case the right hand side is $\Sigma^\infty(G/H)^G_+$. By tom Dieck splitting the left hand side is also equivalent to $\Sigma^\infty(G/H)^G_+$.

□

3 Duality theory for genuine G-spectra

We will use the following simple observation to establish duality results of this section: let $U \hookrightarrow X$ be an open embedding of G-spaces. Then $\overline{X}/(\overline{X} \setminus U)$ is a model for one point compactification of U (where $\overline{X}$ is one point compactification of X). Hence the one point compactification induces a **contravariant** functor from the category of G-spaces and open embeddings to the category of pointed compact G-spaces.

Example 3.1. Let $i\colon Z \hookrightarrow X$ be an embedding of closed smooth G-manifolds. By the tubular neighborhood theorem one can extend i to the open embedding $j\colon \mathrm{Tot}(\nu_{Z/X}) \hookrightarrow X$ of the total space of normal bundle $\nu_{Z/X}$ of Z in X. Hence we obtain the map $\overline{X} \to \overline{\mathrm{Tot}(\nu_{Z/X})} \simeq Z^{\nu_{Z/X}}$ (where we have used that one point compactification of a total space of a bundle E is a model for a Thom space Z^E for compact Z). This is called **Pontryagin-Thom collapse map**.

One point compactification is not a functor from G-spaces, but for a proper map $p\colon X \to Y$ there is an (obvious) map of one point compactifications $\overline{X} \to \overline{Y}$.

Example 3.2. Let X be a closed manifold and E a vector bundle over X. Consider the vector bundle $0 \boxplus E$ over $X \times X$. The diagonal map $X \to X \times X$ induces a proper map of total spaces of bundles $\mathrm{Tot}(E) \to \mathrm{Tot}(0 \boxplus E)$, which induces the map $X^E \to X_+ \wedge X^E$. This is called **Thom diagonal**.

It follows that for any roof $X \overset{j}{\leftarrow} Z \overset{p}{\to} Y$ where j is an open embedding and p is proper, we obtain a map $\overline{X} \to \overline{Y}$.

We will now establish analogs of classical duality theorems in equivariant setting.

Theorem 3.3 (Atiyah duality)**.** *For a closed smooth G-manifold M the suspension spectrum $\Sigma^\infty_G M_+$ and the Thom spectrum of negative tangent bundle $M^{-\mathbb{T}_M}$ are dual to one another.*

Idea of the proof. There is always an embedding of M into some representation V and by the tubular neighborhood theorem we can extend it to the embedding of normal bundle $\nu \hookrightarrow V$. We will prove that

$$(\Sigma^\infty_G M_+)^\vee \simeq \Sigma^{-V}_G M^\nu \tag{1}$$

the result would follow from equivalence $\mathbb{S}^{-V}_G \wedge M^\nu \simeq M^{\nu - V} \simeq M^{-\mathbb{T}_M}$.

To prove (1) we will simply exhibit co-evaluation $\mathbb{S}_G \to \Sigma^\infty_G M_+ \wedge \Sigma^{-V}_G M^\nu$ and evaluation $\Sigma^\infty_G M_+ \wedge \Sigma^{-V}_G M^\nu \to \mathbb{S}_G$ maps. Let $\eta\colon \mathbb{S}^V \to M_+ \wedge M^\nu$ be a composite of the Pontryagin-Thom collapse map and Thom diagonal and the map $\varepsilon\colon \mathbb{S}^V \to M_+ \wedge M^\nu$ is induces by the roof $M \times \nu \xleftarrow{j} M \times V \xrightarrow{p} V$, where j is an open embedding of (trivial) normal bundle of M in $M \times \nu$ and p is the projection on the second factor. It is an exercise to check that $\Sigma^{\infty-V}_G \eta$ and $\Sigma^{\infty-V}_G \varepsilon$ are desired co-evaluation and evaluation maps respectively. $\square$

Remark 3.4. It follows that the Pontryagin-Thom collapse map $\mathbb{S}^V \to \Sigma^V(\Sigma^\infty M_+)^\vee$ is just a V-suspension of the morphism, dual to the canonical map $\Sigma^\infty_{G+} M \to \Sigma^\infty_{G+} *$.

Corollary 3.5. *The suspension spectrum of orbit space G/H is dualizable in Sp^G with the dual equivalent to $G \wedge_H \mathbb{S}_H^{-L(H)}$, where $L(H)$ is the tangent H-representation at the identity coset of G/H.*

Proof. By the Atiyah duality we know that $\Sigma^\infty G/H_+$ is dualizable with dual $(G/H)^{-\mathbb{T}_{G/H}}$, so it is enough to identify the Thom spectrum.

Let N be an H-manifold with the virtual bundle E over it. Then

$$(G \wedge_H N)^{G \wedge_H E} \simeq G \wedge_H N^E$$

because the induction functor commutes with cofibers and suspension spectrum functor.

The result now follows by taking $N = *$ and $E = -L(H)$. $\square$

Corollary 3.6 (Equivariant Spainer-Whitehead duality)**.** *For genuine G-spectrum X the following conditions are equivalent*

1. *X is a retract of finite G-spectrum.*
2. *X is dualizable.*
3. *X is compact.*

Proof. $\underline{1 \Rightarrow 2.}$ Follows from the previous corollary, because dualiable object in stable presentable category are closed under finite (co)limits and retracts.

$\underline{2 \Rightarrow 3.}$ Let Y_α be a filtered diagram of G-spectra. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sp}^G}(X, \varinjlim Y_\alpha) \simeq \mathrm{Hom}_{\mathrm{Sp}^G}(\mathbb{S}, X^\vee \wedge \varinjlim Y_\alpha) \simeq \mathrm{Hom}_{\mathrm{Sp}^G}(\mathbb{S}, \varinjlim (X^\vee \wedge Y_\alpha)) \simeq \\ \simeq \varinjlim \mathrm{Hom}_{\mathrm{Sp}^G}(\mathbb{S}, X^\vee \wedge Y_\alpha) \simeq \varinjlim \mathrm{Hom}_{\mathrm{Sp}^G}(X, Y_\alpha)\end{aligned}$$

where we have used that $\mathbb{S}$ is compact in Sp^G.

$\underline{3 \Rightarrow 1.}$ By theorem 1.18 we can write X as a filtered colimit of finite G-spectra X_α. By compactness the identity morphism

$$1_X \in \pi_0 \mathrm{Hom}_{\mathrm{Sp}^G}(X, X) \simeq \pi_0 \mathrm{Hom}_{\mathrm{Sp}^G}(X, \varinjlim X_\alpha) \simeq \varinjlim \pi_0 \mathrm{Hom}_{\mathrm{Sp}^G}(X, X_\alpha)$$

factors as

$$X \to X_A \to \varinjlim X_\alpha \simeq X$$

for some A, hence X is a retract of finite G-spectrum X_A. $\square$

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