The Kervaire Invariant One Problem, Talk 4
Independent University of Moscow, Fall semester 2016

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The main aim of this lecture is to define the Kervaire invariant. The main reference is W. Browder's paper [Bro69].

1 Secondary cohomological operations

Let $\mathcal{S}$ be the category of spaces or the category of spectra. Let Z_0, Z_1, Z_2 be objects of the category $\mathcal{S}$ and $\varphi\colon Z_0 \to Z_1$, $\psi\colon Z_1 \to Z_2$ be two morphisms in the category $\mathcal{S}$. Define two functors from the category $\mathcal{S}$ to the category of sets.

Definition 1. The functor $S_{Z_0}\colon \mathcal{S} \to Sets$ is defined on objects by the formula:

$$S_{Z_0}(Y) = \{f \in [Y, Z_0] \,|\, \varphi \circ f \sim *\}.$$

The functor $T_{\Omega Z_2}\colon \mathcal{S} \to Ab$ is defined on objects by the formula:

$$T_{\Omega Z_2}(Y) = [Y, \Omega Z_2]/\psi_*[Y, \Omega Z_1].$$

Suppose that the composition $\psi \circ \varphi\colon Z_0 \to Z_2$ is homotopically trivial. Then the following diagram is homotopically commutative:

$$\begin{array}{ccc} Z_0 & \longrightarrow & * \\ \downarrow{\scriptstyle\varphi} & & \downarrow \\ Z_1 & \xrightarrow{\psi} & Z_2 \end{array}$$

Therefore, there exists the natural map $\chi\colon \operatorname{Fib}(\varphi) \to \operatorname{Fib}(* \to Z_2)$. Here $\operatorname{Fib}(f)$ is the homotopy fiber of f. Since $\operatorname{Fib}(* \to Z_2) \simeq \Omega Z_2$, we construct the map $\chi\colon \operatorname{Fib}(\varphi) \to \Omega Z_2$.

Definition 2. The secondary cohomological operation Φ based on the relation $\psi \circ \varphi \sim *$ is a natural transformation $\Phi\colon S_{Z_0} \to T_{\Omega Z_2}$ which is defined on objects by the following rule. Let Y be an object of the category $\mathcal{S}$ and let f belong to the set $S_{Z_0}(Y)$. Since the composition $\varphi \circ f$ is trivial, there exists a lifting $\bar{f}\colon Y \to \operatorname{Fib}(\varphi)$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccccc} & & \operatorname{Fib}(\varphi) & & \\ & \overset{\bar{f}}{\nearrow} & \downarrow & & \\ Y & \xrightarrow{f} & Z_0 & \xrightarrow{\varphi} & Z_1 \end{array}$$

Now define $\Phi_Y(f) = q(\chi \circ \bar{f}) \in T_{\Omega Z_2}(Y)$. Here q is the quotient map from the abelian group $[Y, \Omega Z_2]$ to the group $T_{\Omega Z_2}(Y)$

Remark. The natural transformation Φ is well-defined. Indeed, let $\bar{f}_1$ and $\bar{f}_2$ be two liftings of a map $f \in S_{Z_0}(Y)$. Since $\operatorname{Fib}(\operatorname{Fib}(\varphi) \to Z_0) \simeq \Omega Z_1$, the group $[Y, \Omega Z_1]$ acts transitively on the set of liftings of the map f. Therefore, the difference $\chi \circ \bar{f}_1 - \chi \circ \bar{f}_1$ lies in the subset $\psi_*[Y, \Omega Z_1]$.

Recall that usual (or primary) cohomological operations can be used for detection of non-trivial homotopy classes in $\pi_n(S^m)$. Secondary cohomological operation is good for this purpose too.

Definition 3. Let $n \geq m$ be a pair of integers. We say that $f \in [S^n, S^m]$ is detected by a secondary cohomological operation Φ if the morphism $\Phi_{\mathrm{Cone}(f)} \colon S_{Z_0}(\mathrm{Cone}(f)) \to T_{\Omega Z_2}(\mathrm{Cone}(f))$ is non-trivial.

Remark. This notion is due to the following observation. Assume that objects Z_0, Z_1, Z_2 are Eilenberg-MacLane objects of the category $\mathcal{S}$. Suppose that f is the trivial map, then $\mathrm{Cone}(f) \simeq S^m \vee S^{n+1}$. Since Φ is an natural transformation, $\Phi_{\mathrm{Cone}(f)}$ is zero map. In this case, non-triviality of the map $\Phi_{\mathrm{Cone}(f)}$ says that the map f is non-trivial. I.e. the map f is "detected" by Φ. In the case of arbitrary Z_i, non-triviality of the map $\Phi_{Cone(f)}$ is only some kind of witness that the map f is non-trivial.

Example 1. The element $\eta^2 \in \pi_2^s(S^0)$ is detected by the secondary cohomological operation Φ based on the relation

$$\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^2\mathrm{Sq}^2 = 0.$$

Here $Z_0 = H\mathbb{Z}/2$, $Z_1 = \Sigma H\mathbb{Z}/2 \vee \Sigma^2 H\mathbb{Z}/2$, $Z_2 = \Sigma^4 H\mathbb{Z}/2$, the map φ is equal to $(\mathrm{Sq}^1, \mathrm{Sq}^2)$, and the map ψ is equal to $(\mathrm{Sq}^3, \mathrm{Sq}^2)$. The proof can be found in [Har02] on page 95.

Proposition 1. *Let Φ be a secondary cohomological operation based on the relation $Z_0 \xrightarrow{\varphi} Z_1 \xrightarrow{\psi} Z_2$.*

(i) Let $f \colon A \to B$ be a morphism in $\mathcal{S}$. Suppose that $g \in S_{Z_0}(B)$. Then $g \circ f \in S_{Z_0}(A)$ and $\Phi_A(g \circ f) = \Phi_B(g) \circ f_*$. Here the equality takes place in $T_{\Omega Z_2}(A)$.

(ii) Suppose that $\mathcal{S}$ is the category of spectra Sp. Then for any $Y \in \mathcal{S}$ the set $S_{Z_0}(Y)$ has the natural structure of an abelian group. Moreover, the natural transformation Φ is a natural transformation of functors to the category of abelian groups. It means that for any $Y \in Sp$ the morphism $\Phi_Y \colon S_{Z_0}(Y) \to T_{\Omega Z_2}(Y)$ is a homomorphism of abelian groups.

Proof. Obvious. □

Let Φ be a secondary cohomological operation based on the relation $Z_0 \xrightarrow{\varphi} Z_1 \xrightarrow{\psi} Z_2$, and let $f \colon A \to Z_0$ belong to the set $S_{Z_0}(A)$. Denote by $\mathrm{Cone}(f)$ the cofiber of f. Then $\Phi_A(f)$ can be obtained from the following diagram chase. Consider the diagram:

$$\begin{array}{ccccccccc}
[A, Z_1] & \xleftarrow{f^*} & [Z_0, Z_1] & \xleftarrow{d^*} & [\mathrm{Cone}(f), Z_1] & \xleftarrow{\delta} & [\Sigma A, Z_1] & & \\
 & & \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} & & \\
 & & [Z_0, Z_2] & \xleftarrow{d^*} & [\mathrm{Cone}(f), Z_2] & \xleftarrow{\delta} & [\Sigma A, Z_2] & \xleftarrow{\Sigma f^*} & [\Sigma Z_0, Z_2]
\end{array}$$

All rows in this diagram are exact sequence (of abelian groups or pointed sets). Now consider φ as an element of $[Z_0, Z_1]$. Then $f^*(\phi) \sim *$. So there exists $\bar{\varphi} \in [\mathrm{Cone}(f), Z_2]$ such that $d^*(\bar{\varphi}) = \varphi$. Since the element $d^*\psi_*(\bar{\varphi}) = \psi_*(\varphi)$ is trivial, we have that $\psi_*(\bar{\varphi}) \in \ker(d^*) = \mathrm{Im}(\delta)$. It means that there exists $g \in [\Sigma A, Z_2]$ such that $\delta(g) = \psi_*(\bar{\varphi})$. Such $g \in [\Sigma A, Z_2] \cong [A, \Omega Z_2]$ is a unique up to $\mathrm{Im}\, \psi_* + \mathrm{Im}\, f^*$.

Proposition 2. *In the situation as above, $\Phi_A(f) = g$ in $T_{\Omega Z_2}(A)/\mathrm{Im}\, f^*$.*

Proof. Exercise. □

2 The Kervaire invariant

In order to define the Kervaire invariant of a framed manifold we should first construct a quadratic refinement of the intersection form on the middle cohomology of the manifold.

Let X be a finite spectrum, M be a closed smooth manifold of dimension $2q$. Denote by ν the stable normal bundle of M and denote by $T(\nu)$ the Thom spectrum of ν.

Definition 4. A mock X-orientation of M is a morphism $\eta \colon T(\nu) \to X$.

Example 2. (i) Denote by $h \colon M \to BO$ by the classifying map of ν. We have the map $\tilde{h} : \nu \to \gamma$, where γ is the universal bundle over BO. Notice that $T(\gamma) = MO$ by definition, where MO is the spectrum, which represents the non-oriented cobordism theory. So the classifying map gives the mock MO-orientation $T(\tilde{h}) \colon T(\nu) \to MO$.

(ii) Similarly, any framing (i.e. trivialization of ν) gives a mock S-orientation of M. Here S is the sphere spectrum.

Recall that the category of finite spectra is a closed symmetric monoidal category with respect to the smash product. It means that there exists the duality functor $D\colon (Sp^{\mathrm{fin}})^{op} \to Sp^{\mathrm{fin}}$ such that $\mathrm{Hom}(X \wedge Y, Z) \cong \mathrm{Hom}(X, DY \wedge Z)$. In the case of the category of finite spectra this functor is called the Spanier-Whitehead duality functor.

Let us apply the functor D to a mock X-orientation $\eta\colon T(\nu) \to X$. It gives the map

$$D\eta\colon DX \to DT(\nu).$$

By the Atiyah duality $DT(\nu) \cong \Sigma^{-2q}\Sigma^{\infty} M_+$. So we obtain the map

$$D\eta\colon DX \to \Sigma^{-2q}\Sigma^{\infty} M_+.$$

Denote by K_q the Eilenberg-Maclane space $K(\mathbb{Z}/2, q)$. Then any cohomology class $x \in H^q(M, \mathbb{Z}/2)$ can be represented by the map $\mathbf{x} \in [M_+, K_q]$. Denote by $\Sigma^{\infty}(\mathbf{x}) \in [\Sigma^{\infty}(M_+), \Sigma^{\infty}(K_q)]$ the map which obtained as the result of applying of the infinity suspension functor $\Sigma^{\infty}\colon \mathcal{S} \to Sp$ to the map $\mathbf{x}$.

Denote by $R\colon \Sigma^{-q}\Sigma^{\infty}K_q \to H\mathbb{Z}/2$ the map which represents the tautological cohomological class $\iota_q \in H^q(K_q, \mathbb{Z}/2)$. We also have the map $\mathrm{Sq}^{q+1}\colon H\mathbb{Z}/2 \to \Sigma^{q+1}H\mathbb{Z}/2$ which represents the $(q+1)$-th Steenrod square. All in all, we have the following sequence of maps:

$$DX \xrightarrow{D\eta} \Sigma^{-2q}\Sigma^{\infty} M_+ \xrightarrow{\Sigma^{\infty}(\mathbf{x})} \Sigma^{-2q}\Sigma^{\infty} K_q \xrightarrow{R} \Sigma^{-q}\, H\mathbb{Z}/2 \xrightarrow{\mathrm{Sq}^{q+1}} \Sigma\, H\mathbb{Z}/2\,.$$

Since $\mathrm{Sq}^{q+1}(\iota_q) = 0$ the composition $\mathrm{Sq}^{q+1} \circ R$ is trivial. So this relation defines the secondary cohomological operation Φ. Also notice that the composition $\Sigma^{\infty}(\mathbf{x}) \circ D\eta$ belongs to the group $S_{\Sigma^{-2q}\Sigma^{\infty}K_q}(DX)$ if and only if $x \in H^q(M, \mathbb{Z}/2)$ belongs to the kernel of $D\eta^*\colon H^*(M, \mathbb{Z}/2) \to H^*(DX, \mathbb{Z}/2)$.

Definition 5. Define the map

$$\psi_\eta\colon (\ker D\eta^*)^q \to T_{H\mathbb{Z}/2}(DX) = \frac{H^0(DX, \mathbb{Z}/2)}{\mathrm{Sq}^{q+1}(H^{-q-1}(DX, \mathbb{Z}/2))}$$

by the formula

$$x \mapsto \Phi(\Sigma^{\infty}(\mathbf{x}) \circ D\eta).$$

This map should play a role of a quadratic refinement of the intersection form on $H^q(M, \mathbb{Z}/2)$. However, the target looks pretty big. Let us define a class of spectra for which the target of the map ψ_η is the group $\mathbb{Z}/2$.

Definition 6. A spectrum X is called a Wu-spectrum of level q if X satisfies two conditions:

1) X is a connective spectrum such that $H^0(X, \mathbb{Z}/2) = \mathbb{Z}/2$;

2) $(q+1)$-th Steenrod square Sq^{q+1} acts by zero on $H^{-q-1}(DX, \mathbb{Z}/2)$.

The Steenrod algebra is a Hopf algebra, so there exists the conjugation map $\chi\colon \mathcal{A}_2 \to \mathcal{A}_2$. The last condition is equivalent to the condition:

2′) the operation $\chi(\mathrm{Sq}^{q+1})$ acts by zero on $H^0(X, \mathbb{Z}/2)$.

We need two main examples of Wu-spectra. The first one is the sphere spectrum S and the second one is the Browder spectrum $MO\langle v_{q+1}\rangle$. The following subsection is devoted to define it.

2.1 The Browder spectrum

Let M be a smooth manifold of dimension $2q$. Denote by ν the stable normal bundle of M and denote by $h\colon M \to BO$ the classifying map of ν.

Definition 7. The i-th Wu class $v_i(M)$ of the manifold M is a cohomological class in $H^i(X, \mathbb{Z}/2)$ such that for any $x \in H^{2q-i}(M, \mathbb{Z}/2)$ the following equation is satisfied:

$$v_i(M) \cup x = \chi(\mathrm{Sq}^i)(x).$$

By the Poincare duality, Wu classes of M are well-defined. Moreover, the complete Wu class $V(M) = 1 + v_1(M) + \ldots$ can be defined by the equation:

$$\mathrm{Sq}(V(M)) = W^{-1}(M),$$

where $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \ldots$ is the complete Steenrod square operation and $W(M) = 1 + w_1(M) + \ldots$ is the complete Stiefel-Whitney class of M. From this equation there exists the collection of universal Wu classes $v_i \in H^i(BO, \mathbb{Z}/2)$ such that $v_i(M) = h^*(v_i)$ for any $i \in \mathbb{N}$.

Let $BO\langle v_{q+1}\rangle$ be the homotopy fiber of the map $v_{q+1} \colon BO \to K_{q+1}$. Denote by π the embedding of the fiber $BO\langle v_{q+1}\rangle$ in BO.

Definition 8. A v_{q+1}-orientation of the stable normal bundle ν is a lifting $\bar{h}$ of the map h along the map π:

$$\begin{array}{ccc} & & BO\langle v_{q+1}\rangle \\ & \overset{\bar{h}}{\nearrow} & \downarrow \pi \\ M & \xrightarrow{\ h\ } & BO. \end{array}$$

Remark. By the Poincare duality, the $(q+1)$-th Wu class $v_{q+1}(M)$ is equal to zero for any $2q$-dimensional manifold M. So any $2q$-dimensional manifold is v_{q+1}-orientable.

Denote by γ the universal vector bundle over BO. Then the spectrum MO is the Thom spectrum of γ.

Definition 9. The Browder spectrum $MO\langle v_{q+1}\rangle$ is the Thom spectrum of the vector bundle $\pi^*(\gamma)$ over $BO\langle v_{q+1}\rangle$.

Remark. Any v_{q+1}-orientation of M gives a mock $MO\langle v_{q+1}\rangle$-orientation M. As a consequence, any $2q$-dimensional manifold has a mock $MO\langle v_{q+1}\rangle$-orientation.

Proposition 3. *The Browder spectrum $MO\langle v_{q+1}\rangle$ is a Wu spectrum of level q.*

Proof. Denote by $U \in H^0(MO, \mathbb{Z}/2)$ the Thom class and set $\bar{U} := \pi^* U \in H^0(MO\langle v_{q+1}\rangle, \mathbb{Z}/2)$.

1) Clearly, the spectrum $MO\langle v_{q+1}\rangle$ is connective and $H^0(MO\langle v_{q+1}\rangle, \mathbb{Z}/2) = \mathbb{Z}/2\langle \bar{U}\rangle$.

2) Let us prove that $\chi(\mathrm{Sq}^{q+1})$ acts by zero on $H^0(MO\langle v_{q+1}\rangle, \mathbb{Z}/2)$. Indeed, we have equalities

$$\chi(\mathrm{Sq}^{q+1})\bar{U} = \pi^*(\chi(\mathrm{Sq}^{q+1})U) = \pi^*(v_{q+1} \cup U) = \pi^*(v_{q+1}) \cup \bar{U}.$$

But by definition $\pi^*(v_{q+1}) = 0$.

□

2.2 The Kervaire invariant

Let X be a Wu spectrum of level q and $\eta \colon T(\nu) \to X$ be a mock X-orientation. Above we constructed the mapping $\psi_\eta \colon (\ker D\eta^*)^q \to T_{H\mathbb{Z}/2}(DX) = \mathbb{Z}/2$ by the formula $\psi_\eta(x) = \Phi(\Sigma^\infty(\mathbf{x}) \circ D\eta)$.

Proposition 4. *The mapping ψ_η is a quadratic form on $(\ker D\eta^*)^q$, i.e. for any $x, y \in (\ker D\eta^*)^q$:*

$$\psi_\eta(x + y) = \psi_\eta(x) + \psi_\eta(y) + (D\eta)^*(x \cup y).$$

Proof. The sum $\mathbf{x} + \mathbf{y} \in [M_+, K_q]$ is taken with respect to the H-space structure on K_q. Therefore,

$$\Sigma^\infty(\mathbf{x} + \mathbf{y}) = \Sigma^\infty(\mathbf{x}) + \Sigma^\infty(\mathbf{y}) + h(\mu)(\mathbf{x} \wedge \mathbf{y})\,\Delta.$$

Here

- $\Delta\colon \Sigma^\infty M_+ \to \Sigma^\infty M_+ \wedge \Sigma^\infty M_+$ is the diagonal map,
- $\mathbf{x}\wedge\mathbf{y}\colon \Sigma^\infty M_+ \wedge \Sigma^\infty M_+ \to \Sigma^\infty K_q \wedge \Sigma^\infty K_q$ is the smash product of maps,
- $h(\mu)\colon \Sigma^\infty K_q \wedge \Sigma^\infty K_q \to \Sigma^\infty K_q$ is the map induced by the multiplication on K_q.

Hence,

$$\begin{aligned}\psi_\eta(x+y) &= \Phi((\Sigma^\infty(\mathbf{x}) + \Sigma^\infty(\mathbf{y}) + h(\mu)(\mathbf{x}\wedge\mathbf{y})\,\Delta)\circ D\eta) = \\ &= \Phi(\Sigma^\infty(\mathbf{x})\circ D\eta) + \Phi(\Sigma^\infty(\mathbf{y})\circ D\eta) + \Phi(h(\mu)(\mathbf{x}\wedge\mathbf{y})\,\Delta\circ D\eta) = \\ &= \psi_\eta(x) + \psi_\eta(y) + \Phi(h(\mu)(\mathbf{x}\wedge\mathbf{y})\,\Delta\circ D\eta).\end{aligned}$$

Let us compute $\Phi(h(\mu)(\mathbf{x}\wedge\mathbf{y})\,\Delta\circ D\eta)$. Recall the secondary cohomological operation Φ is based on the relation $\mathrm{Sq}^{q+1}\circ R \sim *$, where

$$\Sigma^{-2q}\Sigma^\infty K_q \xrightarrow{R} \Sigma^{-q}\, H\mathbb{Z}/2 \xrightarrow{\mathrm{Sq}^{q+1}} \Sigma\, H\mathbb{Z}/2\,.$$

Notice that the composition $R\circ h(\mu)$ is trivial, because $H^q(K_q\wedge K_q,\mathbb{Z}/2) = 0$. Hence, by Proposition 1:

$$\Phi(h(\mu)(\mathbf{x}\wedge\mathbf{y})\,\Delta\circ D\eta) = \Phi(h(\mu))\circ(\mathbf{x}\wedge\mathbf{y})\,\Delta\circ D\eta.$$

So it is enough to compute $\Phi(h(\mu))$.

Lemma 1. $\Phi(h(\mu)) = \iota_q\wedge\iota_q \in H^{2q}(K_q\wedge K_q,\mathbb{Z}/2) = T_{H\mathbb{Z}/2}(\Sigma^{-2q}\Sigma^\infty(K_q\wedge K_q))$.

Proof. Let C be a homotopy cofiber of $h(\mu)\colon \Sigma^\infty(K_q\wedge K_q)\to\Sigma^\infty(K_q)$. Consider the diagram

$$\begin{array}{ccccccc} H^q(K_q\wedge K_q,\mathbb{Z}/2) & \xleftarrow{h(\mu)^*} & H^q(K_q,\mathbb{Z}/2) & \xleftarrow{\ f\ } & H^q(C,\mathbb{Z}/2) & & \\ & & \downarrow{\scriptstyle \mathrm{Sq}^{q+1}} & & \downarrow{\scriptstyle \mathrm{Sq}^{q+1}} & & \\ & & H^{2q+1}(K_q,\mathbb{Z}/2) & \xleftarrow{\ g\ } & H^{2q+1}(C,\mathbb{Z}/2) & \xleftarrow{\ \delta\ } & H^{2q}(K_q\wedge K_q,\mathbb{Z}/2) \end{array}$$

Since $h(\mu)^*(\iota_q)=0$, there exists an element $\alpha\in H^q(C,\mathbb{Z}/2)$ such that $f(\alpha)=\iota_q$. Since we have the equality $g(\mathrm{Sq}^{q+1}(\alpha)) = \mathrm{Sq}^{q+1}(f(\alpha)) = \mathrm{Sq}^{q+1}(\iota_q) = 0$, there exists an element $\beta\in H^{2q}(K_q\wedge K_q,\mathbb{Z}/2)$ such that $\delta(\beta) = \mathrm{Sq}^{q+1}(\alpha)$.

By Proposition 2, $\beta = \Phi(h(\mu))$. So let us compute β. Since $H^{2q}(K_q\wedge K_q,\mathbb{Z}/2) = \mathbb{Z}/2\langle\iota_q\wedge\iota_q\rangle$, it is enough to prove that $\beta\neq 0$.

Assume that $\beta = 0$, then $\mathrm{Sq}^{q+1}(\alpha) = 0$. Moreover, the cohomology group $H^q(C,\mathbb{Z}/2)$ is generated by only one element α. So it is enough to check that the map

$$\mathrm{Sq}^{q+1}\colon H^q(C,\mathbb{Z}/2)\to H^{2q+1}(C,\mathbb{Z}/2)$$

is non-zero.

The map $h(\mu)$ is induced by the map of spaces $\tilde{\mu}\colon \Sigma(K_q\wedge K_q)\to\Sigma(K_q\times K_q)\xrightarrow{\Sigma(\mu)}\Sigma K_q$. Denote by $\mathrm{Cone}(\tilde{\mu})$ the homotopy cofiber of $\tilde{\mu}$. Then it is enough to prove that the map

$$\mathrm{Sq}^{q+1}\colon H^{q+1}(\mathrm{Cone}(\tilde{\mu}),\mathbb{Z}/2)\to H^{2q+2}(\mathrm{Cone}(\tilde{\mu}),\mathbb{Z}/2)$$

is non-zero. But we have the fiber sequence

$$\Sigma(K_q\wedge K_q)\xrightarrow{\tilde{\mu}}\Sigma K_q\xrightarrow{\Sigma(\iota_q)} K_{q+1},$$

where $\Sigma(\iota_q)\colon \Sigma\Omega K_{q+1}\to K_{q+1}$ is the counit map of usual pair of adjoint functors. So we have the map $\varepsilon\colon \mathrm{Cone}(\tilde{\mu})\to K_{q+1}$ and this map induces an isomorphism on $H^n(-,\mathbb{Z}/2)$ for any $n\leq 2q+2$. So it is enough to prove that the cohomological operation

$$\mathrm{Sq}^{q+1}\colon H^{q+1}(K_{q+1},\mathbb{Z}/2)\to H^{2q+2}(K_{q+1},\mathbb{Z}/2)$$

is a non-zero map. But this is obvious. $\square$

Using the lemma we obtain that $\psi_\eta(x+y)=\psi_\eta(x)+\psi_\eta(y)+(\iota_q\wedge\iota_q)\circ(\mathbf{x}\wedge\mathbf{y})\,\Delta\circ D\eta$. But the last summand is equal to $D\eta^*(x\cup y)$. So the proposition is proved. □

Let (M,η) be a mock X-orientation such that $D\eta^*(\Sigma^{-2q}[M]^\vee)=\alpha$, where $[M]^\vee\in H^{2q}(M,\mathbb{Z}/2)$ is the fundamental cohomological class, and $\alpha\in H^0(DX,\mathbb{Z}/2)$ is the generator (recall that X is a Wu spectrum, so $H^0(DX,\mathbb{Z}/2)=\mathbb{Z}/2$). For instance, $X=S$ or $X=MO\langle v_{q+1}\rangle$ and η comes from a framing or from a v_{q+1}-orientation, respectively. Then $D\eta^*(x\cup y)=(x\cup y)[M]$ and ψ_η is a quadratic form on $(\ker D\eta^*)^q$ which associated bilinear form is the intersection form.

Let $A\subset(\ker D\eta^*)^q$ be a $\mathbb{Z}/2$-submodule such that the intersection form is a non-degenerate bilinear form on A.

Definition 10. The Kervaire invariant $c(M,\eta,A)$ of the triple (M,η,A) is the Arf invariant of the quadratic form $\psi_\eta|_A$ defined on the $\mathbb{Z}/2$-vector space A.

Definition 11. An X-oriented manifold (M,η) is called an X-oriented boundary if there exists a manifold W such that $M=\partial W$ and for the embedding $i\colon M\to W$ the following composition:

$$DX\xrightarrow{D\eta}\Sigma^{-2q}\Sigma^\infty M_+\xrightarrow{i}\Sigma^{-2q}\Sigma^\infty W_+$$

is trivial.

The following properties show that the Kervaire invariant is a bordism invariant.

Proposition 5. *(i) Let (M_1,η_1,A_1) and (M_2,η_2,A_2) be two triples for which the Kervaire invariant is defined. When the Kervaire invariant of the triple $(M_1\sqcup M_2,\eta_1\vee\eta_2,A_1\oplus A_2)$ is defined and*

$$c(M_1\sqcup M_2,\eta_1\vee\eta_2,A_1\oplus A_2)=c(M_1,\eta_1,A_1)+c(M_2,\eta_2,A_2).$$

(ii) Let $M=\partial W$ be a X-oriented boundary, then $\psi_\eta(\mathrm{Im}(i^*))=0$. Moreover, if the rank of $A\cap\mathrm{Im}(i^*)$ is one half of the rank of A, then $c(M,\eta,A)=0$.

(iii) The quadratic form ψ is natural with respect to morphisms of orientations. It means the following. Let $f\colon X\to Y$ be a map between Wu spectra and (M,η) be an X-orientated manifold. Then the pair $(M,f\circ\eta)$ is a Y-orientated manifold and the following diagram is commutative:

$$\begin{array}{ccc}(\ker(D\eta)^*)^q & \xrightarrow{\psi_\eta} & H^0(DX,\mathbb{Z}/2)\\ \big\downarrow & & \big\downarrow{\scriptstyle Df^*}\\ (\ker(D\eta\circ Df)^*)^q & \xrightarrow{\psi_{f\circ\eta}} & H^0(DY,\mathbb{Z}/2).\end{array}$$

Proof. Obvious. □

Remark. If $X=S$, then the kernel $(\ker D\eta^*)^q$ is equal to $H^q(M,\mathbb{Z}/2)$ for any mock S-orientation $\eta\colon T(\nu)\to S$. Therefore, for any framed manifold (M,η) we can always take all $H^q(M,\mathbb{Z}/2)$ as A. So the Kervaire invariant defines the homomorphism:

$$c\colon\Omega^{fr}_{2q}\to\mathbb{Z}/2,$$
$$(M,\eta)\mapsto c(M,\eta,H^q(M,\mathbb{Z}/2)).$$

2.3 Example of a v_{q+1}-orientation on $S^q\times S^q$ with the Kervaire invariant one

Let (M,η) be a v_{q+1}-oriented manifold of dimension $2q$.

Proposition 6. *If the stable normal bundle ν of the manifold M is trivial, then*

$$(\ker D\eta^*)^q=H^q(M,\mathbb{Z}/2).$$

Proof. Denote by $h\colon M\to BO$ the classifying map of ν and denote by $\tilde h\colon M\to BO\langle v_{q+1}\rangle$ the lifting of h such that the induced map $T(\tilde h)\colon T(\nu)\to MO\langle v_{q+1}\rangle$ is the map η.

Now the map $\pi_*\colon H_q(BO\langle v_{q+1}\rangle,\mathbb{Z}/2)\to H_q(BO,\mathbb{Z}/2)$ is a monomorphism (by the Serre spectral sequence of the fiber sequence $K_q\to BO\langle v_{q+1}\rangle\to BO$). By assumption the map h is the trivial map, so the map

$$\tilde h_*\colon H_q(M,\mathbb{Z}/2)\to H_q(BO\langle v_{q+1}\rangle,\mathbb{Z}/2)$$

is the zero map (because $\pi\circ\tilde h=h$). By the Thom isomorphism we get that the map

$$\eta_*=T(\tilde h)_*\colon H_q(T(\nu),\mathbb{Z}/2)\to H_q(MO\langle v_{q+1}\rangle,\mathbb{Z}/2)$$

is the zero map. So by the Spanier-Whitehead duality the map

$$D\eta^*\colon H^q(M,\mathbb{Z}/2)\to H^q(DMO\langle v_{q+1}\rangle,\mathbb{Z}/2)$$

is the zero map. So the proposition is proved. □

Let us consider $M=S^q\times S^q$ framed in S^{2q+1}, so that it is naturally cobordant to zero. Denote by η_0 this v_{q+1}-orientation. Then, of course, $c(M,\eta_0)=0$. Let $g\colon M\to K_q$ be a map such that $g^*(\iota_q)=1\otimes[S^q]^\vee+[S^q]^\vee\otimes 1$. Recall that the group $[M,K_q]$ acts on the set of liftings of $h: M\to BO$. Denote by η the v_{q+1}-orientation which is obtained by the action of g on the v_{q+1}-orientation η_0.

Proposition 7. *The Kervaire invariant of $(S^q\times S^q,\eta)$ is equal to one.*

Proof. By Proposition 6 the domain of the quadratic form ψ_η is $H^q(S^q\times S^q,\mathbb{Z}/2)$. Denote by x the cohomology class $[S^q]^\vee\otimes 1\in H^q(S^q\times S^q,\mathbb{Z}/2)$ and denote by y the cohomology class $1\otimes[S^q]^\vee$. Let us prove that $\psi_\eta(x)=1$. The proof that $\psi_\eta(y)=1$ will be the same. So the Kervaire invariant $c(M,\eta)=Arf(\psi_\eta)=\psi_\eta(x)\psi_\eta(y)=1$.

By definition $\psi_\eta(x)=\Phi(\Sigma^\infty(\mathbf{x})\circ D\eta)$. The map $\mathbf{x}\colon S^q\times S^q\to K_q$ such that $\mathbf{x}^*(\iota_q)=x$ factors through the projection $p\colon S^q\times S^q\to S^q$, $p(a,b)=a$. Hence the composition $\Sigma^\infty(\mathbf{x})\circ D\eta$ is equal to the composition

$$DMO\langle v_{q+1}\rangle\xrightarrow{D\eta}\Sigma^{-2q}\Sigma^\infty(S^q\times S^q)_+\xrightarrow{\Sigma^\infty p}\Sigma^{-2q}\Sigma^\infty S^q\xrightarrow{\Sigma^\infty r}\Sigma^{-2q}\Sigma^\infty K_q.$$

Here $r\colon S^q\to K_q$ is such that $r^*(\iota_q)=[S^q]^\vee$. Denote by $\widetilde R\colon S^{-q}\to\Sigma^{-q}H\mathbb{Z}/2$ the composition $R\circ\Sigma^\infty r$ (it is just a shift of the unit map). Recall that the secondary cohomological operation Φ is based on the relation $\mathrm{Sq}^{q+1}\circ R\sim *$. Denote by Ψ the secondary cohomological operation based on the relation $\mathrm{Sq}^{q+1}\circ\widetilde R\sim *$. Then

$$\Phi(\Sigma^\infty(\mathbf{x})\circ D\eta)=\Psi(\Sigma^\infty p\circ D\eta).$$

Let us show that $\Psi(\Sigma^\infty p\circ D\eta)\neq 0$. Denote by C_0 the cofiber of the map

$$\Sigma^\infty p\circ D\eta\colon DMO\langle v_{q+1}\rangle\to S^{-q}.$$

Then, by Proposition 2, it is enough to prove that Sq^{q+1} acts by non-zero on $H^{-q}(C_0,\mathbb{Z}/2)$. But the map $\Sigma^\infty p\circ D\eta$ is dual to the map

$$\eta\circ\Sigma^\infty i\colon S^q\to\Sigma^\infty(S^q\times S^q)_+\to MO\langle v_{q+1}\rangle,$$

where $i\colon S^q\to S^q\times S^q$ is the standard embedding $i(a)=(a,b_0)$.

Denote by C_1 the cofiber of the map $\eta\circ\Sigma^\infty i$. Then $\Sigma DC_0\simeq C_1$. By duality we should check that $\chi(\mathrm{Sq}^{q+1})$ acts by non-zero on $H^0(C_1,\mathbb{Z}/2)$.

Now the composition

$$S^q\xrightarrow{\Sigma^\infty i}\Sigma^\infty(S^q\times S^q)_+\to MO\langle v_{q+1}\rangle\xrightarrow{\pi}MO$$

is trivial. So there exists a map $\bar{\pi}\colon C_1 \to MO$. W. Browder proved (see Theorem 5.2 in [Bro69], p.172) that $\bar{\pi}$ can be chosen such that $\bar{\pi}_*$ induces a monomorphism on $H^{q+1}(-, \mathbb{Z}/2)$. But the Steenrod square

$$\chi(\mathrm{Sq}^{q+1})\colon H^0(MO, \mathbb{Z}/2) \to H^{q+1}(MO, \mathbb{Z}/2)$$

is a non-zero map, so the Steenrod square

$$\chi(\mathrm{Sq}^{q+1})\colon H^0(C_1, \mathbb{Z}/2) \to H^{q+1}(C_1, \mathbb{Z}/2)$$

is also a non-zero map.

It means that $\psi_\eta(x) = 1$ and $c(M, \eta) = 1$. □

Proposition 8. *The v_{q+1}-orientation η comes from a framing of $S^q \times S^q$ if and only if $q = 1, 3, 7$.*

Proof. Proposition 5.3 in [Bro69]. □

References

[Bro69] William Browder, *The Kervaire Invariant of Framed Manifolds and its Generalization*, Annals of Mathematics **90** (1969), no. 1, 157–186.

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