The Kervaire Invariant One Problem, Lecture 9, Independent University of Moscow, Fall semester 2016

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1 $\mathcal{C}$-fibrations.

Notation. We will denote by $\mathcal{S}$ the $(\infty,1)$-category of spaces and by Cat_∞ the $(\infty,1)$-category of $(\infty,1)$-categories. Throughout this document we will frequently consider spaces as special kinds of $(\infty,1)$-categories using the obvious inclusion $\mathcal{S} \longrightarrow \mathsf{Cat}_\infty$. For an $(\infty,1)$-category $\mathcal{C} \in \mathsf{Cat}_\infty$ we will denote its core (obtained by throwing out all the noninvertible morphisms in $\mathcal{C}$) by $\mathcal{C}^\simeq \in \mathcal{S}$. We will denote by Sp the $(\infty,1)$-category of spectra.
From now on we will also frequently omit the symbol $(\infty,1)$.

Let $X \in \mathcal{S}$ be a space. What is a vector bundle over X? At least for every point $x \in X$ we should have a vector space $V_x \in \mathsf{Vect}$, where Vect is the category of vector spaces (considered as topologically enriched category made into $(\infty,1)$-category). More then that, given a path from x_1 to x_2 in X we should have a morphism $V_{x_1} \longrightarrow V_{x_2}$ corresponding to this path. The picture should be similar for all the higher homotopies motivating the following

Definition. A **vector bundle** E over a space X is a functor $X \xrightarrow{E} \mathsf{Vect}$.

Even more generally, we can consider the following

Definition. For a space $X \in \mathcal{S}$ and a category $\mathcal{C} \in \mathsf{Cat}_\infty$ define the category $\mathsf{Fib}_{\mathcal{C}}(X) \in \mathsf{Cat}_\infty$ of **$\mathcal{C}$-fibrations over** X simply as the category of functors

$$\mathsf{Fib}_{\mathcal{C}}(X) := \mathsf{Funct}(X, \mathcal{C}).$$

Examples.
1) We have

$$\mathsf{Fib}_{\mathcal{C}}(*) = \mathsf{Funct}(*, \mathcal{C}) \simeq \mathcal{C}$$

2) Considering $S^1 \in \mathcal{S}$ as a Kan simplicial set with only one point $* \in S^1$ we see that specifying a vector bundle $V \in \mathsf{Fib}_{\mathsf{Vect}}(S^1)$ amounts to specifying a vector space $V \in \mathsf{Vect}$ together with an endomorphism $V \xrightarrow{f} V$.

3) For a group G the fibration $(EG \longrightarrow BG) \in \mathsf{Fib}_{\mathcal{S}}(BG)$ is classified by the functor

$$BG \longrightarrow \mathcal{S}$$

which sends the unique point $\star \in BG$ to $G \in \mathcal{S}$ and the arrows in BG are sent to the action of G on itself.

Remarks.
1) Notice that since X is an ∞-groupoid there is an equivalence $X^{\mathsf{op}} \simeq X$ so that we obtain an equivalence

$$\mathsf{Fib}_{\mathcal{C}}(X) = \mathsf{Funct}(X, \mathcal{C}) \simeq \mathsf{Funct}(X^{\mathsf{op}}, \mathcal{C}).$$

2) In the special case when $\mathcal{C} = \mathcal{S}$ the category of spaces we have an equivalence

$$\mathsf{Fib}_{\mathcal{S}}(X) = \mathsf{Funct}(X, \mathcal{S}) \simeq \mathcal{S}_{/X},$$

where $\mathcal{S}_{/X}$ is the category of spaces over X and the last equality holds since it holds in the special case $X \simeq *$ and both of the functors

$$\mathcal{S}^{\mathsf{op}} \xrightarrow{\mathsf{Funct}(\bullet,\mathcal{S}),\ \mathcal{S}_{/\bullet}} \mathsf{Cat}_\infty$$

preserve limits.
In fact, the equivalence above holds not only for the category of spaces but also for arbitrary topos.

Sometimes it is also useful to control fibers of a $\mathcal{C}$-fibration. In order to work with this comfortably, we introduce the following

Notation. For an object $C \in \mathcal{C}$ we will denote by $\mathsf{Fib}^C_{\mathcal{C}}(X)$ the full subcategory of $\mathsf{Fib}_{\mathcal{C}}(X)$ spanned by those $\mathcal{C}$-fibrations $E \in \mathsf{Fib}_{\mathcal{C}}(X)$ whose fiber over any point is equivalent to $C \in \mathcal{C}$. More formally, for every point $x \in X$ there should be an equivalence $E(x) \simeq C$ in $\mathcal{C}$.

Example. Suppose that the space $X \in \mathcal{S}$ is connected and let $E \in \mathsf{Fib}_{\mathcal{C}}(X)$ be a $\mathcal{C}$-fibration over X. Then E actually lies in $\mathsf{Fib}^C_{\mathcal{C}}(X)$, where C is the fiber of E over any point $x \in E$. Indeed, since X is connected, for two arbitrary points $x, y \in X$ there exists a path $x \xrightarrow[\sim]{} y$ in X. Since such a path gives an equivalence between x and y, the induced morphism $E(x) \longrightarrow E(y)$ should be an equivalence so that the fibers $E(x)$ and $E(y)$ are equivalent.

Now to investigate $\mathcal{C}$-fibrations we have the following

Proposition.
1) The $\mathcal{C}$-fibrations assigmnent is actually a functor

$$\mathcal{S}^{\mathsf{op}} \xrightarrow{\mathsf{Fib}_{\mathcal{C}}(\bullet)} \mathsf{Cat}_\infty$$

2) For any $X \in \mathcal{S}$ there is an equivalence

$$\pi_0\,\mathsf{Fib}_{\mathcal{C}}(X) \simeq \pi_0\,\mathsf{Hom}_{\mathcal{S}}\big(X, \bigsqcup_{[C]\in\mathcal{C}^{\simeq}} \mathsf{BAut}_{\mathcal{C}}(C)\big).$$

3) For any $X \in \mathcal{S}$ and $C \in \mathcal{C}$ there is an equivalence

$$\pi_0\,\mathsf{Fib}^C_{\mathcal{C}}(X) \simeq \pi_0\,\mathsf{Hom}_{\mathcal{S}}\big(X, \mathsf{BAut}_{\mathcal{C}}(C)\big).$$

Proof.

1) Given a morphism $X \xrightarrow{f} Y$ the induced functor $\mathsf{Fib}_{\mathcal{C}}(Y) \xrightarrow{f^*} \mathsf{Fib}_{\mathcal{C}}(X)$ is given simply by precomposition with f.

2) We have

$$\pi_0\,\mathsf{Fib}_{\mathcal{C}}(X) \simeq \pi_0\,\mathsf{Funct}(X,\mathcal{C}) \simeq \pi_0\,\mathsf{Hom}_{\mathsf{Cat}_\infty}(X,\mathcal{C}) \simeq$$
$$\simeq \pi_0\,\mathsf{Hom}_{\mathcal{S}}(X,\mathcal{C}^{\simeq}) \simeq \pi_0\,\mathsf{Hom}_{\mathcal{S}}\big(X, \bigsqcup_{[C]\in\mathcal{C}^{\simeq}} \mathsf{BAut}_{\mathcal{C}}(C)\big)$$

as deisred.

3) Similar as (2) above. $\square$

Remarks.
1) Notice that despite from the equivalence above in most cases we have

$$\mathsf{Fib}_{\mathcal{C}}(X) \not\simeq \mathsf{Hom}_{\mathsf{Cat}_\infty}(X, \mathcal{C}^{\simeq}).$$

The easiest way to see this is to consider the special case $X \simeq *$: we then have $\mathsf{Fib}_{\mathcal{C}}(*) \simeq \mathcal{C}$ while $\mathsf{Hom}_{\mathsf{Cat}_\infty}(*, \mathcal{C}^{\simeq}) \simeq \mathcal{C}^{\simeq}$.

2) Supppose that the category $\mathcal{C}$ is good enough. Then given a morphism

$$X \xrightarrow{\quad f \quad} Y$$

in $\mathcal{S}$ there is a sequence of adjunctions $f_! \dashv f^* \dashv f_*$, where the functors $f_!$ and f_* are defined as the left and the right Kan extensions along f respectively (for an arbitrary $\mathcal{C}$ they may not exist).

Examples.
1) Through the equivalence above the nullhomotopic map $X \longrightarrow * \xrightarrow{\mathsf{Id}_C} \mathsf{BAut}_{\mathcal{C}}(C)$ corresponds to the constant $\mathcal{C}$-fibration $\mathsf{Const}_C \in \mathsf{Fib}_{\mathcal{C}}^C(X) \subseteq \mathsf{Fib}_{\mathcal{C}}(X)$.

2) Suppose we are interested in $\mathbb{R}$-vector bundles over a space X of a fixed dimension n up to an equivalence. Another words, we wish to understand the set $\pi_0 \mathsf{Fib}_{\mathsf{Vect}_\mathbb{R}}^{\mathbb{R}^n}(X)$. Then using the proposition above we get

$$\pi_0 \mathsf{Fib}_{\mathcal{C}}^{\mathbb{R}^n}(X) \simeq \pi_0 \mathsf{Hom}_{\mathcal{S}}\left(X, \mathsf{BAut}_{\mathsf{Vect}_\mathbb{R}}(\mathbb{R}^n)\right) \simeq \pi_0 \mathsf{Hom}_{\mathcal{S}}\left(X, \mathsf{BGL}_n(\mathbb{R})\right) \simeq \pi_0 \mathsf{Hom}_{\mathcal{S}}\left(X, \mathsf{BO}_n(\mathbb{R})\right),$$

where the last equivalence follows from the homotopy equivalence $\mathsf{BGL}_n(\mathbb{R}) \simeq \mathsf{BO}_n$ given by the Gram–Schmidt process. Consequently, we see that fibrations over X with the fiber $\mathbb{R}^n \in \mathsf{Vect}$ are classified up to equivalence by the maps

$$X \longrightarrow \mathsf{BO}_n(\mathbb{R})$$

up to a homotopy.

3) A discussion as above leads to the proof that principle G-fibrations over a space $X \in \mathcal{S}$ are classified up to equivalence by the maps $X \longrightarrow BG$ up to homotopy.

4) Let X be a (connected) space and $X_{n+1} \xrightarrow{\varphi_n} X_n$ be the stage of its Postnikov's tower where $n > 1$. Recall that φ_n is a fibration with the fiber over any point being $K(\pi_{n+1}(X), n+1)$. By the discussion above such fibrations are classified up to equivalence by the space

$$\pi_0 \mathsf{Hom}_{\mathcal{S}}\left(X_n, \mathsf{B}\,\mathsf{Aut}_{\mathcal{S}}(K(\pi_{n+1}X, n+1))\right)$$

Notice also that threre is an equivalence of spaces

$$\mathsf{Aut}_{\mathcal{S}}(K(\pi_{n+1}X, n+1)) \simeq \mathsf{Aut}_{\mathsf{Ab}}(\pi_{n+1}X) \ltimes K(\pi_{n+1}X, n+1).$$

Indeed, recall that the $(n+1)$-times loopspace functor gives an equivalence

$$\mathcal{S}_*^{\geq n+1} \xrightarrow[\sim]{\Omega^{n+1}} \mathsf{Alg}_{\mathbb{E}_{n+1}}^{\mathsf{grouplike}}(\mathcal{S})$$

so that we have

$$\mathsf{Hom}_{\mathcal{S}_*}(K(\pi_{n+1}X, n+1), K(\pi_{n+1}X, n+1)) \simeq \mathsf{Hom}_{\mathsf{Alg}_{\mathbb{E}_{n+1}}^{\mathsf{grouplike}}(\mathcal{S})}(\pi_{n+1}X, \pi_{n+1}X) \simeq \mathsf{End}_{\mathsf{Ab}}(\pi_{n+1}X)$$

where the last equivalence holds since $\pi_{n+1}X$ is discrete. A more direct way to get an equivalence above is to calculate homotopy groups of the space $\mathsf{Hom}_{\mathcal{S}_*}(K(\pi_{n+1}X, n+1), K(\pi_{n+1}X, n+1))$ using the fact that Eilenberg-Maclane spaces represent (reduced) cohomology. Consequently, we can rewrite the fibration

$$\begin{array}{ccc} \mathsf{Hom}_{\mathcal{S}_*}(K(\pi_{n+1}X, n+1), K(\pi_{n+1}X, n+1)) & \longrightarrow & \mathsf{Hom}_{\mathcal{S}}(K(\pi_{n+1}X, n+1), K(\pi_{n+1}X, n+1)) \\ \downarrow & & \downarrow{\scriptstyle \mathsf{ev}_*} \\ * & \longrightarrow & K(\pi_{n+1}X, n+1) \end{array}$$

as

$$\begin{array}{ccc} \mathsf{End}_{\mathsf{Ab}}(\pi_{n+1}X) & \longrightarrow & \mathsf{Hom}_{\mathcal{S}}(K(\pi_{n+1}X, n+1), K(\pi_{n+1}X, n+1)) \\ \downarrow & & \downarrow{\scriptstyle \mathsf{ev}_*} \\ * & \longrightarrow & K(\pi_{n+1}X, n+1). \end{array}$$

Now the section $K(\pi_{n+1}X, n+1) \longrightarrow \mathsf{Hom}_{\mathcal{S}}(K(\pi_{n+1}X, n+1), K(\pi_{n+1}X, n+1))$ given by the action of $K(\pi_{n+1}X, n+1)$ on itself by translations gives an equivalence

$$\mathsf{End}_{\mathcal{S}}(K(\pi_{n+1}X, n+1)) \simeq \mathsf{End}_{\mathsf{Ab}}(\pi_{n+1}X) \ltimes K(\pi_{n+1}X, n+1).$$

Consequently, we get

$$\mathsf{Aut}_{\mathcal{S}}(K(\pi_{n+1}X, n{+}1)) \simeq \mathsf{gl}_1\left(\mathsf{End}_{\mathsf{Ab}}(\pi_{n+1}X) \ltimes K(\pi_{n+1}X, n{+}1)\right) \simeq \mathsf{Aut}_{\mathsf{Ab}}(\pi_{n+1}X) \ltimes K(\pi_{n+1}X, n{+}1)$$

as desired.

2 Total objects, sections and orientations.

In the case when $\mathcal{C}$ is good enough (for example, presentable) given a $\mathcal{C}$-fibration $E \in \mathsf{Fib}_{\mathcal{C}}(X)$ over X once can also consider its sections or its total space.

Convention. From now on we will assume that the category $\mathcal{C} \in \mathsf{Cat}_\infty$ is presentable.

Definition. For a space $X \in \mathcal{S}$ define a **sections functor** and a **total object functor**

$$\mathsf{Fib}_{\mathcal{C}}(X) \simeq \mathsf{Funct}(X, \mathcal{C}) \xrightarrow{\Gamma,\ \mathsf{Tot}} \mathcal{C}$$

simply as the limit and colimit functors.

Examples.
1) Let $C \in \mathcal{C}$. Consider the constant fibration with the fiber C:

$$X \xrightarrow{\mathsf{Const}_C} \mathcal{C}.$$

Then the corresponding total object is simply $\mathsf{Tot}(\mathsf{Const}_C) \simeq \mathsf{colim}_X C = X \otimes C$.

2) Consider the case $\mathcal{C} = \mathcal{S}$ is the category of spaces. Then for any space $X \in \mathcal{S}$ under the equivalence $\mathsf{Fib}_{\mathcal{S}}(X) \simeq \mathcal{S}_{/X}$ we have

$$\Gamma(Y \xrightarrow{p} X) \simeq \mathsf{sect}(p)$$

and

$$\mathsf{Tot}(Y \xrightarrow{p} X) \simeq Y,$$

where $\mathsf{sect}(p)$ is the space of sections of p.

Remark. Let $X \xrightarrow{p} *$ be the projection map. Then by formal nonsense we see that for every $E \in \mathsf{Fib}_{\mathcal{C}}(X)$ we have equivalences $\mathsf{Tot}(E) \simeq p_! E$ and $\Gamma(E) \simeq p_* E$. In particular, for any $C \in \mathcal{C}$ we have

$$\mathsf{Hom}_{\mathcal{C}}(\mathsf{Tot}(E), C) \simeq \mathsf{Hom}_{\mathsf{Fib}_{\mathcal{C}}(*)}(p_! E, C) \simeq \mathsf{Hom}_{\mathsf{Fib}_{\mathcal{C}}(X)}(E, p^* C) \simeq \mathsf{Hom}_{\mathsf{Fib}_{\mathcal{C}}(X)}(E, \mathsf{Const}_C)$$

and similarly

$$\mathsf{Hom}_{\mathcal{C}}(C, \Gamma(E)) \simeq \mathsf{Hom}_{\mathsf{Fib}_{\mathcal{C}}(*)}(C, p_* E) \simeq \mathsf{Hom}_{\mathsf{Fib}_{\mathcal{C}}(E)}(p^* C, E) \simeq \mathsf{Hom}_{\mathsf{Fib}_{\mathcal{C}}(X)}(\mathsf{Const}_C, E).$$

Now for the sake of topology we introduce the following

Definition. For $E \in \mathsf{Fib}_{\mathcal{C}}(X)$ and $C \in \mathcal{C}$ we define a **space of C-orientations of E** denoted by $\mathsf{Orient}_C(E) \in \mathcal{S}$ as the pullback

$$\begin{array}{ccc} \mathsf{Orient}_C(E) & \longrightarrow & \mathsf{Hom}_{\mathsf{Fib}_{\mathcal{C}}(X)^{\simeq}}(E, \mathsf{Const}_C) \\ \downarrow & & \downarrow \\ \mathsf{Hom}_{\mathcal{C}}(\mathsf{Tot}(E), C) & \xrightarrow[\sim]{} & \mathsf{Hom}_{\mathsf{Fib}_{\mathcal{C}}(X)}(E, \mathsf{Const}_C) \end{array}$$

in $\mathcal{S}$.

Remark. Directly by definition we see that $E \in \mathsf{Fib}^C_{\mathcal{C}}(X)$ is C-orientable (that is, the space $\mathsf{Orient}_C(E)$ is nonempty) iff the functor $X \xrightarrow{E} \mathcal{C}$ is equivalent to a constant map to $C \in \mathcal{C}$, that is, iff the corresponding map of spaces $X \longrightarrow \mathsf{BAut}_{\mathcal{C}}(C)$ is nullhomotopic.

Now given an oriented fibration it is quite easy to describe its total space: namely, we have the following

Proposition. A point $t \in \mathsf{Orient}_C(E)$ determines an equivalence

$$\mathsf{Tot}(E) \xrightarrow[\sim]{t'} X \otimes C$$

in $\mathcal{C}$.

Proof. Consider the projection map $X \xrightarrow{p} *$ and let $t' \in \mathsf{Hom}_{\mathsf{Fib}_{\mathcal{C}^{\simeq}}(X)}(E, p^* C)$ be the point which corresponds to t. We then obtain a point

$$p_!(t') \in \mathsf{Hom}_{\mathcal{C}^{\simeq}}(p_! E, p_! \mathsf{Const}_C) \simeq \mathsf{Hom}_{\mathcal{C}^{\simeq}}(\mathsf{Tot}(E), \mathsf{Tot}(\mathsf{Const}_C)) \simeq \mathsf{Hom}_{\mathcal{C}^{\simeq}}(\mathsf{Tot}(E), X \otimes C)$$

as desired. □

Example. In the special case $\mathcal{C} := \mathsf{Mod}(R)$ the category of modules over some commutative ring spectrum $R \in \mathsf{CAlg}(\mathsf{Sp})$ and $C := R \in \mathsf{Mod}(R)$ is the free module the proposition above states that for an R-oriented $E \in \mathsf{Fib}_{\mathsf{Mod}(R)}(X)$ we obtain an equivalence or R-module spectra

$$\mathsf{Tot}(E) = \Sigma^\infty X_+ \otimes R$$

which is frequently called a **Thom isomorphism theorem**.

We end this section with the ring structure theorem. To get an intuition, recall that a lax monoidal functor between two discrete (ordinary) monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ such that for any $X, Y \in \mathcal{C}$ we are given natural morphisms

$$F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$$

and a morphism

$$I_{\mathcal{D}} \longrightarrow F(I_{\mathcal{C}})$$

which are compatible with the associativity constraints in $\mathcal{C}$ and $\mathcal{D}$. If the category $\mathcal{D}$ is cocomplete and the monoidal structure on $\mathcal{D}$ preserves colimits by each variable separately, then the colimit $B := \mathsf{colim}(F)$ admist a structure of an associative algebra in $\mathcal{D}$, where the multiplication is given by

$$B \otimes B \simeq (\mathsf{colim}_{X\in\mathcal{C}} F(X)) \otimes (\mathsf{colim}_{Y\in\mathcal{C}} F(Y)) \simeq$$

$$\simeq \mathsf{colim}_{X,Y\in\mathcal{C}} F(X) \otimes F(Y) \longrightarrow \mathsf{colim}_{X,Y\in\mathcal{C}} F(X\otimes Y) \longrightarrow B.$$

Now the algebra $B \in \mathsf{Alg}(\mathcal{D})$ in fact admits a universal property. In order to see this, recall that for every associative algebra $A \in \mathsf{Alg}(\mathcal{D})$ the category $\mathcal{D}_{/A}$ also admits a monoidal structure given by

$$(D_1 \xrightarrow{f} A) \otimes (D_2 \xrightarrow{g} A) := (D_1 \otimes D_2 \xrightarrow{f\otimes g} A \otimes A \longrightarrow A)$$

so that the natural projection functor $\mathcal{D}_{/A} \xrightarrow{\pi} \mathcal{D}$ is monoidal.

Now a morphism $B \xrightarrow{f} A$ in $\mathsf{Alg}(\mathcal{D})$ of algebras gives a lax monoidal lift G

$$\begin{array}{ccc} & & \mathcal{D}_{/A} \\ & \overset{G}{\nearrow} & \downarrow \pi \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

given by

$$G(X) := \big(F(X) \longrightarrow \mathsf{colim}_{X\in\mathcal{C}} F(X) = B \xrightarrow{f} A\big)$$

for $X \in \mathcal{C}$. Conversely, since the projection functor $\mathcal{D}_{/A} \longrightarrow \mathcal{D}$ preserves colimits we see that a lax monoidal lift G gives us a morphism $B = \mathsf{colim}_{X\in\mathcal{C}} F(X) \simeq \mathsf{colim}_{X\in\mathcal{C}} G(X) \xrightarrow{f} A$ of algebras. One can check that this is a 1-to-1 correspondence.
The discussion above motivates the following

Proposition. Let $\mathcal{C}, \mathcal{D}$ be (symmetric) monoidal $(\infty, 1)$-categories such that $\mathcal{D}$ is presentable and the monoidal structure on $\mathcal{D}$ preserves colimits by each variable separately. Then given a lax (symmetric) monoidal functor

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

the colimit $\mathsf{colim}(F) \in \mathcal{D}$ admits a (commutative) algebra structure such that for any other (commutative) algebra A in $\mathcal{D}$ the space of (commutative) algebra morphisms $\mathsf{colim}(F) \longrightarrow A$ in naturally equivalent to the space of lax (symmetric) monoidal lifts

$$\begin{array}{ccc} & & \mathcal{D}_{/A} \\ & \overset{G}{\nearrow} & \downarrow \pi \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D}. \end{array}$$

Proof. Theorems 2.8 and 2.13 in [ACB]. □

Corollary. Let $X \in \mathsf{CAlg}(\mathsf{Sp})$ be a commutative monoid in spaces and $\mathcal{C} \in \mathsf{CAlg}(\mathrm{Pr}^{\mathsf{L}}_{\infty})$ be a presentable symmetric monoidal $(\infty, 1)$-category whose monoidal structure preserves colimits in each variable separately. Then in the case when a fibration

$$X \xrightarrow{E} \mathcal{C}$$

is a symmetric monoidal functor (here we consider X as a symmetric monoidal category via the inclusion $\mathsf{CAlg}(\mathcal{S}) \longrightarrow \mathsf{CAlg}(\mathsf{Cat}_\infty) = \mathsf{Cat}_\infty^{\mathsf{sym}}$) the total space $\mathsf{Tot}(E) \in \mathcal{C}$ admits a structure of a commutative algebra object in $\mathcal{C}$, which is, moreover, universal in the sense of the discussion above.

Remarks.
1) A similar statement holds, in fact, when we work with algebras over an arbitrary ∞-operad $\mathcal{O}$.

2) We refer the reader to [ACB] for a discussion of how one can pursue the universal property above further in the case when X is in addition grouplike.

3 Cobordisms.

In this section we use the theory of fibrations developed above to define the cobordism spectra we will need further in our course. We start with the following

Notation. Let

$$\mathsf{CAlg}^{\mathsf{grouplike}}(\mathcal{S}) =: \mathsf{Ab}_\infty \xrightarrow{\quad i \quad} \mathsf{CMon}_\infty := \mathsf{CAlg}(\mathcal{S})$$

be the natural inclusion of grouplike commutative monoids intro all commutative monoids. We will denote its left adjoint called **completion** by K and its right adjoint by gl_1 so that we get an adjunction $\mathsf{K} \dashv i \dashv \mathsf{gl}_1$.

Remark. Recall that grouplike $\mathbb{E}_n$-algebras in $\mathcal{S}$ are in 1-to-1 correspondence with n-times loopspaces. Consequently, we see that the functor

$$\mathsf{Sp}_{\geq 0} \xrightarrow{\quad \Omega^\infty \quad} \mathsf{Ab}_\infty$$

is an equivalence, where $\mathsf{Sp}_{\geq 0}$ is the full subcategory of spectra spanned by connected spectra.

Examples.
1) For a space $X \in \mathcal{S}$ let $\mathsf{Free}_{\mathbb{E}_\infty}(X) \simeq \coprod_{n\geq 0} X^{\times n}/\Sigma_n$ be the free commutative monoid on X (notice that we take the factor above in the ∞-categorical sense). Since the completion of the free monoid on X should be the free abelian group on X, which is due to the remark above is equivalent to the delooping of the suspension spectrum of X, we get an equivalence

$$\mathsf{K}(\coprod_{n\geq 0} X^{\times n}/\Sigma_n) \simeq \Omega^\infty\Sigma^\infty X_+.$$

In particular, in the special case when $X = *$ is the point we get

$$\Omega^\infty\mathbb{S} \simeq \mathsf{K}(\coprod_{n\geq 0} */\Sigma_n) \simeq \mathsf{K}(\coprod_{n\geq 0} B\Sigma_n) \simeq \mathsf{K}(\mathsf{Fin}^{\simeq}),$$

where $\mathbb{S} \in \mathsf{Sp}_{\geq 0}$ is the sphere spectrum.

2) Consider the category $\mathsf{Vect}_\mathbb{R}$ of vector spaces over $\mathbb{R}$ (considered as a topologically enriched category made into $(\infty, 1)$-category). The direct sum of vector spaces endows $\mathsf{Vect}_\mathbb{R}$ with the structure of symmetric monoidal category, and hence we get a commutative monoid $\mathsf{Vect}_\mathbb{R}^{\simeq} \in \mathsf{CMon}_\infty$ in spaces. Notice that since

$$\mathsf{Aut}_{\mathsf{Vect}_\mathbb{R}}(\mathbb{R}^n) \simeq \mathsf{GL}_n(\mathbb{R}) \simeq \mathsf{O}_n$$

we get an equivalence $\mathsf{Vect}_\mathbb{R}^{\simeq} \simeq \coprod_{n\geq 0} \mathsf{BO}_n$. Its completion $\mathsf{K}(\coprod_{n\geq 0} \mathsf{BO}_n) \simeq \mathbb{Z} \times \mathsf{BO} \in \mathsf{Ab}_\infty$ is equivalent to the product of integers and the classifying space of the infinite orthogonal group.

3) Let $\mathsf{Spheres}_* \subset \mathcal{S}_*$ be the full subcategory of $\mathcal{S}_*$ spanned by spheres. Similar as above, smash product of spheres endows $\mathsf{Spheres}_*$ with the structure of symmetric monoidal category and hence we get a commutative monoin $\mathsf{Spheres}_*^{\simeq} \in \mathsf{CMon}_\infty$. Since

$$\mathsf{Aut}_{\mathsf{Spheres}_*}(S^n) \simeq \mathsf{Aut}_{\mathcal{S}_*}(S^n) \simeq \Omega^n S^n \times_{\mathbb{Z}} \{\pm 1\}$$

we get an equivalence $\mathsf{Spheres}_*^{\simeq} \simeq \coprod_{n\geq 0} \mathsf{B}(\Omega^n S^n \times_{\mathbb{Z}} \{\pm1\})$. Its completion $\mathsf{K}(\coprod_{n\geq 0} \mathsf{B}(\Omega^n S^n \times_{\mathbb{Z}} \{\pm1\})) \simeq \mathsf{BAut}_{\mathsf{Sp}}(\mathbb{S}) \times \mathbb{Z}$ is the classifying space of the space of automorphisms of the sphere spectrum.

The completion functor allows us to define a very important morphism:

Definition. Since one-point compactification functor

$$\mathsf{Vect}_{\mathbb{R}}^{\mathsf{inj}} \longrightarrow \mathsf{Spheres}$$

is symmetric monoidal, taking underlying groupoids we get a morphism

$$\coprod_{n\geq 0} \mathsf{BO}_n \simeq \mathsf{Vect}_{\mathbb{R}}^{\simeq} \xrightarrow{\quad S^\bullet \quad} \mathsf{Spheres}^{\simeq} \simeq \coprod_{n\geq 0} \mathsf{B}(\Omega^n S^n \times_{\mathbb{Z}} \{\pm 1\})$$

of commutative monoids. Applying the group completion functor $\mathsf{CMon}_\infty \xrightarrow{\quad \mathsf{K} \quad} \mathsf{Ab}_\infty$ we obtain a map

$$\mathsf{BO} \times \mathbb{Z} \simeq \mathsf{K}(\mathsf{Vect}_{\mathbb{R}}^{\simeq}) \xrightarrow{\quad J \quad} \mathsf{K}(\mathsf{Spheres}^{\simeq}) \simeq \mathsf{BAut}_{\mathsf{Sp}}(\mathbb{S}) \times \mathbb{Z}.$$

in Ab_∞ (or, equivalently, a map of connective spectra) called a **J-homomorphism**.

Remarks.
1) More directly the J-homomorphism can be constructed as follows: composing the obvious map $\mathsf{O}_n \longrightarrow \mathsf{Aut}_{\mathcal{S}}(\mathbb{R}^n)$ with the morphism $\mathsf{Aut}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \mathsf{Aut}_{\mathcal{S}_*}(S^n)$ induced by one-point compactification we get a map

$$\mathsf{O}_n \longrightarrow \mathsf{Aut}_{\mathcal{S}_*}(S^n) \simeq \Omega^n S^n \times_{\mathbb{Z}} \{\pm1\} \subset \mathsf{Hom}_{\mathcal{S}_*}(S^n, S^n) \simeq \Omega^n S^n.$$

Taking colimit as $n \to \infty$ (notice that we use here that the maps above are compatible with the obvious inclusions) we obtain a map

$$\mathsf{O} \longrightarrow \mathsf{Aut}_{\mathsf{Sp}}(\mathbb{S}) \simeq \Omega^\infty \mathbb{S} \times_{\mathbb{Z}} \{\pm1\} \subset \mathsf{Hom}_{\mathsf{Sp}}(\mathbb{S}, \mathbb{S}) \simeq \Omega^\infty \mathbb{S}$$

which can be delooped to get a map

$$\mathsf{BO} \xrightarrow{\quad J \quad} \mathsf{BAut}_{\mathsf{Sp}}(\mathbb{S})$$

in $\mathcal{S}$. Nevertheless, we prefer the definition given previously since it constructs J-homomorphism as a morphism of infinite loop spaces (that is, as a morphism in Ab_∞).

2) A bit more interesting construction of the J-homomorphism can be obtained as a special case of the cobordism hypothethis which gives a precise description of the free symmetric monoidal (∞, n)-category with duals on a single object on which a group $G \subseteq \mathsf{O}$ acts.

One of the most important applications of the J-homomorphism is the following

Definition. Given a vector bundle $E \in \mathsf{Fib}_{\mathsf{Vect}}(X)$ over a space $X \in \mathcal{S}$ define its **spherization** $\mathsf{Sph}(E) \in \mathsf{Fib}_{\mathsf{Sp}}^{\mathbb{S}}(X)$ as the fibration which corresponds to the composition

$$X \xrightarrow{\quad E \quad} \coprod_{n\geq 0} \mathsf{BO}_n \longrightarrow \mathsf{BO} \times \mathbb{Z} \xrightarrow{\quad J \quad} \mathsf{BAut}_{\mathsf{Sp}}(\mathbb{S}) \times \mathbb{Z} \subset \mathsf{Sp}$$

where the inclusion $\mathsf{BAut}_{\mathsf{Sp}}(\mathbb{S}) \times \mathbb{Z} \subset \mathsf{Sp}$ maps a pair $(\star, n)$ to $\mathbb{S}^n$. The total space $\mathsf{Tot}(\mathsf{Sph}(E)) \in \mathsf{Sp}$ is called a **Thom spectrum** and will be further denoted by $\mathsf{Th}(E)$.

Remarks.
1) Since by the construction J-homomorphism is a map in Ab_∞, we wee that if X is a commutative monoid and the map $X \xrightarrow{E} \mathsf{BO}$ is also lax symmetric monoidal, then the whole composition $\mathsf{Sph}(E)$ is also lax monoidal so that by the discussion above the spectrum $\mathsf{Th}(E) \in \mathsf{Sp}$ admits the structure of a commutative ring spectrum (and, moreover, has a universal property discussed above).

2) Let $E \in \mathsf{Fib}_{\mathsf{Vect}}(X)$ and $R \in \mathsf{Sp}$ be a spectrum. Then the composition

$$X \xrightarrow{E} \coprod_{n\geq 0} \mathsf{BO}_n \longrightarrow \mathsf{BO} \times \mathbb{Z} \xrightarrow{\mathsf{J}} \mathsf{BAut}_{\mathsf{Sp}}(\mathbb{S}) \times \mathbb{Z} \subset \mathsf{Sp} \xrightarrow{\bullet \otimes R} \mathsf{Mod}(R)$$

gives a fibration of R-modules which we will denote by $\mathsf{Sph}_R(E) \in \mathsf{Fib}_{\mathsf{Mod}(R)}(X)$. We instantly see that $\mathsf{Sph}_R(E)$ is R-orientable iff the composition above is nullhomotopic (or, equivalently, is equivalent to the constant functor).

Now one of the most important cases of the definition above is provided by the following

Definition. Let $G \subseteq \mathsf{O}$ be a subgroup of the infinite orthogonal group. We then introduce a G**-cobordisms spectrum** denoted by $MG \in \mathsf{Sp}$ simply as

$$MG := \mathsf{Th}(EG) = \mathsf{Tot}(\mathsf{Sph}(EG)) = \mathsf{Tot}\left(\mathsf{B}G \longrightarrow \mathsf{BO} \times \mathbb{Z} \xrightarrow{\mathsf{J}} \mathsf{BAut}_{\mathsf{Sp}}(\mathbb{S}) \times \mathbb{Z} \subset \mathsf{Sp}\right).$$

Another words, we may write

$$MG := \mathbb{S}_{\mathsf{h}\, G},$$

where G acts on the sphere spectrum $\mathbb{S}$ via the J-homomorphism.

Examples.
1) In the case of the identity map $\mathsf{O} \xrightarrow{\mathsf{Id}_{\mathsf{O}}} \mathsf{O}$ the corresponding spectrum MO is called a **real cobordisms spectrum**. We instantly see that MO is a commutative ring spectrum. Moreover, it is instant that there is an equivalence

$$H/2 \otimes MO \simeq H/2 \otimes BO.$$

Consequently, we see that

$$H/2_*(MO) \simeq H/2_*(BO) \simeq \mathbb{Z}/2[w_1, w_2, ...]$$

generated by the Stiefel-Whitney classes.
In fact, a much stronger result holds: namely, there is an equivalence of spectra

$$MO \simeq \bigoplus_{MO_*} H/2,$$

where $MO_* = \mathbb{Z}/2[w_1, w_2, ...]$ generated by Stiefel-Whitney classes.

2) In the case of the obvious map $\mathsf{U} \longrightarrow \mathsf{O}$ the corresponding spectrum MU is called a **complex cobordisms spectrum**. We instantly see that MU is a commutative ring spectrum. Moreover, it is instant that there is an equivalence

$$H \otimes MU \simeq H \otimes BU.$$

Consequently, we see that

$$H_*(MU) \simeq H_*(BU) \simeq \mathbb{Z}[c_1, c_2, ...]$$

generated by the Chern classes. Another important feature of the complex cobordism spectrum is the **Quillen theorem**, which states that the Hopf algebroid (MU_*, MU_*MU) is the Hopf algebroid which classifies formal group laws and strict isomorphisms.

4) Consider any H-map $\Omega^2 S^3 \longrightarrow \mathsf{BO}$ with nonzero first Stiefel-Whitney class. It is a theorem of Mahowald that in this case the Thom spectrum of the corresponding fibration of spectra over $\Omega^2 S^3$ is precisely $H/2$. In fact, many different spectra can be realized as the Thom spectrum of some fibration.

References

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