COMBINATORIAL KNOT INVARIANTS THAT DETECT TREFOILS

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ABSTRACT

We construct a series of combinatorial quandle-like knot invariants. We colour regions of a knot diagram rather than lines and assign a *weight* to each colouring. Sets of these weights are the invariants we construct (colourings and weights depend on several parameters).

Using these invariants, we prove that left and right trefoils are not isotopic using this invariant (in a particular case).

Keywords: Classical knots, Knot invariants, Quandles, Left and right trefoils

Mathematics Subject Classification: 75M25

1. Introduction

Left and right trefoils were first detected by Max Dehn [1]. The invariants used in the present article to detect trefoils are similar to quandle invariants introduced by Matveev and Joyce [2], [3], but quandles themselves do not detect trefoils. Quandles give us almost complete knot invariant. Quandles detect any pair of distinct knots except for those which can be obtained one from another by double involution, i. e. simultaneous orientation change of the knot and the ambient space $\mathbb{R}^3$. The approach used in the present paper is similar to Fenn, Rourke and Sanderson's one [4], [5], [6]. But they use framed knots and racks instead of quandles. Also their approach uses some topological calculations with homology and homotopy of the rack space and the extended rack space.

In the present article we construct a series of combinatorial knot invariants based on colourings of regions of a knot diagram by elements of some finite ring R. These colourings satisfy a condition at crossings that we call *admissibility.* This condition, in particular, uniquely determines the colour of one of four regions nearby the crossing if the other three colours are known. Thus, it is usually enough to set colours of a small number of regions to obtain a complete colouring. If R is small, the invariant can be calculated easily. In particular, trefoils are detected by the invariant with $R = \mathbb{Z}_3$.

Admissible colourings themselves are similar to quandles. In fact, each admissible colouring of regions yields a line colouring by quandle colours. But, instead of a simple count of admissible colourings, we introduce a function on an admissible colouring and call this function *weight*. The set of weights of all admissible colourings is the invariant we construct. This enables us to distinguish pairs of knots with equal numbers of admissible colourings.

Of course, there are a lot of possible ways to detect trefoils. For example, we will discuss the relationships between the constructed invariant, quandle invariants and more complex quandle cohomology invariants ([10], [11], [12]). In a particular case when we are detecting trefoils, the invariant coinsides with a particular quandle cohomology invariant, $\eta_1 + \eta_2 + \eta_3 + \eta_5$ in [10]. Our goal was to make a simple invariant of knots coloured in some way that could detect trefoils.

The invariant can detect knots other than the trefoils. For example, it can detect the knot 9_1 and its mirror image if $R = \mathbb{Z}_9$.

2. Construction of the invariant

Fix a classical link diagram L and a crossing X of it. We denote the adjacent regions by a, b, c, d as shown in Fig. 1. Namely, denote by a the bottom-left region, we consider the overcrossing line oriented from left to right. Other regions follow clockwise starting from a.

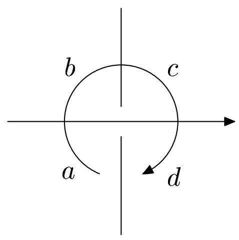

Fig. 1. Notations of the regions near the crossing.

We colour regions by elements of some finite ring R. Fix an invertible element $p \in R$. We call a colouring of L *admissible* if for each crossing X of L we have $pa + b - c - pd = 0$. We get a set of admissible colourings A.

Consider the following 4 functions:

$$\begin{aligned} w_1(a,b,c,d) &= (a-c)(b-d)(a-d)(a+d), \\ w_2(a,b,c,d) &= (a-c)(b-d)(a-d)(b+c), \\ w_3(a,b,c,d) &= (a-c)(b-d)(b-c)(a+d), \\ w_4(a,b,c,d) &= (a-c)(b-d)(b-c)(b+c), \end{aligned} \tag{2.1}$$

and polynomials:

$$\begin{aligned} f_1(p) &= (2p+1)(p^2+1), \\ f_2(p) &= -(p+2)(p^2+1), \\ f_3(p) &= -p(2p+1)(p^2+1), \\ f_4(p) &= p(p+2)(p^2+1). \end{aligned} \tag{2.2}$$

Fix arbitrary polynomials x_1, x_2, x_3, x_4 over the ground ring R. Consider the function $w = x_1w_1 + x_2w_2 + x_3w_3 + x_4w_4$ and call it the *weight* of the crossing. Set the polynomial $f = x_1f_1 + x_2f_2 + x_3f_3 + x_4f_4$. With each admissible colouring we associate the sum of weights $w(a,b,c,d)$ over all crossings and call it the *weight* of a colouring.

Thus, for the link diagram L, we get a set of weighted colourings A (and thus, a set of weights $w(A)$).

Theorem 2.1. *If p is a root of f then this set of weights of colourings is a knot invariant.*

Proof. We shall show that this set is invariant under Reidemeister moves. In all three cases we will assume that we have some admissible colouring of the diagram before the move and construct an admissible colouring of the diagram after the move. This will give us a one-to-one correspondence between the colourings of two diagrams.

Move 1. Consider four variants of the first Reidemeister move, where a single line oriented from left to right or from right to left (not shown) becomes a twist forming positive or negative crossing (see Fig. 2). Because p is invertible, the colour of the newborn small region is completely defined if we want the colouring of the new diagram to be admissible. (Colours of all other regions remain unchanged). So, we have one-to-one correspondence between the colourings of two diagrams before and after performing the Reidemeister move. Regions a and c (or b and d) of this crossing are parts of the same domain so the function w equals 0 for this new crossing. Hence, colourings of the former diagram correspond to the colourings of the latter diagram with the same weights.

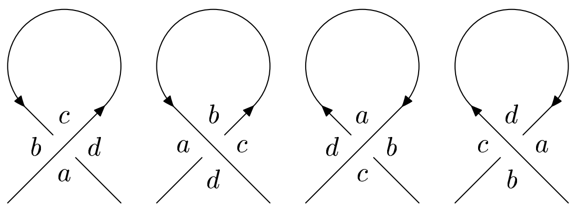

Fig. 2. Reidemeister move 1.

Move 2. Consider the second Reidemeister move such that for the diagram before performing the move we have two close lines, one (upper in the diagram) of which is oriented from left to right or from right to left (see Fig. 3, left part), and for the diagram after performing the move we have these two lines forming two crossings (see Fig. 3, right part).

Assume we have some admissible colouring of the former diagram where regions are coloured by elements $x, y, z \in R$. Leave all the regions of the latter diagram except for the newborn small region coloured their former colours. Let t be the colour of the newborn small region. Then the admissibility at the first and at the second crossings yields the equations $pz + y - x - pt = 0$ and $pt + x - y - pz = 0$. Because p is invertible, these equations uniquely determine (the same) value of t. (In the second case, admissibility yields the equations $px + t - z - py = 0$ and $py + z - t - px = 0$, and again the values of t we obtain are equal).

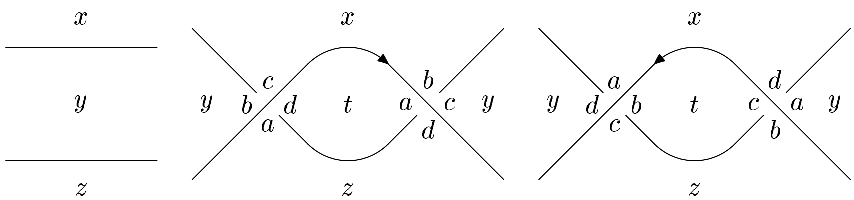

Fig. 3. Reidemeister move 2.

Two newborn crossings add two summands to the weight of the colouring. They are $w(z, y, x, t)$ and $w(t, y, x, z)$ (or $w(x, t, z, y)$ and $w(y, z, t, x)$ in the second case). But all the functions w_i satisfy the condition $w(a, b, c, d) = -w(d, c, b, a)$, so the weight of the former colouring equals the weight of the latter one.

Move 3. It is known that (see [7], [8]) it is sufficient to consider only one case of the third Reidemeister move, where two topmost lines form a positive crossing. Consider an admissible colouring of the former diagram Let its four regions be coloured by elements $a_2, b_3, a_1, b_2 \in R$ (see Fig. 4). Then two other 'exterior' regions of the former diagram are coloured $pb_3 + a_2 - pb_2$ and $a_2 - p^2a_1 + p^2b_2$. The 'interior' region of the former diagram is coloured $pb_3 + a_2 - pa_1$. After performing the Reidemeister move, if the four first regions are coloured a_2, b_3, a_1, b_2, admissibility conditions make two other exterior regions be coloured $pb_3 + a_2 - pb_2$ and $a_2 - p^2a_1 + p^2b_2$ again. The interior region is coloured $pa_1 + b_3 - pb_2$. So every admissible colouring of the former diagram corresponds to an admissible colouring of the latter diagram and vice versa.

Before the Reidemeister move, three former crossings add three summands to the weight. They are $w(b_3, a_2, pb_3 + a_2 - pa_1, a_1)$, $w(a_1, pb_3 + a_2 - pa_1, pb_3 + a_2 - pb_2, b_2)$ and $w(pb_3 + a_2 - pa_1, a_2, a_2 - p^2a_1 + p^2b_2, pb_3 + a_2 - pb_2)$. After performing the

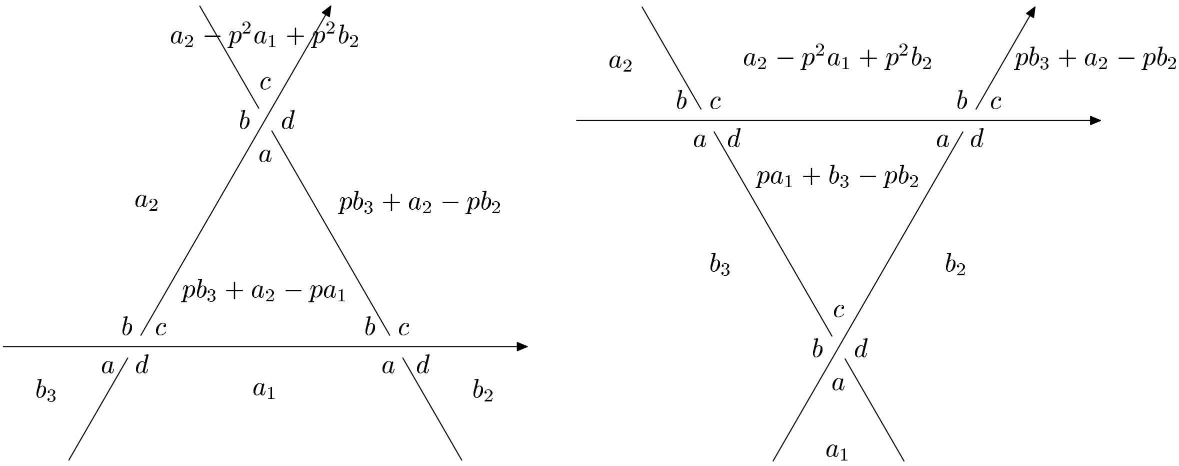

Fig. 4. Reidemeister move 3.

Reidemeister move, we have three new summands: $w(a_1, b_3, pa_1 + b_3 - pb_2, b_2)$, $w(b_3, a_2, a_2 - p^2a_1 + p^2b_2, pa_1 + b_3 - pb_2)$ and $w(pa_1 + b_3 - pb_2, a_2 - p^2a_1 + p^2b_2, pb_3 + a_2 - pb_2, b_2)$. The weight will remain unchanged if and only if

$$\begin{aligned}
&w(b_3, a_2, pb_3 + a_2 - pa_1, a_1) + \\
&w(a_1, pb_3 + a_2 - pa_1, pb_3 + a_2 - pb_2, b_2) + \\
&w(pb_3 + a_2 - pa_1, a_2, a_2 - p^2a_1 + p^2b_2, pb_3 + a_2 - pb_2) - \\
&w(a_1, b_3, pa_1 + b_3 - pb_2, b_2) - \\
&w(b_3, a_2, a_2 - p^2a_1 + p^2b_2, pa_1 + b_3 - pb_2) - \\
&w(pa_1 + b_3 - pb_2, a_2 - p^2a_1 + p^2b_2, pb_3 + a_2 - pb_2, b_2) = 0. \qquad (2.3)
\end{aligned}$$

In order to prove (2.3), first check it for $w = w_i$ where $i = 1, 2, 3, 4$. We will use the fact that $f(p) = 0$.

A calculation (see Appendix) shows that for all i, if $w = w_i$, the sum (2.3) equals $f_i(p)g(p, a_1, a_2, b_2, b_3)$, where g does not depend on i. Hence, if $w = \sum x_i w_i$, the sum (2.3) equals $(\sum x_i(p)f_i(p))g(p, a_1, a_2, b_2, b_3) = f(p)g(p, a_1, a_2, b_2, b_3)$, and this is 0 if p is a root of f. Again, the former weight equals the latter one.

We have proved that for any Reidemeister move, every admissible colouring of the diagram before the move corresponds to the colouring of the diagram after the move, and the weights of these corresponding colorings are the same. Hence, the set of weights is invariant under Reidemeister moves. Hence, it is a knot invariant, and this is what was to be proved. □

Discussion of the construction. The condition $pa + b - c - pd = 0$ comes from the second Reidemeister move. There we have a small newborn region, whose colour is defined by two admissibility conditions at the two newborn crossings. These two conditions should coincide. If we assume that the admissibility condition at a crossing is a linear equation in a, b, c, d, the coincidence can be achieved if this equation is either symmetric or antisymmetric with respect to simultaneous exchange $a \leftrightarrow d, b \leftrightarrow c$. The weights of these crossings should be opposite, and this holds if $w(a, b, c, d) = -w(d, c, b, a)$.

After the first Reidemeister move we can get a single newborn crossing whose weight should be 0. One of the ways to achieve this is to make the equalities $w(a, b, a, d) = 0$ and $w(a, b, c, b) = 0$ hold for all $a, b, c, d \in R$. Assume now that w is a polynomial in a, b, c, d. The simpler the polynomial, the more the possibility that the weight of the diagram is invariant under the third Reidemeister move. But too simple polynomials might turn to detect nothing. We can consider some polynomial of total degree 4, like w_1.

The equation (2.3) is not an identity if w equals w_1. But in this case, the left part of (2.3) can be factorized, and some of the factors depend on p only and does not depend on a, b, c, d. These factors form f_1. Three more functions, w_2, w_3 and w_4, give us similar factors f_2, f_3 and f_4. Observation that the second factor of the left part of (2.3) is always (for all w_i) the same leads us to the general construction $w = \sum x_i w_i$, where x_i are arbitrary polynomials from $R[p]$.

Unfortunately, for most of the rings R the invariant turns to be useless (for a given knot). For example, there always exist a set of trivial admissible colourings, where colours of all the regions are the same. Moreover, if R is a field, the set of admissible colourings is a linear space. In this case, it seems to be true that there always exists a two-dimensional space of trivial admissible colourings. (For the unknot, where there are no crossings, we can colour interior and exterior of a circle in two arbitrary colours). If a knot diagram has n regions, it has $n-2$ crossings. Therefore we have $n-2$ admissibility conditions. If all these conditions are linearly independent, the space of admissible colourings is 2-dimensional. They can be made dependent by choosing the characteristic of the field, and this might yield a non-trivial admissible colouring.

Admissible colourings themselves are similar to quandles. Here we explain how to construct a colouring by quandle colours from an admissible colouring of regions. Consider a diagram line. Let a be the colour of the region at right of it and b be the colour of the region at left of it. Colour this line by $pa + b$. The admissibility condition at crossing may be rewritten as $pa + b = pc + d$, and this makes the definition of the line colour correct. Then, consider a crossing and denote the colour of the overcrossing line by B, the colour of the line between a and d by A and the colour of the line between b and c by C. Then $B = pa + b = pd + c$, A equals either $pa+d$ or $pd+a$, C equals either $pb+c$ or $pc+b$, respectively. Thus, in the former case $-pA+(1+p)B = -p^2a-pd+pa+b+p^2a+pb = pb+(pa+b-pd) = pb+c = C$, and in the latter case $-pA+(1+p)B = -pa-p^2d+pd+c+p^2d+pc = pc+(-pa+pd+c) = pc + b = C$. The quandle operation is: $A \circ B = -pA + (1 + p)B$. And vice versa, if we have lines of the diagram coloured by elements of quandle and know the colour of one of the regions, we can colour other regions using the rule above.

However, weights of crossings cannot be rewritten in terms of A, B, C only. So, the invariant constructed might detect some pairs of knots that quandles cannot detect.

We can say more if $p + 1$ is invertible as well as p. In this case we can colour a line between a region of colour a and a region of colour b by $A = \lambda(pa + b)$, where $\lambda = (1 + p)^{-1}$. Obviously, quandle conditions at crossings are still satisfied. Moreover, $a \circ A = -pa + (1+p)\lambda(pa + b) = b$, and this leads to a standard quandle colouring of lines and regions of the diagram (see [11]). We can consider a quandle cocycle $\theta(x, y, z) = w((p + 1)y - px, (p + 1)z - p(p + 1)y + p^2x, (p + 1)z - px, x)$. If $pa + b - c - pd = 0$, $w(a, b, c, d) = \theta(d, \lambda(pd + a), \lambda(pd + c))$, and we can see that for a positive crossing the weight w coincides to a cocyle weight θ (see fig. 5). For a negative crossing, the cocycle weight is $-\theta(a, \lambda(pa+d), \lambda(pa+b)) = -w(d, c, b, a) = w(a, b, c, d)$, i. e. it also coinsides to the weight w.

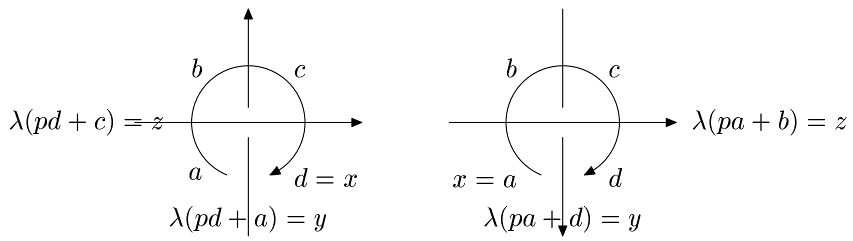

Fig. 5. Cocycle weight at the crossing.

For arbitrary $x, y, z, t \in R$, set $b_2 = x, a_1 = (p+1)t - pb_2, b_3 = (p+1)y - pa_1, a_2 = (p + 1)z - pb_3$. Then $x = b_2, t = \lambda(pb_2 + a_1), y = \lambda(pa_1 + b_3), z = \lambda(pb_3 + a_2)$, and the cocycle condition

$$\begin{aligned}&\theta(x, z, t) - \\ &\theta(x, y, t) + \\ &\theta(x, y, z) - \\ &\theta(x \circ y, z, t) + \\ &\theta(x \circ z, y \circ z, t) - \\ &\theta(x \circ t, y \circ t, z \circ t) = 0\end{aligned} \tag{2.4}$$

immediately follows from 2.3. The conditions $\theta(x, x, y) = 0$ and $\theta(x, y, y) = 0$ are also satsfied. Indeed, $\theta(x, x, y) = w(x, (p + 1)y - px, (p + 1)y - px, x) = 0$ since w_1 and w_2 equal zero if $a = d$ and w_3 and w_4 equal zero if $b = c$. $\theta(x, y, y) = w((p + 1)y - px, p^2x - (p^2 - 1)y, (p + 1)y - px, x) = 0$ since $w(a, b, a, d) = 0$ for all

$a, b, d \in R$.

Threrfore, any w function leads to some cocycle θ. But we do not know whether this cocycle is zero in the cohomology group.

Remark 2.2. We may assume that all the colourings of diagram have the infinite exterior region of colour 0 (or to any other colour of R), because the corresponding colourings of any two diagrams, one of which is result of a Reidemeister move applied to the other, have their exterior regions coloured the same colour.

Also we can consider *long knots* that can be 'constructed' from (compact) knots in the usual way.

It is known that (see [9]) two (compact) knots are isotopic iff two long knots obtained from them are isotopic. A knot diagram in $\mathbb{R}_2$ can also be made of a diagram of an isotopic knot using Reidemeister moves. A diagram of a long knot has two infinite exterior regions. This yields the following remark:

Remark 2.3. Theorem 2.1 holds for long knot diagrams too. For a long knot diagram, we may assume that all the colourings have the *two* infinite exterior regions of colours a_1, a_2, where a_1, a_2 are *any* prefixed elements of R.

3. Example (the left and right trefoils)

Consider two trefoil diagrams shown in Fig. 6. Let us prove they are not isotopic. Set the ground ring be $\mathbb{Z}_3$, whose elements will be denoted by 0, 1, 2. Set $p = 1$ and $w = w_1$.

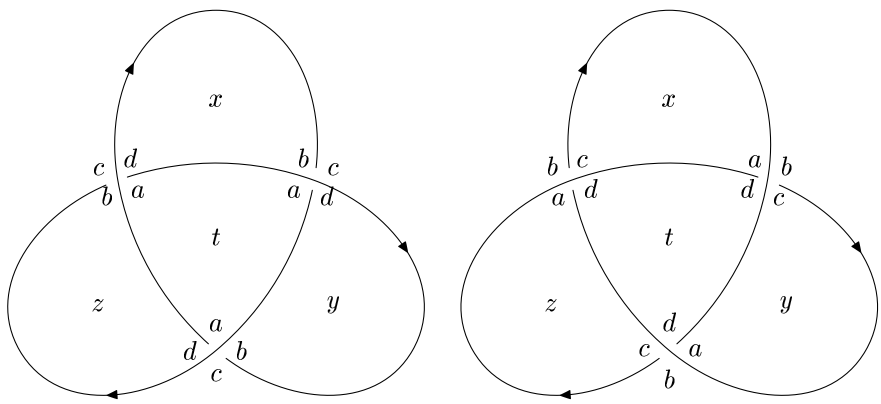

Fig. 6. The left and right trefoils.

Consider the left trefoil. If the region t is coloured 0, then all the regions x, y, z are coloured 0, and this yields three colourings of weight 0. If t is coloured 1, the colours of x, y, z form a cyclic permutation of 0, 1, 2, respectively (three possible colourings). Two of the crossings (one with $a = 1, b = 0, c = 0, d = 1$ and another with $a = 1, b = 1, c = 0, d = 2$) are of weight 0, and the last crossing (with $a = 1, b = 2, c = 0, d = 0$) is of weight $(1 - 0)(2 - 0)(1 - 0)(1 + 0) = 2$. So these three colourings are of weight 2 each. Finally, if t is coloured 2, the colours of x, y, z form a cyclic permutation of 1, 0, 2, respectively. Two of the crossings (one with $a = 2, b = 0, c = 0, d = 2$ and the other with $a = 2, b = 2, c = 0, d = 1$) are of weight 0, and the last crossing (with $a = 2, b = 1, c = 0, d = 0$) is of weight $(2 - 0)(1 - 0)(2 - 0)(2 + 0) = 2$. So, we have three more colourings of weight 2. Hence, there are three colourings of weight 0 and six colourings of weight 2.

Now consider the right trefoil. If we find a colouring of weight 1 (with the exterior region coloured 0), trefoils will be distinguished. Let region t be coloured 1, then the regions x, y, z have colours 0, 1, 2 ordered counter-clockwise. Two of the crossings (one with $a = 1, b = 0, c = 0, d = 1$ and the other with $a = 2, b = 0, c = 1, d = 1$) are of weight 0, and the last crossing (with $a = 0$, $b = 0$, $c = 2$, $d = 1$) is of weight $(0 - 2)(0 - 1)(0 - 1)(0 + 1) = 1$, and this distinguishes the trefoils. (Actually, there are three colourings of weight 0 and six colourings of weight 1 of the right trefoil).

Acknowledgements

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Appendix

Here we will show that if $w = w_i$, the sum (2.3) equals $f_i(p)g(p, a_1, a_2, b_2, b_3)$, where g does not depend on i. (In fact, these calculations were performed using Mathemathical Explorer). Indeed, if $w = w_1$, after expanding, collecting similar terms and factorization, we obtain: $w_1(b_3, a_2, pb_3 + a_2 - pa_1, a_1) + w_1(a_1, pb_3 + a_2 - pa_1, pb_3 + a_2 - pb_2, b_2) + w_1(pb_3 + a_2 - pa_1, a_2, a_2 - p^2a_1 + p^2b_2, pb_3 + a_2 - pb_2) - w_1(a_1, b_3, pa_1 + b_3 - pb_2, b_2) - w_1(b_3, a_2, a_2 - p^2a_1 + p^2b_2, pa_1 + b_3 - pb_2) - w_1(pa_1 + b_3 - pb_2, a_2 - p^2a_1 + p^2b_2, pb_3 + a_2 - pb_2, b_2) = 2a_1b_2^3p^5 - 4a_1^2b_2^2p^5 + 2a_1^2b_3^2p^5 + 2b_2^2b_3^2p^5 - 4a_1b_2b_3^2p^5 + 2a_1^3b_2p^5 - 2a_1^3b_3p^5 - 2b_2^3b_3p^5 + 2a_1b_2^2b_3p^5 + 2a_1^2b_2b_3p^5 + a_1b_2^3p^4 - 2a_2b_2^3p^4 + 2a_1b_3^3p^4 - 2b_2b_3^3p^4 + 4a_1a_2b_2^2p^4 - 5a_1^2b_3^2p^4 + b_2^2b_3^2p^4 + 4a_1b_2b_3^2p^4 - a_1^3b_2p^4 - 2a_1^2a_2b_2p^4 + a_1^3b_3p^4 + b_2^3b_3p^4 - 7a_1b_2^2b_3p^4 + 2a_2b_2^2b_3p^4 + 2a_1^2a_2b_3p^4 + 5a_1^2b_2b_3p^4 - 4a_1a_2b_2b_3p^4 + 2a_1b_2^3p^3 - a_2b_2^3p^3 - a_1b_3^3p^3 + b_2b_3^3p^3 - 3a_1^2b_2^2p^3 + a_1^2b_3^2p^3 + 2a_1a_2b_3^2p^3 - a_1b_2b_3^2p^3 - 2a_2b_2b_3^2p^3 + a_1^3b_2p^3 + a_1^2a_2b_2p^3 - a_1^3b_3p^3 - b_2^3b_3p^3 + 3a_2b_2^2b_3p^3 - a_1^2a_2b_3p^3 + 2a_1^2b_2b_3p^3 - 2a_1a_2b_2b_3p^3 + a_1b_2^3p^2 - 2a_2b_2^3p^2 + a_1b_3^3p^2 - b_2b_3^3p^2 + 3a_1a_2b_2^2p^2 - 4a_1^2b_3^2p^2 + a_1a_2b_3^2p^2 + 4a_1b_2b_3^2p^2 - a_2b_2b_3^2p^2 - a_1^3b_2p^2 - a_1^2a_2b_2p^2 + a_1^3b_3p^2 + b_2^3b_3p^2 - 6a_1b_2^2b_3p^2 + 3a_2b_2^2b_3p^2 + a_1^2a_2b_3p^2 + 4a_1^2b_2b_3p^2 - 4a_1a_2b_2b_3p^2 - a_2b_2^3p - a_1b_3^3p + b_2b_3^3p + a_1^2b_2^2p - a_1^2b_3^2p - 2b_2^2b_3^2p + 2a_1a_2b_3^2p + 3a_1b_2b_3^2p - 2a_2b_2b_3^2p - a_1^3b_2p + a_1^2a_2b_2p + a_1^3b_3p + b_2^3b_3p - 2a_1b_2^2b_3p + 3a_2b_2^2b_3p - a_1^2a_2b_3p - 2a_1a_2b_2b_3p - a_1b_3^3 + b_2b_3^3 - a_1a_2b_2^2 + a_1^2b_3^2 - b_2^2b_3^2 + a_1a_2b_3^2 - a_2b_2b_3^2 + a_1^2a_2b_2 + a_1b_2^2b_3 + a_2b_2^2b_3 - a_1^2a_2b_3 - a_1^2b_2b_3 = (p^2+1)(2p+1)(a_1 - b_2)(b_2 - b_3)(2p+1)(pa_1 - a_1 + b_3 - b_2p)(-a_2 + b_3 + a_1p - b_3p) = f_1(p)g(p, a_1, a_2, b_2, b_3)$, where $g(p, a_1, a_2, b_2, b_3) = (a_1 - b_2)(b_2 - b_3)(2p + 1)(pa_1 - a_1 + b_3 - b_2p)(-a_2 + b_3 + a_1p - b_3p)$.

Performing the same operations with w_2 yields: $w_2(b_3, a_2, pb_3 + a_2 - pa_1, a_1) + w_2(a_1, pb_3 + a_2 - pa_1, pb_3 + a_2 - pb_2, b_2) + w_2(pb_3 + a_2 - pa_1, a_2, a_2 - p^2a_1 + p^2b_2, pb_3 + a_2 - pb_2) - w_2(a_1, b_3, pa_1 + b_3 - pb_2, b_2) - w_2(b_3, a_2, a_2 - p^2a_1 + p^2b_2, pa_1 + b_3 - pb_2) - w_2(pa_1 + b_3 - pb_2, a_2 - p^2a_1 + p^2b_2, pb_3 + a_2 - pb_2, b_2) = -a_1b_2^3p^5 + 2a_1^2b_2^2p^5 - a_1^2b_3^2p^5 - b_2^2b_3^2p^5 + 2a_1b_2b_3^2p^5 - a_1^3b_2p^5 + a_1^3b_3p^5 + b_2^3b_3p^5 - a_1b_2^2b_3p^5 - a_1^2b_2b_3p^5 - 2a_1b_2^3p^4 + a_2b_2^3p^4 - a_1b_3^3p^4 + b_2b_3^3p^4 + 3a_1^2b_2^2p^4 - 2a_1a_2b_2^2p^4 + a_1^2b_3^2p^4 - 2b_2^2b_3^2p^4 + a_1b_2b_3^2p^4 - a_1^3b_2p^4 + a_1^2a_2b_2p^4 + a_1^3b_3p^4 + b_2^3b_3p^4 + 2a_1b_2^2b_3p^4 - a_2b_2^2b_3p^4 - a_1^2a_2b_3p^4 - 4a_1^2b_2b_3p^4 + 2a_1a_2b_2b_3p^4 - a_1b_2^3p^3 + 2a_2b_2^3p^3 - a_1b_3^3p^3 + b_2b_3^3p^3 - 3a_1a_2b_2^2p^3 + 4a_1^2b_3^2p^3 - a_1a_2b_3^2p^3 - 4a_1b_2b_3^2p^3 + a_2b_2b_3^2p^3 + a_1^3b_2p^3 + a_1^2a_2b_2p^3 - a_1^3b_3p^3 - b_2^3b_3p^3 + 6a_1b_2^2b_3p^3 - 3a_2b_2^2b_3p^3 - a_1^2a_2b_3p^3 - 4a_1^2b_2b_3p^3 + 4a_1a_2b_2b_3p^3 - 2a_1b_2^3p^2 + a_2b_2^3p^2 + a_1b_3^3p^2 - b_2b_3^3p^2 + 3a_1^2b_2^2p^2 - a_1^2b_3^2p^2 - 2a_1a_2b_3^2p^2 + a_1b_2b_3^2p^2 + 2a_2b_2b_3^2p^2 - a_1^3b_2p^2 - a_1^2a_2b_2p^2 + a_1^3b_3p^2 + b_2^3b_3p^2 - 3a_2b_2^2b_3p^2 + a_1^2a_2b_3p^2 - 2a_1^2b_2b_3p^2 + 2a_1a_2b_2b_3p^2 + 2a_2b_2^3p - a_1b_3^3p + b_2b_3^3p - 2a_1^2b_2^2p - 3a_1a_2b_2^2p + 5a_1^2b_3^2p + b_2^2b_3^2p - a_1a_2b_3^2p - 6a_1b_2b_3^2p + a_2b_2b_3^2p + 2a_1^3b_2p + a_1^2a_2b_2p - 2a_1^3b_3p - 2b_2^3b_3p + 7a_1b_2^2b_3p - 3a_2b_2^2b_3p - a_1^2a_2b_3p - 3a_1^2b_2b_3p + 4a_1a_2b_2b_3p + 2a_1b_3^3 - 2b_2b_3^3 + 2a_1a_2b_2^2 - 2a_1^2b_3^2 +$

$2b_2^2b_3^2 - 2a_1a_2b_3^2 + 2a_2b_2b_3^2 - 2a_1^2a_2b_2 - 2a_1b_2^2b_3 - 2a_2b_2^2b_3 + 2a_1^2a_2b_3 + 2a_1^2b_2b_3 = -(p+2)(p^2+1)(a_1-b_2)(b_2-b_3)(pa_1-a_1+b_3-b_2p)(-a_2+b_3+a_1p-b_3p) = f_2(p)g(p), a_1, a_2, b_2, b_3)$, where $g(p,a_1,a_2,b_2,b_3)$ is the same.

For w_3:

$w_3(b_3,a_2,pb_3+a_2-pa_1,a_1) + w_3(a_1,pb_3+a_2-pa_1,pb_3+a_2-pb_2,b_2) + w_3(pb_3+a_2-pa_1,a_2,a_2-p^2a_1+p^2b_2,pb_3+a_2-pb_2) - w_3(a_1,b_3,pa_1+b_3-pb_2,b_2) - w_3(b_3,a_2,a_2-p^2a_1+p^2b_2,pa_1+b_3-pb_2) - w_3(pa_1+b_3-pb_2,a_2-p^2a_1+p^2b_2,pb_3+a_2-pb_2,b_2) = -2a_1b_2^3p^6 + 4a_1^2b_2^2p^6 - 2a_1^2b_3^2p^6 - 2b_2^2b_3^2p^6 + 4a_1b_2b_3^2p^6 - 2a_1^3b_2p^6 + 2a_1^3b_3p^6 + 2b_2^3b_3p^6 - 2a_1b_2^2b_3p^6 - 2a_1^2b_2b_3p^6 - a_1b_3^3p^5 + 2a_2b_3^3p^5 - 2a_1b_3^3p^5 + 2b_2b_3^3p^5 - 4a_1a_2b_2^2p^5 + 5a_1^2b_3^2p^5 - b_2^2b_3^2p^5 - 4a_1b_2b_3^2p^5 + a_1^3b_2p^5 + 2a_1^2a_2b_2p^5 - a_1^3b_3p^5 - b_2^3b_3p^5 + 7a_1b_2^2b_3p^5 - 2a_2b_2^2b_3p^5 - 2a_1^2a_2b_3p^5 - 5a_1^2b_2b_3p^5 + 4a_1a_2b_2b_3p^5 - 2a_1b_2^3p^4 + a_2b_2^3p^4 + a_1b_3^3p^4 - b_2b_3^3p^4 + 3a_1^2b_2^2p^4 - a_1^2b_3^2p^4 - 2a_1a_2b_3^2p^4 + a_1b_2b_3^2p^4 + 2a_2b_2b_3^2p^4 - a_1^3b_2p^4 - a_1^2a_2b_2p^4 + a_1^3b_3p^4 + b_2^3b_3p^4 - 3a_2b_2^2b_3p^4 + a_1^2a_2b_3p^4 - 2a_1^2b_2b_3p^4 + 2a_1a_2b_2b_3p^4 - a_1b_2^3p^3 + 2a_2b_2^3p^3 - a_1b_3^3p^3 + b_2b_3^3p^3 - 3a_1a_2b_2^2p^3 + 4a_1^2b_3^2p^3 - a_1a_2b_3^2p^3 - 4a_1b_2b_3^2p^3 + a_2b_2b_3^2p^3 + a_1^3b_2p^3 + a_1^2a_2b_2p^3 - a_1^3b_3p^3 - b_2^3b_3p^3 + 6a_1b_2^2b_3p^3 - 3a_2b_2^2b_3p^3 - a_1^2a_2b_3p^3 - 4a_1^2b_2b_3p^3 + 4a_1a_2b_2b_3p^3 + a_2b_3^3p^2 + a_1b_3^3p^2 - b_2b_3^3p^2 - a_1^2b_2^2p^2 + a_1^2b_3^2p^2 + 2b_2^2b_3^2p^2 - 2a_1a_2b_2^2p^2 - 3a_1b_2b_3^2p^2 + 2a_2b_2b_3^2p^2 + a_1^3b_2p^2 - a_1^2a_2b_2p^2 - a_1^3b_3p^2 - b_2^3b_3p^2 + 2a_1b_2^2b_3p^2 - 3a_2b_2^2b_3p^2 + a_1^2a_2b_3p^2 + 2a_1a_2b_2b_3p^2 + a_1b_3^3p - b_2b_3^3p + a_1a_2b_2^2p - a_1^2b_3^2p + b_2^2b_3^2p - a_1a_2b_3^2p + a_2b_2b_3^2p - a_1^2a_2b_2p - a_1b_2^2b_3p - a_2b_2^2b_3p + a_1^2a_2b_3p + a_1^2b_2b_3p = -p(2p+1)(p^2+1)(a_1-b_2)(b_2-b_3)(pa_1-a_1+b_3-b_2p)(-a_2+b_3+a_1p-b_3p) = f_3(p)g(p,a_1,a_2,b_2,b_3)$, where $g(p,a_1,a_2,b_2,b_3)$ is the same as in the previous cases.

For w_4:

$w_4(b_3,a_2,pb_3+a_2-pa_1,a_1) + w_4(a_1,pb_3+a_2-pa_1,pb_3+a_2-pb_2,b_2) + w_4(pb_3+a_2-pa_1,a_2,a_2-p^2a_1+p^2b_2,pb_3+a_2-pb_2) - w_4(a_1,b_3,pa_1+b_3-pb_2,b_2) - w_4(b_3,a_2,a_2-p^2a_1+p^2b_2,pa_1+b_3-pb_2) - w_4(pa_1+b_3-pb_2,a_2-p^2a_1+p^2b_2,pb_3+a_2-pb_2,b_2) = a_1b_2^3p^6 - 2a_1^2b_2^2p^6 + a_1^2b_3^2p^6 + b_2^2b_3^2p^6 - 2a_1b_2b_3^2p^6 + a_1^3b_2p^6 - a_1^3b_3p^6 - b_2^3b_3p^6 + a_1b_2^2b_3p^6 + a_1^2b_2b_3p^6 + 2a_1b_2^3p^5 - a_2b_2^3p^5 + a_1b_3^3p^5 - b_2b_3^3p^5 - 3a_1^2b_2^2p^5 + 2a_1a_2b_2^2p^5 - a_1^2b_3^2p^5 + 2b_2^2b_3^2p^5 - a_1b_2b_3^2p^5 + a_1^3b_2p^5 - a_1^2a_2b_2p^5 - a_1^3b_3p^5 - b_2^3b_3p^5 - 2a_1b_2^2b_3p^5 + a_2b_2^2b_3p^5 + a_1^2a_2b_3p^5 + 4a_1^2b_2b_3p^5 - 2a_1a_2b_2b_3p^5 + a_1b_2^3p^4 - 2a_2b_2^3p^4 + a_1b_3^3p^4 - b_2b_3^3p^4 + 3a_1a_2b_2^2p^4 - 4a_1^2b_3^2p^4 + a_1a_2b_3^2p^4 + 4a_1b_2b_3^2p^4 - a_2b_2b_3^2p^4 - a_1^3b_2p^4 - a_1^2a_2b_2p^4 + a_1^3b_3p^4 + b_2^3b_3p^4 - 6a_1b_2^2b_3p^4 + 3a_2b_2^2b_3p^4 + a_1^2a_2b_3p^4 + 4a_1^2b_2b_3p^4 - 4a_1a_2b_2b_3p^4 + 2a_1b_2^3p^3 - a_2b_2^3p^3 - a_1b_3^3p^3 + b_2b_3^3p^3 - 3a_1^2b_2^2p^3 + a_1^2b_3^2p^3 + 2a_1a_2b_3^2p^3 - a_1b_2b_3^2p^3 - 2a_2b_2b_3^2p^3 + a_1^3b_2p^3 + a_1^2a_2b_2p^3 - a_1^3b_3p^3 - b_2^3b_3p^3 + 3a_2b_2^2b_3p^3 - a_1^2a_2b_3p^3 + 2a_1^2b_2b_3p^3 - 2a_1a_2b_2b_3p^3 - 2a_2b_2^3p^2 + a_1b_3^3p^2 - b_2b_3^3p^2 + 2a_1^2b_2^2p^2 + 3a_1a_2b_2^2p^2 - 5a_1^2b_3^2p^2 - b_2^2b_3^2p^2 + a_1a_2b_3^2p^2 + 6a_1b_2b_3^2p^2 - a_2b_2b_3^2p^2 - 2a_1^3b_2p^2 - a_1^2a_2b_2p^2 + 2a_1^3b_3p^2 + 2b_2^3b_3p^2 - 7a_1b_2^2b_3p^2 + 3a_2b_2^2b_3p^2 + a_1^2a_2b_3p^2 + 3a_1^2b_2b_3p^2 - 4a_1a_2b_2b_3p^2 - 2a_1b_3^3p + 2b_2b_3^3p - 2a_1a_2b_2^2p + 2a_1^2b_3^2p - 2b_2^2b_3^2p + 2a_1a_2b_3^2p - 2a_2b_2b_3^2p + 2a_1^2a_2b_2p + 2a_1b_2^2b_3p + 2a_2b_2^2b_3p - 2a_1^2a_2b_3p - 2a_1^2b_2b_3p = p(p+2)(p^2+1)(a_1-b_2)(b_2-b_3)(pa_1-a_1+b_3-b_2p)(-a_2+b_3+a_1p-b_3p) = f_4(p)g(p,a_1,a_2,b_2,b_3)$.