An estimate of canonical dimension of groups based on Schubert calculus

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Abstract

We sketch the proof of a connection between the canonical (0-)dimension of semisimple split simply connected groups and cohomology of their full flag varieties. Using this connection, we get a new estimate of the canonical (0-)dimension of simply connected split exceptional groups of type E understood as a group.

1 Introduction

To define the canonical (0-)dimension of an algebraic group understood as a group, we first need to define the canonical (0-)dimension of a scheme understood as a scheme (which is a different definition). Roughly speaking, the canonical (0-)dimension of a scheme is a number indicating how hard it is to get a rational point in the scheme. The canonical (0-)dimension of an algebraic group shows how hard it is to get rational points in torsors related to the group.

To be more precise, let us fix some conventions and give some definitions. We speak of algebraic schemes and use stacks project as the source of basic definitions. All schemes in the present text are of finite type over a field and separated. The base field is arbitrary.

Speaking of canonical dimension of schemes, there are two closely related notions in the literature: the *canonical 0-dimension of a scheme* defined in [14] and the *canonical dimension of a scheme* defined in [9]. These two definitions are not known to be always equivalent, but they are equivalent for two particular classes of schemes: for smooth complete schemes and for torsors of split reductive groups (see [13, Theorem 1.16, Remark 1.17, and Example 1.18]). The definition from [14] looks more motivated, so we are going to use it.

Definition 1.1 ([14, Section 4a, first paragraph of Section 4b, and the last paragraph of Section 2a])**.** Given a scheme X over a field K, the canonical 0-dimension of X understood as a scheme (notation: $\mathrm{cd}_0(X)$) is:

$$\mathrm{cd}_0(X) = \max_{\substack{L=\text{a field containing } K\\ X_L \text{ has a rational point}}} \ \min_{\substack{L_0=\text{a subfield of } L,\ K\subseteq L_0\\ X_{L_0} \text{ still has a rational point}}} \operatorname{trdeg}_K L_0.$$

A bit less formally, canonical dimension can be explained as follows. Suppose we have expanded the base field K to L, and got a rational point in X_L. How large can L be, compared to K? In general, it can be very large, this is unbounded. A related question with a finite answer is: how many algebraically independent generators do we have to keep, at worst (for the worst L), to still have a rational point after scalar extension (*not necessarily the same* rational point that we found after expanding scalars to L)? This number of generators is the canonical dimension of X. For more properties of canonical dimension, see [14] in the case of general X and [8] in the case of smooth projective X.

We have underlined above that we want to get a rational point over a field between K and L, but not necessarily the same rational point. If we demanded to get the same rational point, we would get

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the definition of the *essential 0-dimension* of a scheme, which is known to coincide with the (standard in algebraic geometry) dimension, see [13, Proposition 1.2]. This can be viewed as a motivation for the word "dimension". (But essential dimension is not only defined for schemes, and in broader generality it becomes a much more nontrivial notion.)

Another motivation for canonical (0-)dimension comes from *incompressible varieties*, but this motivation is only valid for the canonical (0-)dimension of smooth complete schemes. The definition that we are going to give next, the canonical 0-dimension of an algebraic group, and that will be used in the main theorem of this text, does not involve the canonical (0-)dimension of smooth complete schemes, so this motivation will be useless for us. One can find details for this motivation in [8, Section 2].

The second object we need to define before we can define the canonical 0-dimension of a group is a torsor of a group. All algebraic groups in this text are affine. All reductive, semisimple, and simple groups in this text are smooth. Torsors of algebraic groups (over a point) are, informally speaking, homogeneous spaces that are "as large as the group itself". This notion is mostly interesting over non algebraically closed fields.

Definition 1.2 ([14, Section 3a])**.** Given an algebraic group G, a *G-torsor over a point* (or simply a *G-torsor*) is a scheme E with an action $\varphi\colon G \times E \to E$ such that $(\varphi, \mathrm{pr}_2)\colon G \times E \to E \times E$, where pr_2 is the projection to the second factor, is an isomorphism.

It is known that all torsors of affine algebraic groups over a point are affine.

Finally, the canonical 0-dimension of an algebraic group understood as a group measures how hard it is to get rational points in torsors, informally speaking, related to the group. Precisely:

Definition 1.3 ([14, Section 4g])**.** given an algebraic group G over a field F, the canonical 0-dimension of G understood as a group (notation: $\mathfrak{cd}_0(G)$) is

$$\mathfrak{cd}_0(G) = \max_{K=\text{a field containing } F} \; \max_{E=\text{a } G_K\text{-torsor}} \mathrm{cd}_0(E).$$

The definition of canonical dimension of an algebraic group understood as a group in [9, Introduction] repeats this definition almost exactly, with the only difference being that instead of $\mathrm{cd}_0(E)$ it uses the definition of canonical dimension of E understood as a scheme from the paper [9] itself. But as we already mentioned above, it is known that these two notions are known to be equivalent for torsors of split reductive groups. So, Definition 1.3 is also equivalent to the definition of canonical dimension of a group from [9, Introduction] for split reductive groups. All groups whose canonical dimension we are going to estimate in this text are split reductive (and even simply connected semisimple), so these results also estimate the canonical dimension in the sense of [9, Introduction].

To formulate the main goal of this text precisely, we need to introduce some more notation and terminology. Given a split semisimple algebraic group G and a Borel subgroup B, the corresponding Weyl group W, and the element $w_0 \in W$ of maximal length, for each $w \in W$ we denote the Schubert variety $\overline{Bw_0w^{-1}B/B} \subseteq G/B$ by Z_w. This Z_w is a Schubert divisor if and only if w is a simple reflection, and we denote all Schubert divisors by $D_1, \ldots, D_r$.

It is known that the classes $[Z_w] \in \mathrm{CH}(G/B)$ for all $w \in W$ form a free set of generators of $\mathrm{CH}(G/B)$ as of an abelian group. We say that a product of classes of Schubert divisors $[D_1]^{n_1} \ldots [D_r]^{n_r}$ is *multiplicity-free* if there exists $w \in W$ such that the coefficient at $[Z_w]$ in the decomposition of $[D_1]^{n_1} \ldots [D_r]^{n_r}$ into a linear combination of Schubert classes equals 1.

Now we can formulate the goal of this text precisely. Our goal is to sketch the proof of the following theorem.

Theorem 1.4. *Let G be a split semisimple simply connected algebraic group over an arbitrary field, let B be a Borel subgroup, let r be the rank of G, and let $D_1, \ldots, D_r \subset G/B$ be the Schubert divisors corresponding to the r simple roots of G. If $[D_1]^{n_1} \ldots [D_r]^{n_r}$ is a multiplicity-free product of Schubert divisors, then $\mathfrak{cd}_0(G) \leq \dim(G/B) - n_1 - \ldots - n_r$.*

As a corollary of this theorem and [5, Theorem 11.5], we will immediately get the following:

Corollary 1.5. *Let G be a split semisimple simply connected algebraic group of type E_r. Then $\mathfrak{cd}_0(G) \leq$ 17, 37, or 86 for $r =$ 6, 7, or 8, respectively.*

The the most difficult part of estimating the canonical dimension of simply connected split groups of type E_r (and in obtaining Corollary 1.5) was actually to understand which products of Schubert divisors are multiplicity-free (and this was understood in [5] by the author). The description of multiplicity-free products of Schubert divisors in [5] is explicit enough to find the maximal degree of such a multiplicity-free product precisely. However, for the canonical dimension we still get only an estimate from above, because Theorem 1.4 can only produce upper estimates anyway.

The part of the argument establishing relation between Schubert calculus and canonical dimension (in other words, the proof of Theorem 1.4 itself) was known to the experts (or at least they believed that the argument is doable this way). However, we were unable to find an exposition suitable for more general mathematical audience. The present paper contains such an exposition. In this text, we are going to follow the ideas of several proofs from [10], where canonical dimension was related to cohomology of flag varieties of orthogonal groups (more precisely, orthogonal Grassmannians, not full flag varieties).

Speaking of the canonical dimension of simply connected split groups of other types, in types A_r and C_r the canonical dimension is known to be zero. For types B_r and D_r, the canonical dimension was estimated (and computed exactly if r is a power of 2) by N. Karpenko in [10]. In type D_r, even though the maximal degree of a multiplicity-free product of Schubert divisors is also found precisely in [5], the resulting estimate of the canonical dimension from Theorem 1.4 turns out to be the same as Karpenko's estimate $\le (r-1)(r-2)/2$. For type G_2, the canonical dimension (of a split simply connected group) is known and equals 3, see [1, Example 10.7]. For type F_4, no nontrivial upper bounds on the canonical dimension are known.

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2 Preparation 1: Recall of basic Galois descent theory

We always denote by $\mathrm{id}_X \colon X \to X$, where X is a scheme, the identity map.

To start proving Theorem 1.4, we first need to define the quotient of a torsor modulo a Borel subgroup. The definition we are going to use is not very intuitive, but it is used in papers on canonical dimension (for example, in [9]).

Definition 2.1. Let G be a semisimple split simply connected algebraic group over a field K, let B be a Borel subgroup, and let E be a G-torsor. The *quotient of the torsor modulo the Borel subgroup* (notation: E/B) is the categorical quotient (see [16, Definition 0.5]; "categorical" is in the category of all separated schemes of finite type over K) of $E \times G/B$ modulo the diagonal action of G.

In fact, it can be proved that such a quotient is also a categorical quotient of E modulo B, but we will not use this. The existence of such a categorical quotient $(E \times G/B)/G$ is known, is stated in [6, Proposition 12.2], and can be proved using Galois descent theory. We will need an explicit construction for E/B, and we will recall it below. It is known that such a quotient E/B is smooth, absolutely irreducible, and projective.

Given this definition, we can say that the first and the most technically difficult step in proving Theorem 1.4 is to prove the following proposition.

Proposition 2.2. *Let G be a semisimple split simply connected algebraic group over a field K, let B be a Borel subgroup, and let E be a G-torsor. Let K_1 be an extension of K. Then the map of Picard groups induced by field extension* $\operatorname{Pic}(E/B) \to \operatorname{Pic}((E/B)_{K_1})$ *is an isomorphism.*

The proof of this proposition makes a lot of use of Galois descent theory. We will need two versions of this theory: for vector spaces and for schemes.

The version for vector spaces is quite simple. Suppose we have a finite Galois extension of fields $K \subseteq L$ with Galois group Γ.

Definition 2.3. Let V and W be two L-vector spaces, and let $\sigma \in \Gamma$. A map (of sets) $f\colon V \to W$ is called *σ-semilinear* if $f(a_1v_1 + a_2v_2) = \sigma(a_1)f(v_1) + \sigma(a_2)f(v_2)$ for all $a_1, a_2 \in L$ and $v_1, v_2 \in V$.

Definition 2.4. Let V be an L-vector space. A *semirepresentation* of Γ on V is an action $\psi\colon \Gamma \times V \to V$ on V **understood as a set** such that for each $\sigma \in \Gamma$, the map $\psi|_{\{\sigma\}\times V}\colon V \to V$ is σ-semilinear.

Example 2.5. Let U be a K-vector space. Then we can define a Γ-semirepresentation on $V = L \otimes_K U$ by the formula $\sigma(a \otimes u) = \sigma(a) \otimes u$ for all $a \in L$ and $u \in U$: the formula defines a K-bilinear map, so it can be extended to the whole $L \otimes_K U$.

This semirepresentation will be called the *standard representation* of Γ on $L \otimes_K U$.

Given a semirepresentation of Γ on an L-vector space V, we can define the *dual* semirepresentation of Γ on V^* by the formula $(\sigma f)(v) = \sigma(f(\sigma^{-1}(v)))$ for all $\sigma \in \Gamma$, $f \in V^*$, and $v \in V$. A direct computation shows that this action indeed produces elements of V^* out of elements of V^*, and one more direct computation shows that this is a semirepresentation. We can further induce a semirepresentation of Γ on the symmetric algebra $S^\bullet(V^*)$ by saying that $\sigma(fg) = (\sigma f)(\sigma g)$.

We will need the following well-known fact about semirepresentations, which is sometimes called Hilbert's Theorem 90.

Theorem 2.6. *Suppose we have a representation of* Γ *on an L-vector space V. Then V^Γ is a K-vector space, and the (obvious) map $L \otimes_K V^\Gamma \to V$, $a \otimes v \mapsto av$, is an isomorphism.* □

Now let us recall the basic notions and facts of Galois descent theory for schemes. We will need three categories. The first category, $\mathcal{S}ch_K$ is the category of (separated and of finite type, as everywhere in the text) schemes over a field K.

To define the second category, suppose we have two fields, $K \subseteq L$. First, we need to recall the definition of the functor of restriction of scalars from $\mathcal{S}ch_L$ to $\mathcal{S}ch_K$ (notation: $-|_K$). If X is an object of $\mathcal{S}ch_L$, we say that X *with scalars restricted from L to K* is the scheme that has the same topological space as X, the same ring of regular functions on each open subset *as an abstract ring*, but for the algebra structure, we view this ring as a K-algebra rather than an L-algebra (the multiplication by elements of K is given by the embedding $K \subseteq L$). We denote this scheme by $X|_K$. And if $f \in \operatorname{Mor}_{\mathcal{S}ch_L}(X, Y)$, then one can check directly that the same map of topological spaces as in f, together with the same map of abstract rings for each open subset of Y (= each open subset of $Y|_K$) as in f, satisfies the definition of a morphism of K-schemes from $X|_K$ to $Y|_K$.

Now we can say that the second category, which we will call *the category of K, L-schemes* (notation: $\mathcal{S}ch_{K,L}$), has schemes over L as objects, and the set $\operatorname{Mor}_{\mathcal{S}ch_{K,L}}(X, Y)$, where X and Y are L-schemes, is the set of morphisms of K-schemes from $X|_K$ to $Y|_K$.

The third category will be introduced a bit later.

Example 2.7. Let $K \subseteq L$ be a finite Galois extension of fields, and let $\sigma \in \operatorname{Gal}(L/K)$. Let $X = \operatorname{Spec} L$. Then $K[X|_K] = L$, and $\sigma^{-1}\colon L \to L$ is an automorphism of this K-algebra. It defines the dual automorphism of the K-scheme $X|_K$, which we denote by $\sigma_* \in \operatorname{Mor}_{\mathcal{S}ch_{K,L}}(\operatorname{Spec} L, \operatorname{Spec} L)$.

We keep the notation σ_* until the end of the text.

Definition 2.8. Let $K \subseteq L$ be a finite Galois extension of fields with Galois group Γ, and let $\sigma \in \Gamma$. A morphism $f\colon X \to Y$ in $\mathcal{S}ch_{K,L}$ is called *σ-semilinear* if the following diagram (in $\mathcal{S}ch_{K,L}$) is commutative:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
\operatorname{Spec} L \arrow[r, "\sigma_*"] & \operatorname{Spec} L
\end{tikzcd}$$

The vertical arrows are the restrictions of scalars of the structure morphisms.

Clearly, under the conditions of this definition, if f (resp. g) is a σ (resp. τ)-semilinear morphism, then $g \circ f$ is a $\tau\sigma$-semilinear morphism. It is also clear that 1-semilinear morphisms are exactly the restrictions of scalars of the morphisms in $\mathcal{S}ch_L$.

Definition 2.9. Let $K \subseteq L$ be a finite Galois extension of fields with Galois group Γ. We will say that we have a *Galois-semiaction* of Γ on an L-scheme X (or that Γ *Galois-semiacts* on X) if we have an action $\psi\colon \Gamma \times X|_K \to X|_K$ (here Γ is understood as an algebraic group over K) such that for each $\sigma \in \Gamma$, the automorphism $\psi_\sigma = \psi|_{\{\sigma\}\times X|_K}$ of $X|_K$, understood as an automorphism of X in $\mathcal{S}ch_{K,L}$, is σ-semilinear.

We say that a finite affine open covering of X is *Γ-stable* if Γ preserves (normalizes) each of these open sets.

Example 2.10. Let V be a L-vector space equipped with a semirepresentation of Γ. Then, informally speaking one can "extend this semirepresentation to a semiaction of Γ on V understood as a scheme".

More formally, consider the dual semirepresentation of Γ on V^* and the induced semirepresentation on $S^\bullet(V^*)$. Then for each $\sigma \in \Gamma$, the action of σ^{-1} on $S^\bullet(V^*)$ is a σ^{-1}-semilinear map of vector spaces $S^\bullet(V^*) \to S^\bullet(V^*)$, and a direct check shows that the dual morphism of schemes $\operatorname{Spec}(S^\bullet(V^*))|_K \to \operatorname{Spec}(S^\bullet(V^*))|_K$ is a σ-semilinear morphism in $\mathcal{S}ch_{K,L}$. Another direct check shows that these semilinear morphisms together (for all $\sigma \in \Gamma$) form a Galois-semiaction on $\operatorname{Spec}(S^\bullet(V^*))$ (which is a formal way of "viewing V as a scheme") and that on rational points, this Galois-semiaction coincides with the original semirepresentation of Γ on V understood as a vector space.

We will say that this Galois-semiaction is *induced* by the original semirepresentation of Γ on V.

Definition 2.11. Let V be a L-vector space equipped with a semirepresentation of Γ. Let X be a subscheme of V preserved (normalized) by the induced Galois-semiaction on V understood as a scheme (denote this Galois-semiaction by $\psi\colon \Gamma \times V|_K \to V|_K$). Then we also call the restriction of ψ onto X (formally, $\psi|_{\Gamma\times X|_K}$) the *induced Galois-semiaction* on X.

Similarly, if X is defined by a homogeneous ideal in $S^\bullet(V^*)$, then this induced Galois-semiaction can be extended in the obvious way to the projectivization $\mathbf{P}(X)$. The resulting Galois-semiaction will also be called the Galois-semiaction on $\mathbf{P}(X)$ *induced* by the semirepresentation of Γ on V.

Now we are ready to define the third category we need to formulate basic facts of Galois descent theory. Given a finite Galois extension of fields $K \subseteq L$ with Galois group Γ, we define the category of *stable L-schemes with semiaction of* Γ (notation: $\mathcal{S}t\mathcal{S}ch_{L,\Gamma}$). Its objects are pairs (X, ψ), where X is an L-scheme, and $\psi\colon \Gamma \times X|_K \to X|_K$ is a Galois-semiaction such that X admits a Γ-stable finite affine open covering. The morphisms are morphisms in $\mathcal{S}ch_L$ that become Γ-equivariant after the restriction of scalars to K.

Now recall that if a finite group acts on a scheme (now this is going to be a scheme over the smaller field, K), and there is a stable finite affine open covering for this action, then the categorical quotient always exists, and can be constructed, for example, as the orbit space of the action.

So, for a *finite* Galois extension $K \subseteq L$ with group Γ, we can define the *Galois descent functor* $\operatorname{Dec}_K\colon \mathcal{S}t\mathcal{S}ch_{L,\Gamma} \to \mathcal{S}ch_K$ as follows: an object (X, ψ) is mapped to the categorical quotient X/Γ, and the morphisms are mapped using the universal property of the categorical quotient.

We can also define the *Galois upgrade functor* $\cdot_{L,\Gamma}\colon \mathcal{S}ch_K \to \mathcal{S}t\mathcal{S}ch_{L,\Gamma}$. On the objects, it maps a K-scheme Y to (Y_L, φ), where the semiaction φ is defined on the affine charts as follows: if U is an open affine chart of Y, $\sigma \in \Gamma$, then $\varphi(\Gamma, (U_L)|_K) = (U_L)|_K$ (recall that the restriction of scalars does not change the topological space). And if $f \otimes \lambda \in L[U_L] = K[U] \otimes L$, then $(\varphi|_{\{\sigma\}\times(U_L)|_K})^*(f \otimes \lambda) = f \otimes \sigma^{-1}(\lambda)$. On the morphisms, the Galois upgrade functor is just extension of scalars.

Remark 2.12. *Let U be a K-vector space. Then $U_{L,\Gamma}$ is canonically isomorphic to $(L \otimes_K U, \psi)$, where ψ is the Galois-semiaction on $L \otimes_K U$ understood as a scheme induced (Example 2.10) by the standard semirepresentation (Example 2.5) of Γ on $L \otimes_K U$.*

Similarly, if $X \subseteq U$ (resp. $X \subseteq \mathbf{P}(U)$) is a subscheme, then X_L can be canonically embedded into $L \otimes_K U$ (resp. $\mathbf{P}(L \otimes_K U)$), and the semiaction on $X_{L,\Gamma}$ is also induced by the standard semirepresentation on $L \otimes_K U$.

Using the Galois descent and upgrade functors, let us state the main theorem of Galois descent theory

Theorem 2.13. *Let $K \subseteq L$ be a Galois extension with Galois group Γ. The Galois descent and upgrade functors are mutually quasiinverse equivalences of categories $\mathcal{S}ch_K \leftrightarrow \mathcal{S}t\mathcal{S}ch_{L,\Gamma}$.*

Proof. Well-known. For a proof one can see, for example, [18, §V.4.20, Proposition 12 and its proof], although the terminology there is a bit different. Instead of actions of Γ by semilinear automorphisms, the terminology there is based on families of varieties (where each variety is obtained by "twisting" by the corresponding element of Γ) and families of morphisms (over L, in the standard sense) between these varieties. The functoriality is not proved there, but it easily follows from the explicit construction of Dec_K using orbit spaces. □

So, using this theorem, instead of studying schemes over K (they may not have rational points or be otherwise not so nice), we can now study varieties over a larger field L (which must be a finite Galois extension of K, but otherwise we can choose it freely, for example so that our schemes over K become nicer when we extend scalars to L). But, to work with torsors and to prove Proposition 2.2, we will need a few more facts from general Galois descent theory.

First, let us immediately prove a corollary of Theorem 2.13 about semiactions induced by semirepresentations.

Corollary 2.14. *Let V be an L-vector space with a semirepresentation of Γ, and let $X \subseteq \mathbb{P}(V)$ be an irreducible and reduced subscheme with the induced Galois-semiaction ψ. Let $D \in \mathrm{CH}^1(X)$ be the class of (any) hyperplane section of X.*

Then, after the identification $(X, \psi) \cong (\mathrm{Dec}_K(X, \psi))_{L,\Gamma}$, D belongs to the image of the scalar extension map $\mathrm{CH}^1(\mathrm{Dec}_K(X, \psi)) \to \mathrm{CH}^1(X)$.

Proof. Consider the dual semirepresentation of Γ on V^*. By Theorem 2.6, there exists a nonzero function $f \in (V^*)^\Gamma$. Then the vanishing locus of f in X is a Γ-invariant hyperplane section. Denote this hyperplane section by Y.

It follows from the explicit construction of Dec_K using orbit spaces that the Galois descent of the embedding of Y into X is still an embedding of a closed subscheme. By Theorem 2.13, this subscheme becomes Y after the extension of scalars back to L. □

Second, to work with actions of algebraic groups over K using Theorem 2.13, we need to understand how direct products work in $\mathcal{S}ch_{K,L}$ and in $\mathcal{S}t\mathcal{S}ch_{L,\Gamma}$. The direct products in $\mathcal{S}ch_L$ and in $\mathcal{S}ch_{K,L}$ are different. However, the following lemma shows that direct products from $\mathcal{S}ch_L$ are useful in $\mathcal{S}ch_{K,L}$ if we work with semilinear morphisms.

Lemma 2.15. *Let $K \subseteq L$ be a Galois extension of fields with Galois group Γ. Let X and Y be L-schemes, let Z be their product in $\mathcal{S}ch_L$, and let $p_1 \in \mathrm{Mor}_{\mathcal{S}ch_L}(Z, X)$ and $p_2 \in \mathrm{Mor}_{\mathcal{S}ch_L}(Z, Y)$ be the standard projections. Then for every L-scheme T, for every $\sigma \in \Gamma$, and for every two σ-semilinear morphisms $f\colon T \to X$ and $g\colon T \to Y$ there exists a unique σ-semilinear morphism $h\colon T \to Z$ such that $p_1|_K \circ h = f$ and $p_2|_K \circ h = g$.*

Proof. Easy to see. Details omitted. □

Suppose, for K, L, Γ, X, Y, Z, p_1, and p_2 as in the lemma, we have two σ-semilinear morphisms: $f \in \mathrm{Mor}_{\mathcal{S}ch_{K,L}}(A, X)$ and $g \in \mathrm{Mor}_{\mathcal{S}ch_{K,L}}(B, Y)$. Let C be the product of A and B in $\mathcal{S}ch_L$, and let $q_1 \in \mathrm{Mor}_{\mathcal{S}ch_L}(C, A)$ and $q_2 \in \mathrm{Mor}_{\mathcal{S}ch_L}(C, B)$ be the standard projections. In this case we will denote by $f \times g \in \mathrm{Mor}_{\mathcal{S}ch_{K,L}}(C, Z)$ the unique σ-semilinear morphism such that $p_1|_K \circ (f \times g) = f \circ q_1|_K$ and $p_2|_K \circ (f \times g) = f \circ q_2|_K$. Informally speaking, this is a straightforward way to build a morphism $A \times B \to X \times Y$ out of morphisms $A \to X$ and $B \to Y$.

After we have this lemma, it is easy to construct a Galois-semiaction on a product of two L-schemes X and Y out of two semiactions on X and on Y. Precisely, if $\psi_1 \colon \Gamma \times X|_K \to X|_K$ and $\psi_2 \colon \Gamma \times Y|_K \to Y|_K$ are two Galois-semiactions, then the new semiaction on $Z = X \times Y$ (the product in $\mathcal{S}ch_L$), which we will call *the product of semiactions* and denote $\psi_1 \times \psi_2$, is defined as follows: $(\psi_1 \times \psi_2)|_{\{\sigma\} \times Z|_K} = (\psi_1)|_{\{\sigma\} \times X|_K} \times (\psi_2)|_{\{\sigma\} \times Y|_K}$. (In fact, $(Z, \psi_1 \times \psi_2)$ will then be the product of (X, ψ_1) and (Y, ψ_2) in $\mathcal{S}t\mathcal{S}ch_{L,\Gamma}$, but we will not need this.) Then a direct check shows that $(Z, \psi_1 \times \psi_2)$ is the product of (X, ψ_1) and (Y, ψ_2) in $\mathcal{S}t\mathcal{S}ch_{L,\Gamma}$.

Using this description of products, we can say, for example, the following.

Example 2.16. Let K, L, and Γ are as above, let G be an algebraic group over K, let X be a scheme over K, and let $\varphi \colon G \times X \to X$ be an action. Then, if we denote by ψ_1 and ψ_2 the Galois-semiactions such that $G_{L,\Gamma} = (G_L, \psi_1)$ and $X_{L,\Gamma} = (X_L, \psi_2)$, then φ_L is Γ-equivariant for the actions $\psi_1 \times \psi_2$ and ψ_2.

This finishes the part of theory of Galois descent that we need.

3 Preparation 2: Isomorphism of Picard groups under scalar extension

Now let us apply Galois descent theory to torsors and prove Proposition 2.2. We will need a few more preliminary steps.

First, if (E, φ) is a torsor of an algebraic group G, and if e is a rational point of E, then we denote the map $\varphi|_{G \times \{e\}} \colon G \to E$ by triv_e. Clearly, this is an isomorphism. We keep this notation until the end of the text. Recall also that a torsor is called *trivial* if it has a rational point.

Then, we need a lemma.

Lemma 3.1. *Let (E, φ) be a torsor of a* smooth *algebraic group G over a field K. Then there exists a finite Galois extension L of K such that (E_L, φ_L) is a trivial G_L-torsor.*

Idea of the proof. Clearly, E is smooth. Smooth schemes obtain a rational point after scalar extension to a separable closure ([17, Prop. 3.2.20]). □

So, instead of studying a torsor without rational points, we can make a finite Galois extension of scalars and study a torsor with a rational point and with a compatible Galois-semiaction.

From now on, we fix until the end of this section: a split semisimple simply connected algebraic group G over a field K, a Borel subgroup B of G, a maximal torus T of G contained in B, a G-torsor (E, φ), a finite Galois extension L of K such that E_L has a rational point, and a rational point $e \in E_L$. Denote $\Gamma = \mathrm{Gal}(L/K)$. It is known that G_L is also split semisimple, and that B_L is a Borel subgroup.

Denote by $\mathrm{inv} \colon G_L \to G_L$ the inversion map. Denote the action map $G_L \times (G/B)_L \to (G/B)_L$ by ξ, and for each individual element (rational point) g of G_L, denote by ξ_g the action of this element g on $(G/B)_L$ (in other words, $\xi_g = \xi|_{\{g\} \times (G/B)_L}$). Denote by ψ_1, $\overline{\psi}_1$, ψ_2, the semiactions such that $G_{L,\Gamma} = (G_L, \psi_1)$, $(G/B)_{L,\Gamma} = ((G/B)_L, \overline{\psi}_1)$, and $E_{L,\Gamma} = (E_L, \psi_2)$.

Recall that E/B is defined as a categorical quotient $(E \times G/B)/G$. Now we need to recall an explicit construction for G/B and for E/B.

For a strongly dominant weight λ of G, denote the corresponding representation of G by V_λ. If $v_\lambda \in V_\lambda$ is a highest weight vector, then it is known that the stabilizer of $\ell = \mathrm{Span}(v_\lambda)$ is B, and that G/B can be constructed as $G\ell \subseteq \mathbf{P}(V_\lambda)$ (this orbit is known to be closed).

As well as with G/B, we will also have a separate construction for E/B for each strongly dominant weight λ of G. First, note that the above construction for G/B obviously commutes with the field extension, so $(G/B)_L$ can be constructed as $G_L \operatorname{Span}(1 \otimes v_\lambda) \subseteq \mathbf{P}(L \otimes_K V_\lambda)$. So, the action $\overline{\psi}_1$ is induced by the standard semirepresentation of Γ on $L \otimes_K V_\lambda$.

We are going to construct E/B as the Galois descent of $(G/B)_L$ equipped with a specific Galois-semiaction (most likely different from $\overline{\psi}_1$). First, let us denote by p the following map from $E_L \times (G/B)_L \to (G/B)_L$:

$$p = \xi \circ ((\mathrm{inv} \circ \mathrm{triv}_e^{-1}) \times \mathrm{id}_{(G/B)_L}) \tag{3.2}$$

In other words, we first isomorphically map E_L to G_L, then invert G_L (during these maps, $(G/B)_L$ stays untouched), and then we act by G_L on $(G/B)_L$.

Lemma 3.3. *The variety $(G/B)_L$ together with the map p is a categorical quotient of $E_L \times (G/B)_L$ modulo the diagonal action of G_L.*

Proof. The G_L-equivariance of p is a direct computation. The universal property is an easy diagram chase. ☐

However, the map p is not equivariant for the semiactions $\psi_2 \times \overline{\psi}_1$ and $\overline{\psi}_1$. Let us introduce a new semiaction $\overline{\psi}_2$ on $(G/B)_L$. Namely, for each $\sigma \in \Gamma$, set

$$\overline{\psi}_2|_{\{\sigma\}\times(G/B)_L} = \xi_{(\mathrm{triv}_e^{-1}(\psi_2(\sigma,e)))^{-1}} \circ \overline{\psi}_1|_{\{\sigma\}\times(G/B)_L} \tag{3.4}$$

(The fact that this is a semiaction needs to be checked, but this is a computation using the Γ-equivariance of the maps φ_L and and ξ and the discussion after Definition 2.8. In terms of Galois cohomology, which we didn't recall here, the formula for $\overline{\psi}_2$ can be formulated as "$\overline{\psi}_2$ is obtained from $\overline{\psi}_1$ by twisting by the cocycle $\sigma \mapsto \xi_{(\mathrm{triv}_e^{-1}(\psi_2(\sigma,e)))^{-1}}$ from $H^1(\Gamma, \mathrm{Aut}((G/B)_L))$".)

Lemma 3.5. *The map p is equivariant for the semiactions $\psi_2 \times \overline{\psi}_1$ and $\overline{\psi}_2$.*

Moreover, let $q\colon (E_L \times (G/B)_L, \psi_2 \times \overline{\psi}_1) \to (X, \psi_3)$ be another G_L-invariant and Γ-equivariant morphism (where X is an arbitrary L-scheme with a Galois-semiaction ψ_3). Then the unique map $r\colon (G/B)_L \to X$ from the universal property of the categorical quotient is actually Γ-equivariant for the semiactions $\overline{\psi}_2$ and ψ_3.

Proof. Direct computation. The first statement again uses the Γ-equivariance of the maps φ_L (for the semiactions $\psi_1 \times \psi_2$ and ψ_2) and ξ (for the semiactions $\psi_1 \times \overline{\psi}_1$ and $\overline{\psi}_1$). The second statement uses the uniqueness of the map in the universal property of a categorical quotient. Details omitted. ☐

Proposition 3.6. *The scheme $\mathrm{Dec}_K((G/B)_L, \overline{\psi}_2)$ with the map $\mathrm{Dec}_K(p)$ is a categorical quotient of $E \times (G/B)$ modulo the diagonal action of G. Therefore, it can be used as E/B, and $(E/B)_L$ then becomes isomorphic to $(G/B)_L$.*

Proof. Follows from Lemmas 3.3 and 3.5. Also uses Theorem 2.13. ☐

Remark 3.7. *It follows from this construction that if E itself is trivial, then we can take $L = K$ and see that E/B is isomorphic to G/B.*

Now, after we have recalled an explicit construction of E/B, let us prove the surjectivity in Proposition 2.2 for $K_1 = L$ (the field we have fixed). We start with the following easy lemma.

Lemma 3.8. *For each strongly dominant weight λ and for the corresponding embedding $(G/B)_L \hookrightarrow \mathbf{P}(L \otimes_K V_\lambda)$, the semiaction $\overline{\psi}_2$ is induced by a semirepresentation of Γ on $L \otimes_K V_\lambda$.*

Proof. Denote the standard semirepresentation of Γ on $L \otimes_K V_\lambda$ by $\widetilde{\psi}_1$. Denote the action of an element (a rational point) g of G on $L \otimes_K V_\lambda$ by Ξ_g.

For each $\sigma \in \Gamma$, denote

$$\widetilde{\psi}_{2,\sigma} = \Xi_{(\mathrm{triv}_e^{-1}(\psi_2(\sigma,e)))^{-1}} \circ \widetilde{\psi}_1|_{\{\sigma\}\times(L\otimes_K V_\lambda)} \tag{3.9}$$

One more direct computation, this time using the Γ-equivariance of the representation map $G_L \times (L \otimes_K V_\lambda) \to L \otimes_K V_\lambda$ and of the action map φ_L, shows that $\widetilde{\psi}_{2,\sigma}$ is a σ-semilinear map from $L \otimes_K V_\lambda$ to itself for each $\sigma \in \Gamma$, and that all these maps together, for all $\sigma \in \Gamma$, form a semirepresentation of Γ on $L \otimes_K V_\lambda$. Then it is clear from the formulas 3.4 and 3.9 that this semirepresentation induces the semiaction $\overline{\psi}_2$. ☐

Now, the last piece of theory we need to prove the surjectivity in Proposition 2.2 for $K_1 = L$ is the following explicit description of the Picard group of a flag variety.

Theorem 3.10. *For G, B, and T as above, denote the weight lattice of G_L by Λ. For each strongly dominant weight $\lambda \in \Lambda$, denote by $\mathcal{L}_\lambda$ the pullback of the anticanonical bundle under the embedding $G_L/B_L \hookrightarrow \mathbf{P}(L \otimes_K V_\lambda)$ described above. Then:*

1. *The notation $\mathcal{L}_\lambda$ and the map $\lambda \mapsto \mathcal{L}_\lambda$ (which we have so far defined for strongly dominant weights λ only) can be extended to a group homomorphism $\Lambda \to \operatorname{Pic}(G_L/B_L)$. Moreover, this group homomorphism is actually an isomorphism.*
2. *In terms of this notation, if λ_i is the ith fundamental weight, then the vanishing locus of (any) global section of $\mathcal{L}_\lambda$ is $(D_i)_L$, where D_i is the divisor described in the Introduction.*

Proof. Well-known. ☐

Proposition 3.11. *Let E, B, and L be as above. Then the map of Picard groups induced by field extension $\operatorname{Pic}(E/B) \to \operatorname{Pic}((E/B)_L)$ is surjective.*

Proof. Follows from Corollary 2.14, Lemma 3.8, and Theorem 3.10.

More accurately, we also need the fact that for any smooth and absolutely connected scheme X, the isomorphism $\operatorname{Pic}(X) \to \mathrm{CH}^1(X)$ commutes with extension of scalars (this is well-known), and the fact that the construction of G/B also commutes with extension of scalars (this follows directly from the construction, as we have already mentioned). ☐

Now we will need to recall a well-known result about Picard and Brauer groups. First, note that for any Galois-semiaction on an irreducible scheme Y there is a straightforward way to extend this semiaction to an action on the set of open subsets of Y, on the field of rational functions on Y, and therefore on the Picard group of Y.

Now let us state the result about Picard and Brauer groups. For any two fields $K' \subseteq L'$, denote $\operatorname{Br}_{L'}(K') = \ker(\cdot \otimes_{K'} L' \colon \operatorname{Br}(K') \to \operatorname{Br}(L'))$.

Lemma 3.12. *Let X be a complete smooth absolutely connected scheme over a field K', and let L' be a finite Galois extension of K'. Let $\Gamma' = \operatorname{Gal}(L'/K')$. Then:*

1. *The image of the map $\cdot_{L'} \colon \operatorname{Pic}(X) \to \operatorname{Pic}(X_{L'})$ is contained in $\operatorname{Pic}(X_{L'})^{\Gamma'}$.*
2. *There is an exact sequence*

$$0 \to \operatorname{Pic}(X) \xrightarrow{\cdot_{L'}} \operatorname{Pic}(X_{L'})^{\Gamma'} \to \operatorname{Br}_{L'}(K') \xrightarrow{\cdot \otimes K'(X)} \operatorname{Br}(K'(X))$$

Proof. Well-known. Follows from exact sequences

$$1 \to L'^* \to L'(X_{L'})^* \to L'(X_{L'})^*/L'^* \to 1$$

and

$$1 \to L'(X_{L'})^*/L'^* \to \operatorname{Div}(X_{L'}) \to \operatorname{Pic}(X_{L'}) \to 1.$$

☐

Now we are ready to prove Proposition 2.2 in the whole generality.

Lemma 3.13. *Proposition 2.2 is true if E is a trivial torsor.*

Proof. Follows from Remark 3.7 and the explicit description of $\operatorname{Pic}(G/B)$ (like Theorem 3.10, but over an arbitrary field instead of L). ☐

Lemma 3.14. *Proposition 2.2 is true when K_1 equals L (the field we fixed earlier in this section).*

Proof. The injectivity follows[1] from Lemma 3.12 (2) for $K' = K$, $L' = L$. The surjectivity is Proposition 3.11. ☐

Idea of proof of Proposition 2.2 in the general case. We omit the details regarding commutativity of the diagrams of Picard groups for consecutive field extensions. First, prove the proposition for K_1 containing L using Lemma 3.13 for the torsor E_L and for the extension K_1/L.

Then, for a completely arbitrary K_1 containing K, we first find a finite Galois extension L_1 of K_1 for the G_{K_1}-torsor E_{K_1} in the same way as we found and fixed L for K, G, and E. Since L is a finite Galois extension of K, we can construct a field L_2 admitting embeddings of L and L_1. By the previous step for E_{K_1} instead of E, $\operatorname{Pic}((E/B)_{K_1}) \cong \operatorname{Pic}((E/B)_{L_2})$. By the previous step for the original E, $\operatorname{Pic}(E/B) \cong \operatorname{Pic}((E/B)_{L_2})$. Therefore, $\operatorname{Pic}(E/B) \to \operatorname{Pic}((E/B)_{K_1})$ is an isomorphism. ☐

4 Estimate of canonical dimension

The next steps of the proof of Theorem 1.4 follows the idea of proof of [10, Proposition 5.1].

First, we will need a result from [9]. To formulate it, let us start with recalling a definition from [9]. Let X be a scheme over an arbitrary field F. The *determination function associated with* X (see [9, Section 2]) is the following functor from the category of all fields containing F to the category consisting of $\varnothing$ and a fixed one-element set $\{0\}$: A field F_1 is mapped to $\{0\}$ if and only if X_{F_1} has a rational point, otherwise $F_1 \mapsto \varnothing$.

Also, recall that an algebraic group H over K is called *special* if all torsors of all groups H_{K_1}, where K_1 is a field extension of K, are trivial. It is known (see, for example, [11, Section 3 and Theorem 2.1]) that B is special.

Now, with these two definitions, we can say that the following lemma becomes a particular case of [9, Lemma 6.5], namely, for the special group P there being equal to B:

Lemma 4.1. *For any field extension K'/K, $E_{K'}$ has a rational point if and only if $(E/B)_{K'}$ has a rational point.* ☐

Then, we will need a well-known fact about the Chow ring of a smooth scheme.

Proposition 4.2. *Let X be a smooth scheme over a field K, and let L be an extension of K. The map of Chow rings $\mathrm{CH}_L\colon \mathrm{CH}(X) \to \mathrm{CH}(X_L)$, $[Y] \mapsto [Y_L]$ for each irreducible*[2] *and reduced subscheme Y of X is well-defined and is a morphism of rings.*

The isomorphism $\operatorname{Pic}(X) \to \mathrm{CH}^1(X)$ commutes with extension of scalars.

Proof. Well-known. ☐

We will also need the following theorem. It is stated in [10, Theorem 2.3] and follows from [7, Corollary 12.2], the preceding commutative diagram, and the definition of distinguished varieties in [7]. More precisely, this definition implies that in the particular case of the commutative diagram, the distinguished varieties are subvarieties of the intersection of supports of the cycles. Recall that a cycle (a formal linear combination of irreducible subvarieties) is called *nonnegative* if the coefficients in this linear combination are nonnegative, and an element of the Chow ring is called *nonnegative* if it can be represented by a nonnegative cycle.

Theorem 4.3. *Let X be a smooth scheme over an arbitrary field K such that the tangent bundle is generated by global sections. Let α and β be nonnegative elements of $\mathrm{CH}(X)$. If α (resp. β) is represented by a nonnegative cycle with support on $A \subseteq X$ (resp. $B \subseteq X$), then $\alpha\beta$ can be represented by a nonnegative cycle with support on $A \cap B$.* ☐

We need two more facts from [10]:

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[1] The idea of using the exact sequence of Brauer and Picard groups to prove the isomorphism between Picard groups is present in [10, Proof of Theorem 1.4].

[2] We will not need this, but the map defined this way actually maps the class of any subscheme Y of X to $[Y_L] \in \mathrm{CH}(X_L)$.

Lemma 4.4 ([10, Remark 2.4]). *Let G be a split simple simply connected algebraic group over an arbitrary field K, let B be a Borel subgroup of G, let E be a G-torsor. Then the tangent bundle of E/B is generated by global sections.* ☐

Lemma 4.5 ([10, Corollary 2.2]). *Let X be a smooth absolutely irreducible scheme over an arbitrary field K, and let L be an extension of K. Let $\alpha \in \mathrm{CH}^1(X)$. If $\mathrm{CH}_L(\alpha) \in \mathrm{CH}^1(X_L)$ is nonnegative, then $\alpha \in \mathrm{CH}^1(X)$ is nonnegative.* ☐

The following proposition is like Proposition 5.1 in [10], but in a different situation. It is known that if an algebraic group G over a field F is semisimple, split, and simply connected, and B is a Borel subgroup, then for every extension K of F, G_K is also semisimple, split, and simply connected, and B_K is a Borel subgroup.

Proposition 4.6. *Let G be a semisimple split simply connected algebraic group over an arbitrary field F. Let B be a Borel subgroup, and let $D_1, \dots, D_r \subset G/B$ be the Schubert divisors. Suppose that the product $[D_1]^{n_1} \dots [D_r]^{n_r}$ is multiplicity-free.*

Let K be a field extension of F, and let E be a G_K-torsor. Then there exists a closed, irreducible, and reduced subscheme Y of E/B_K of codimension $n_1 + \ldots + n_r$ such that $Y_{K(E/B_K)}$ has a rational point.

Proof. Denote $X = E/B_K$ and $L = K(X)$. Write

$$[D_1]^{n_1}[D_2]^{n_2} \dots [D_r]^{n_r} = \sum C_{w,n_1,\dots,n_r}[Z_w].$$

Fix an element $v \in W$ such that $C_{v,n_1,\dots,n_r} = 1$. Set $v' = vw_0$. Then it follows from [4, §3.3, Proposition 1a] that $[D_1]^{n_1} \dots [D_r]^{n_r}[Z_{v'}] = [\mathrm{pt}]$. By Proposition 4.2, we have $[(D_1)_L]^{n_1} \dots [(D_r)_L]^{n_r}[(Z_{v'})_L] = [\mathrm{pt}] \in \mathrm{CH}((G/B)_L)$.

It is easy to see that X_L has a rational point. By Lemma 4.1, E_L also has a rational point. Then by Remark 4.4, X_L is isomorphic to $(G/B)_L$. Fix one such isomorphism (it depends on the choice of a rational point of E_L) and denote it by $f\colon X_L \to (G/B)_L$.

Denote the composition $f_* \circ \mathrm{CH}_L\colon \mathrm{CH}(X) \to \mathrm{CH}(X_L) \to \mathrm{CH}((G/B)_L)$ by g. Denote $g_1 = g|_{\mathrm{CH}^1(X)}$. By Proposition 2.2 (and by Proposition 4.2), g_1 is an isomorphism between $\mathrm{CH}^1(X)$ and $\mathrm{CH}^1((G/B)_L)$ For each i ($1 \le i \le r$), denote $\alpha_i = g_1^{-1}((D_1)_L) \in \mathrm{CH}^1(X)$. By Lemma 4.5, these are nonnegative classes (although we don't claim that each α_i is representable by a single irreducible and reduced divisor).

By Theorem 4.3, the class $\alpha_1^{n_1} \dots \alpha_r^{n_r}$ is nonnegative. Choose irreducible subvarieties $Y_i \subseteq X$ of codimension $n_1 + \ldots + n_r$ such that $\alpha_1^{n_1} \dots \alpha_r^{n_r}$ can be written as their linear combination with nonnegative coefficients. Denote these coefficients by $c_i \ge 0$:

$$\alpha_1^{n_1} \dots \alpha_r^{n_r} = \sum c_i[Y_i].$$

It is clear from the definitions that for each i, $g([Y_i])$ is a linear combination of the irreducible components of $f((Y_i)_L)$ with nonnegative coefficients. Since g is a morphism of rings (Proposition 4.2), we have

$$g\left(\sum c_i[Y_i]\right)[(Z_{v'})_L] = [(D_1)_L]^{n_1} \dots [(D_r)_L]^{n_r}[(Z_{v'})_L] = [\mathrm{pt}].$$

On the other hand, $g(\sum c_i[Y_i])[(Z_{v'})_L] = \sum (c_i g([Y_i])[(Z_{v'})_L])$, and by Theorem 4.3, each $g([Y_i])[(Z_{v'})_L]$ is (can be written as) a linear combination of (reduced) 0-dimensional subvarieties (i. e. closed points) of $f((Y_i)_L) \cap (Z_{v'})_L$ with nonnegative coefficients.

So, a rational point of $(G/B)_L$ is equivalent in the Chow ring to a linear combination of some closed points with nonnegative coefficients. Then it follows from the well-definedness of the degree map $\mathrm{CH}^{\dim(G/B)}((G/B)_L) \to \mathbb{Z}$ (see [7, Definition 1.4]) that the linear combination actually consists of just one point with coefficient 1, and this point is rational. Recall that this was a point in some intersection $f((Y_i)_L) \cap (Z_{v'})_L$. In particular, we see that for one of the schemes Y_i, $(Y_i)_L$ has a rational point, and we can set $Y = Y_i$.

(We don't need this, but for this index i we also get $c_i = 1$, and for all other indices i we get $g([Y_i])[(Z_{v'})_L] = 0$ or $c_i = 0$.) ☐

(Last steps of the) Proof of Theorem 1.4. Let F be the base field of G. It is known that for any extension K of F, G_K is also semisimple, split, and simply connected, and B_K is a Borel subgroup of G_K.

We are going to use some results from [9]. As we already mentioned in the Introduction, the definitions of canonical dimension in [9] are not literally the same as here, so for simplicity of notation, we write cd without subscript for the canonical dimension of a scheme in the sense of [9, Section 2] and $\mathfrak{cd}$ (also without subscript) for the canonical dimension of a group in the sense of [9, Introduction].

As we also mentioned in Introduction, if E is a torsor of a split reductive group, then $\mathrm{cd}_0(E) = \mathrm{cd}(E)$ by [13, Theorem 1.16 and Example 1.18]. Therefore it follows from the statements of Definition 1.3 and the definition of $\mathfrak{cd}$ in [9, Introduction], that $\mathfrak{cd}(G) = \mathfrak{cd}_0(G)$ since G_K is (in particular) a split reductive group for any extension K of F.

Until the end of this paragraph, let K be an extension of F, let E be a G_K-torsor. Denote $L = K(E/B_K)$. By Proposition 4.6, there exists a subscheme $Y \subseteq E/B_K$ of codimension $n_1 + \ldots + n_r$ such that Y_L has a rational point. By [9, Corollary 4.7], we have[3] $\mathrm{cd}(E/B_K) \leq \dim(G/B) - n_1 - \ldots - n_r$.

Now, [9, discussion after Lemma 6.7] says that $\mathfrak{cd}(G)$ can be computed as the supremum of $\mathrm{cd}(E/B_K)$ for all extensions K of F and all G_K-torsors E. Therefore, $\mathfrak{cd}(G) \leq \dim(G/B) - n_1 - \ldots - n_r$ and also $\mathfrak{cd}_0(G) \leq \dim(G/B) - n_1 - \ldots - n_r$. □

References

[1] G. Berhuy, Z. Reichstein, *On the notion of canonical dimension for algebraic groups*, Adv. Math. **198**:2 (2005), 128–171.

[2] A. Borel, J.-P. Serre, *Théorèmes de finitude en cohomologie galoisienne*, Commentarii Mathematici Helvetici, **39** (1964), 111–164.

[3] A. Borel, J. Tits, *Groupes réductifs*, Pub. Math. I.H.E.S. **27** (1965), 55–150.

[4] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Annales scientifiques de l'É. N. S. 4^e série, **7**:1 (1974), 53–88.

[5] R. Devyatov, *Multiplicity-free products of Schubert divisors*, preprint arXiv:1711.02058 [math.AG], 6 Nov 2017.

[6] M. Florence, *On the essential dimension of cyclic p-groups*, Invent. Math., **171**:1 (2008), 175–189.

[7] W. Fulton, *Intersection theory*, Springer, New York, 1998.

[8] N. Karpenko, *Canonical Dimension*, in: R. Bhatia, A. Pal, G Rangarajan, V Srinivas, M Vanninathan (Eds.), *Proceedings of the International Congress of Mathematicians 2010, Hyderabad, India, 19–27 August 2010, Vol. 2*, Hindustan Book Agency, New Delhi, 2010, 146–161.

[9] N. A. Karpenko, A. S. Merkurjev, *Canonical p-dimension of algebraic groups*, Adv. Math. **205**:2 (2006), 410–433.

[10] N. Karpenko, *A bound for canonical dimension of the (semi)spinor groups*, Duke Math. J. **133**:2 (2006), 391–404.

[11] N. Karpenko, *On generically split generic flag varieties*, Bull. London Math. Soc. **50**:3 (2018), 496–508.

[12] A. S. Merkurjev, J.-P. Tignol, *The multipliers of similitudes and the Brauer group of homogeneous varieties*, J. reine angew. Math. **1995**:461 (1995), 13–47.

[13] A. S. Merkurjev, *Essential dimension*, in: R. Baeza, W. K. Chan, D. W. Hoffmann, and R. Schulze-Pillot (Eds.), *Quadratic Forms—Algebra, Arithmetic, and Geometry*, Contemporary Mathematics **493**, AMS, Providence, RI, 2009, 299–325.

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[3] We don't need this fact directly, but since E/B_K is smooth and projective, by [13, Theorem 1.16 and Remark 1.17], we have $\mathrm{cd}(E/B_K) = \mathrm{cd}_0(E/B_K)$. So, we could write "$\mathrm{cd}_0(E/B_K)$" here.

[14] A. S. Merkurjev, *Essential dimension: a survey*, Transformation Groups **18**:2 (2013), 415–481.

[15] J. S. Milne, *Algebraic Groups*, Cambridge Studies in Advanced Mathematics **170**, Cambridge University Press, Cambridge, 2017.

[16] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, Ergebnisse der Mathematikund ihrer Grenzgebiete 2. Folge **34**, Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo Hong Kong Barcelona Budapest, 1994.

[17] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics **6**, Oxford University Press, Oxford, 2002.

[18] J.-P. Serre, *Algebraic Groups and Class Fields*, Graduate Texts in Mathematics **117**, Springer-Verlag, New York, 1988.

[19] J.-P. Serre, *Galois Cohomology*, Springer Monographs in Mathematics, Springer-Verlag, Berlin Heidelberg New York, 2002.

[20] J. Tits, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque*, J. reine angew. Math. **1971**:247 (1971), 196–220.