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Generically Transitive Actions on Multiple Flag Varieties

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Let G be a semisimple algebraic group whose decomposition into the product of simple components does not contain simple groups of type A, and $P \subseteq G$ be a parabolic subgroup. Extending the results of Popov [8], we enumerate all triples (G, P, n) such that (a) there exists an open G-orbit on the *multiple flag variety* $G/P \times G/P \times \ldots \times G/P$ (n factors), (b) the number of G-orbits on the multiple flag variety is finite.

1 Introduction

Let G be a semisimple connected algebraic group over an algebraically closed field of characteristic zero, and $P \subseteq G$ be a parabolic subgroup. One easily checks that G-orbits on $G/P \times G/P$ are in bijection with P-orbits on G/P. The Bruhat decomposition of G implies that the number of P-orbits on G/P is finite and that these orbits are enumerated by a subset in the Weyl group W corresponding to G. In particular, there is an open G-orbit on $G/P \times G/P$. So we come to the following questions: for which G, P and $n \geq 3$ is there an open G-orbit on the *multiple flag variety* $(G/P)^n := G/P \times G/P \times \ldots \times G/P$? For which G, P and n is the number of orbits finite?

First, let $\pi\colon \widetilde{G} \to G$ be a simply connected cover. Then π induces a bijection between parabolic subgroups $P \subseteq G$ and $\widetilde{P} \subseteq \widetilde{G}$, namely $\widetilde{P} = \pi^{-1}(P)$, and an isomorphism $\widetilde{G}/\widetilde{P} \to G/P$. Also, $\widetilde{G}/\widetilde{P}$ may be considered as G-variety since $\operatorname{Ker}\pi$ acts trivially on it. In this sense the isomorphism is G-equivariant. Therefore, we may replace G with its simply connected cover and vice versa. Moreover, any connected and simply connected semisimple group G is isomorphic to $G^{(1)} \times \ldots \times G^{(k)}$, where $G^{(i)}$ are simple, and for every parabolic subgroup $P \subseteq G$ there exist parabolic subgroups $P^{(i)} \subseteq G^{(i)}$ such that $G/P \cong G^{(1)}/P^{(1)} \times \ldots \times G^{(k)}/P^{(k)}$. The product of groups $G^{(1)} \times \ldots \times G^{(k)}$ acts on the product of varieties $(G/P)^n \cong (G^{(1)}/P^{(1)})^n \times \ldots \times (G^{(k)}/P^{(k)})^n$ componentwise. Therefore, the G-orbits on $(G/P)^n$ are products of $G^{(i)}$-orbits on $(G^{(i)}/P^{(i)})^n$. A G-orbit on $(G/P)^n$ is open if and only if it is a product of k open orbits on $(G^{(i)}/P^{(i)})^n$, and the number of G-orbits on $(G/P)^n$ is finite if for

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every i the number of $G^{(i)}$-orbits on $(G^{(i)}/P^{(i)})^n$ is finite. Hence in the sequel we may assume that G is simple. Recall that we may replace any simple group G with its simply connected cover and vice versa, hence we may replace it by any group locally isomorphic to G. So we may consider only one simple group of each type.

The classification of multiple flag varieties with an open G-orbit for maximal subgroups P was given by Popov in [8]. We need some notation to formulate his result. Fix a maximal torus in G and an associated simple root system $\{\alpha_1, \dots, \alpha_l\}$ of the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$. We enumerate simple roots as in [4]. Let $P_i \subset G$ be the maximal parabolic subgroup corresponding to the simple root α_i.

Theorem 1.1. [8, Theorem 3] Let G be a simple algebraic group. The diagonal G-action on the multiple flag variety $(G/P_i)^n$ is generically transitive if and only if $n \leq 2$ or (G, n, i) is an entry in Table 1:

Table 1. Generically transitive actions for maximal parabolic subgroups

Type of G	(n, i)
A_l	$n < \frac{(l+1)^2}{i(l+1-i)}$
$B_l, l \geq 3$	$n = 3, i = 1, l$
$C_l, l \geq 2$	$n = 3, i = 1, l$
$D_l, l \geq 4$	$n = 3, i = 1, l-1, l$
E_6	$n = 3, 4, i = 1, 6$
E_7	$n = 3, i = 7$

□

In [8], the following question was posed: for which non-maximal parabolic subgroups $P \subset G$ is there an open G-orbit in $(G/P)^n$? We solve this problem for all simple groups except for those of type A_l.

Denote the intersection $P_{i_1} \cap \ldots \cap P_{i_s}$ by $P_{i_1,\dots,i_s}$. It is easy to see that $P_{i_1,\dots,i_s}$ is a parabolic subgroup and that every parabolic subgroup is conjugated to some $P_{i_1,\dots,i_s}$.

Theorem 1.2. Let G be a simple algebraic group that is not locally isomorphic to SL_{l+1}, $P \subset G$ be a non-maximal parabolic subgroup and $n \geq 3$. Then the diagonal G-action on the multiple flag variety $(G/P_i)^n$ is generically transitive if and only if $n = 3$ and (G, P) is one of the pairs in Table 2:

Table 2. Generically transitive actions for non-maximal parabolic subgroups

Type of G	P
$D_l, l \geq 5$ is odd	$P_{1,l-1}, P_{1,l}$
$D_l, l \geq 4$ is even	$P_{1,l-1}, P_{1,l}, P_{l-1,l}$

□

The case A_l requires a separate investigation because even in the case of maximal parabolic subgroups, there are much more pairs (P, n) for which there is an open G-orbit in $(G/P)^n$, and the list of all non-maximal subgroups with this property may be much more complicated.

Now let us consider actions with a finite number of orbits. Recall that a G-variety X is called *spherical* if a Borel subgroup $B \subseteq G$ acts on X with an open orbit. It is well-known that the number of B-orbits on a spherical variety is finite, see [2, 10]. Equivalently, the number of G-orbits on $G/B \times X$ is finite if X is spherical. Therefore, if $P \subseteq G$ is a parabolic subgroup and X is a spherical G-variety, then the number of G-orbits on $G/P \times X$ is finite. The classification of all pairs of parabolic subgroups (P, Q) such that $G/P \times G/Q$ is spherical is given in [5, 9]. According to this classification, if (G, P_i) is an entry in Table 1, then $G/P_i \times G/P_i$ is spherical and hence the number of G-orbits on $G/P_i \times G/P_i \times G/P_i$ is finite. In the last section we prove that the number of G-obits on $(G/P)^n$ is infinite if $n \geq 4$. We also check directly that if (G, P) is an entry in Table 2, then the number of G-orbits on $G/P \times G/P \times G/P$ is infinite. Finally, from [6, Theorem 2.2] we see that if the flag variety $GL_{l+1}/P^{(1)} \times GL_{l+1}/P^{(2)} \times GL_{l+1}/P^{(3)}$, where $P^{(i)}$ are parabolic, has a finite number of GL_{l+1}-orbits, then at least one of these parabolic subgroups is maximal. This result can be applied to SL_{l+1} directly as well since the central torus of GL_{l+1} is a subgroup of all parabolic subgroups and acts trivially on the flag variety. Thus we come to the following result.

Theorem 1.3. Let G be a simple algebraic group, $P \subset G$ be a parabolic subgroup and $n \geq 3$. The following properties are equivalent.

1. The number of G-orbits on $(G/P)^n$ is finite.
2. $n = 3$, P is maximal, and there is an open G-orbit on $G/P \times G/P \times G/P$.
3. $n = 3$, and $G/P \times G/P$ is spherical.

□

Corollary 1.4. Let $n \geq 3$. The number of G-orbits on $(G/P)^n$ is finite if and only if $n = 3$ and (G, P) is one of the pairs listed in Table 3:

Table 3. Actions with finite numbers of orbits

Type of G	P
A_l	any maximal
$B_l, l \geq 3$	P_1, P_l
$C_l, l \geq 2$	P_1, P_l
$D_l, l \geq 4$	P_1, P_{l-1}, P_l
E_6	P_1, P_6
E_7	P_7

□

Let us mention a more general result for classical groups. Let $P^{(1)}, \ldots, P^{(n)}$ be parabolic subgroups in G. We call the variety $G/P^{(1)} \times \ldots \times G/P^{(n)}$ a *generalized multiple flag variety*. The classification of all generalized multiple flag varieties with a finite number of G-orbits is given in [6] for $G = SL_{l+1}$ and in [7] for $G = Sp_{2l}$.

Proofs of Theorems 1.2 and 1.3 use methods developed in [8]. In several cases for $G = SO_{2l}$ the existence of an open orbit is checked directly.

2 Preliminaries

Let G be a connected simple algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic zero and $\mathfrak{g} = \operatorname{Lie} G$. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. These data determine a root system Φ of $\mathfrak{g}$, a positive root subsystem Φ^+ and a system of simple roots $\Delta \subseteq \Phi^+$, $\Delta = \{\alpha_1, \dots, \alpha_l\}$. Choose corresponding Chevalley generators $\{x_i, y_i, h_i\}$ of $\mathfrak{g}$. We have $[h, x_i] = \alpha(h)x_i,\ [h, y_i] = -\alpha(h)y_i$ for all $h \in \mathfrak{t} = \operatorname{Lie} T$ and $h_i = [x_i, y_i]$.

Let $I = \{\alpha_{i_1}, \dots, \alpha_{i_s}\} \subseteq \Delta$ be a subset. The Lie algebra of the parabolic subgroup $P_I := P_{i_1,\dots,i_s}$ is

$$\mathfrak{p} = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha,$$

where $\mathfrak{b} = \operatorname{Lie} B$ and $\Phi_I \subseteq \Phi^-$ denotes the set of the negative roots such that their decomposition into the sum of simple roots *does not* contain the roots α_i, $i \in I$. For example, $P_\Delta = B$ and $P_\varnothing = G$. It is known [3, Theorem 30.1] that if a parabolic group P contains B, then $P = P_I$ for some $I \subseteq \Delta$. Therefore any parabolic subgroup $P \subseteq G$ is conjugate to some P_I. If $P = P_I$ for some $I \subseteq \Delta$, we denote by P^- the parabolic subgroup whose Lie algebra is

$$\mathfrak{p}^- = \mathfrak{t} \oplus \bigoplus_{\alpha \in -\Phi_I \cup \Phi^-} \mathfrak{g}_\alpha.$$

Denote the weight lattice of T by $\mathfrak{X}(T)$. Let $\mathfrak{X}^+(T)$ be the subsemigroup of dominant weights with respect to B. Assume first that G is simply connected. Then $\mathfrak{X}^+$ is generated by the fundamental weights $\pi_1, \dots, \pi_l$. Given a dominant weight λ, denote the simple G-module with the highest weight λ by $V(\lambda)$. If G is not simply connected, we may consider a simply connected cover $p\colon \widetilde{G} \to G$, the dominant weight semigroup $\mathfrak{X}^+(p^{-1}(T))$ and the highest weight $\widetilde{G}$-module $V(\lambda)$.

Let G be a simple group and $P = P_{i_1,\dots,i_s}$ be a parabolic subgroup. Notice that if there is an open G-orbit on $(G/P)^n$, then there exists an open G-orbit on $(G/P_i)^n$ for all $i \in \{i_1, \dots, i_s\}$. Indeed, since $P \subseteq P_i$, one has the surjective G-equivariant map $G/P \to G/P_i$, $gP \mapsto gP_i$. It induces the surjective G-equivariant map $\varphi\colon (G/P)^n \to (G/P_i)^n$, and the image of an open G-orbit on $(G/P)^n$ under φ is an open G-orbit on $(G/P_i)^n$. Similarly, if G acts on $(G/P)^n$ with an open orbit and $m < n$, then G acts on $(G/P)^m$ with an open orbit.

Theorem 1.1 leaves us very few cases of non-maximal parabolic groups to consider. Namely, if $n > 3$ and G is of type B_l, C_l or D_l, then G never acts on $(G/P)^n$ with an open orbit. If $n = 3$ and G is of type B_l or C_l, it suffices to consider $P = P_{1,l}$, and we show that there is no open orbit in this case. If $n = 3$ and G is of type D_l, an open orbit may exist only if $P = P_I$ where $I \subseteq \{\alpha_1, \alpha_{l-1}, \alpha_l\}$. So there are four cases to consider. We reduce the case $P_{1,l-1}$ to the case $P_{1,l}$. If G is of type E_6, the only parabolic group we should consider is $P = P_{1,6}$. We show that there is no open orbit for $n = 3$. For G of type E_7, if there existed an open G-orbit on $(G/P)^n$ for $n \geq 3$, then the only maximal parabolic subgroup containing P would be P_7, but in this case P would be maximal itself. If G is of type E_8, F_4 or G_2, an open orbit never exists for maximal parabolic subgroups whenever $n \geq 3$, so there are no cases to consider.

Given a group G acting on an irreducible variety X with an open orbit, according to [8] we denote the maximal n such that there is an open G-orbit on X^n by $\mathrm{gtd}(G : X)$. If G acts on X^n with an open orbit, we say that the action $G : X$ is *generically n-transitive.*

We make use of the following fact proved by Popov.

Proposition 2.1. [8, Corollary 1 (ii) of Proposition 2] Let G be a simple algebraic group, P be a parabolic subgroup, P^- be an opposite parabolic subgroup, $L = P \cap P^-$ be the corresponding Levi subgroup and $\mathfrak{u}^-$ be the Lie algebra of the unipotent radical of P^-. If P is conjugate to P^-, then $\mathrm{gtd}(G : G/P) = 2 + \mathrm{gtd}(L : \mathfrak{u}^-) = 2 + \mathrm{gtd}(L : \mathfrak{u})$. □

In particular cases we will see that L is isomorphic to a well-known reductive group (for example, to $\mathbb{K}^* \times GL_{l-1}$), and that $\mathfrak{u}$ (or $\mathfrak{u}^-$) is isomorphic to a direct sum of well-known L-modules. The exact list of these modules depends on how the isomorphism between L and the reductive group was chosen. For the semisimple part of L, in terms of Dynkin diagrams, it depends on the way we identify the Dynkin diagram obtained from the Dynkin diagram of G (with the enumeration of the vertices as in [4]) by removing the vertices corresponding to P and the Dynkin diagram of the semisimple part of the well-known reductive group. We prefer to identify them so that the order of vertices is the same in both cases. We still have to choose the isomorphism between the centers of L and of the reductive group (despite it is already fixed on the intersection of the center and the semisimple part, which is a finite group, we still can have several possibilities) and to choose whether to use $\mathfrak{u}$ and $\mathfrak{u}^-$. Among these possibilities, we prefer the one that leads to a simpler description of an L-module. For example, we prefer to have more tautological modules than the dual ones or more positive $\mathbb{K}^*$-weights.

We suppose that the group SO_{2l} acts in the $2l$-dimensional space and preserves the bilinear form whose matrix with respect to a standard basis is

$$Q = \begin{pmatrix} & & & & 1 \\ & 0 & & \iddots & \\ & & 1 & & \\ & & 1 & & \\ & \iddots & & 0 & \\ 1 & & & & \end{pmatrix}.$$

When we deal with a tautological GL_l-module V, we always assume that we have chosen a basis of V denoted by $e_1, \ldots, e_l$, unless stated otherwise. Similarly, we denote the corresponding basis of $V \wedge V$ by $\{e_i \wedge e_j\}$ $(1 \leq i < j \leq l)$, we denote the basis of V^* by $\{e_i^*\}$, etc. If this does not lead to an ambiguity, when we have several tautological GL_l-modules, we denote a basis of each of them by $e_1, \ldots, e_l$.

We denote the l-dimensional projective space by $\mathbf{P}^l$ and the Grassmannian of k-dimensional subspaces in $\mathbb{K}^l$ by $\mathrm{Gr}(k, l)$.

We denote the $l \times l$ identity matrix by id_l. If $A_1, \ldots, A_k$ are square matrices, we denote the block diagonal matrix with blocks $A_1, \ldots, A_k$ by $\mathrm{diag}(A_1, \ldots, A_k)$. If V is a vector space with a symmetric bilinear form, $V_1 \subseteq V$ and $V_2 \subseteq V$ are subspaces with prefixed bases and such that the bilinear form establishes a non-degenerate pairing between V_1 and V_2, and A is the matrix of a linear operator on V_1, then we denote the matrix of the adjoint operator on V_2 by A^*.

3 Existence of an Open Orbit

3.1 Groups of type B_l, $l \geq 3$

By Theorem 1.1, it is sufficient to consider the case $P = P_{1,l}$. The Dynkin diagram B_l has no automorphisms, hence P is conjugate to P^-. So we may apply Proposition 2.1, and it suffices to check that $\mathrm{gtd}(L : \mathfrak{u}) = 0$, i. e. L acts on $\mathfrak{u}$ with no open orbit.

Let $G = SO_{2l+1}$. Then $L = \mathbb{K}^* \times GL_{l-1}$ and the L-module $\mathfrak{u}$ can be decomposed into the direct sum $V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$. Here V_1 is a GL_{l-1}-module $(\mathbb{K}^{l-1})^*$ dual to the tautological one and its $\mathbb{K}^*$-weight is 1, V_2 is a trivial one-dimensional GL_{l-1}-module of weight 1, V_3 is a tautological GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight 0, V_4 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight 1, V_5 is a GL_{l-1}-module $\Lambda^2\mathbb{K}^{l-1}$ of weight 0. According to this decomposition, we denote the components of a vector $u \in \mathfrak{u}$ by u_1, u_2, u_3, u_4, u_5.

Notice that there exists a GL_{l-1}-invariant pairing between V_1 and V_4, whose $\mathbb{K}^*$-weight is 2. Therefore the rational function

$$\frac{(u_1, u_4)}{u_2^2}$$

is a non-constant invariant for $L : \mathfrak{u}$, and the action of G on G/P is not generically 3-transitive.

3.2 Groups of type C_l, $l \geq 2$

This case is completely similar to the previous one, and again the only thing we should do is to prove that there is no open L-orbit on $\mathfrak{u}$, where $L = P \cap P^-$ is a Levi subgroup of $P = P_{1,l}$ and $\mathfrak{u}$ is the Lie algebra of the unipotent radical of P.

Let $G = Sp_{2l}$. Then $L = \mathbb{K}^* \times GL_{l-1}$ and the L-module $\mathfrak{u}$ can be written as $V_1 \oplus V_2 \oplus V_3 \oplus V_4$. Here V_1 is a GL_{l-1}-module $(\mathbb{K}^{l-1})^*$ and its $\mathbb{K}^*$-weight is 1, V_2 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight 1, V_3 is a GL_{l-1}-module $S^2\mathbb{K}^{l-1}$ of weight 0, V_4 is a trivial GL_{l-1}-module of weight 2. According to this decomposition, we denote the components of a vector $u \in \mathfrak{u}$ by u_1, u_2, u_3, u_4.

We see that there exists a GL_{l-1}-invariant pairing between V_1 and V_2 with $\mathbb{K}^*$-weight is 2. Therefore we have a rational invariant

$$\frac{(u_1, u_2)}{u_4}$$

for $L : \mathfrak{u}$, and the action of G on G/P is not generically 3-transitive.

3.3 Groups of type D_l, $l \geq 4$

This time we should consider the following four cases of parabolic subgroups: $P = P_{1,l-1}, P_{1,l}, P_{l-1,l}, P_{1,l-1,l}$. One easily checks that P and P^- are conjugate except for the cases $P = P_{1,l}$, l odd, and $P = P_{1,l-1}$, l odd.

Let $G = SO_{2l}$. There exists a diagram automorphism of G that interchanges α_{l-1} and α_l. It preserves the maximal torus and the Borel subgroup and interchanges $P_{1,l-1}$ and $P_{1,l}$. Therefore, the actions $G : G/P_{1,l-1}$ and $G : G/P_{1,l}$ are either generically 3-transitive or not generically 3-transitive simultaneously.

In what follows we always suppose that B is the group of all upper-triangular matrices in G (according to the action of G in the tautological representation), and P_l is the group of all $g \in G$ that preserve the subspace $\langle e_1, \ldots, e_l \rangle$. This eliminates the ambiguity in choosing the numbering of the two last simple roots swapped by the automorphism of the diagram D_l.

3.3.1 $P = P_{l-1,l}$

In this case, P and P^- are conjugate, and we have to find $\mathrm{gtd}(L : \mathfrak{u})$.

The Levi subgroup L is isomorphic to $\mathbb{K}^* \times GL_{l-1}$ and the L-module $\mathfrak{u}$ is isomorphic to $V_1 \oplus V_2 \oplus V_3$, where V_1 is a GL_{l-1}-module $\Lambda^2 \mathbb{K}^{l-1}$ and its $\mathbb{K}^*$-weight is 0, V_2 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight 1, V_3 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight -1. We denote the components of a vector $u \in \mathfrak{u}$ by u_1, u_2, u_3.

Let l be even. We prove that there is an open L-orbit on $\mathfrak{u}$ as follows. We start with a point $(u_1, u_2, u_3) \in \mathfrak{u}$. During the proof at each step we impose an open L-invariant condition on (u_1, u_2, u_3) and prove that if the conditions are satisfied, the point (u_1, u_2, u_3) can be brought to a smaller subset of $\mathfrak{u}$. (The elements of this subset also satisfy the conditions since the conditions are L-invariant.) Finally this subset becomes one point p (that does not depend on (u_1, u_2, u_3)) and we notice that p satisfies all the open conditions we will have imposed. This guarantees that the conditions define a non-empty open L-invariant subset in $\mathfrak{u}$, i. e. an open L-orbit.

Since $l - 1$ is odd, the rank of a generic element $w \in V_1$ is $l - 2$. Consider the set of all $w \in V_1$ such that $\mathrm{rk}\, w = l - 2$. This is an L-invariant subset, so in the sequel we assume that $\mathrm{rk}\, u_1 = l - 2$. Then one can apply to $(u_1, u_2, u_3) \in \mathfrak{u}$ an element of GL_{l-1} that brings u_1 to the following form: $u_1' = e_1 \wedge e_2 + e_3 \wedge e_4 + \ldots + e_{l-3} \wedge e_{l-2}$. Denote the images of u_2 and u_3 under this action by u_2' and u_3'.

Any element $w \in V_1$ of rank $l - 2$ gives rise to (degenerate) skew-symmetric forms on V_2^* and on V_3^*, and their kernels are of dimension 1. Consider the subspace $X_w = (\mathrm{Ker}\, w)^\perp \subset V_2$ where all the functions from the kernel of this form vanish. Similarly denote $Y_w = (\mathrm{Ker}\, w)^\perp \subset V_2$. The conditions $u_2 \notin X_{u_1}$ and $u_3 \notin Y_{u_1}$ are open and L-invariant, and we assume in the sequel that they are satisfied. Then $u_2' \notin X_{u_1'}$, its last coordinate is non-zero, and there is a matrix of the form

$$\begin{pmatrix} & & & * \\ & \mathrm{id}_{l-2} & & \vdots \\ & & & * \\ 0 & \ldots & 0 & \lambda \end{pmatrix}, \quad \lambda \neq 0. \tag{$*$}$$

that moves u_2' to $u_2'' = e_{l-1}$. Notice that all such elements of GL_{l-1} preserve u_1'. Denote the image of u_3' under the action of this element by u_3''.

Now, the subgroup of L of the elements of the form

$$(\mathrm{diag}(A, \lambda), \lambda^{-1}), \quad A \in Sp_{l-2}, \lambda \neq 0$$

keeps u_1' and u_2'' unchanged. We have assumed that $u_3 \notin Y_{u_1}$, so now we have $u_3'' \notin Y_{u_1'}$, and the last coordinate of u_3'' cannot be zero. Then, by an appropriate choice of λ, u_3'' can be brought to an element $u_3''' \in V_3$ whose last coordinate is 1.

Now we impose the last L-invariant open condition on (u_1, u_2, u_3). Namely, we require that u_2 is not a multiple of u_3. Then u_3''' is not a multiple of u_2'', and at least one of the first $l-2$ coordinates of u_3''' is non-zero. Since Sp_{l-2} acts transitively on $\mathbb{K}^{l-2} \setminus 0$, it is possible to bring the vector formed by the first $l-2$ coordinates of u_3''' to e_1. Finally, we have brought u_1 to $e_1 \wedge e_2 + e_3 \wedge e_4 + \ldots + e_{l-3} \wedge e_{l-2}$, u_2 to e_{l-1}, u_3 to $e_1 + e_{l-1}$. These elements satisfy the open conditions we have introduced, hence they define an open L-orbit in $\mathfrak{u}$.

Let l be odd. Then a generic element $u_1 \in V_1$ gives rise to a non-degenerate skew-symmetric bilinear form on the GL_{l-1}-module $(\mathbb{K}^{l-1})^*$. Furthermore, one can consider the corresponding skew-symmetric form on the tautological GL_{l-1}-module. This form is obtained by matrix inversion and we denote it by u_1^{-1}. The following function is a rational L-invariant:

$$u_1^{-1}(u_2, u_3).$$

Thus the action of G on G/P is not generically 3-transitive.

3.3.2 $P = P_{1,l}$

In this case, Proposition 2.1 applies if and only if l is even.

Let l be even. It is sufficient to prove that there is an open L-orbit on $\mathfrak{u}$. The proof is organized as in the previous case for even l.

Again $L = \mathbb{K}^* \times GL_{l-1}$, and the L-module $\mathfrak{u}$ can be decomposed into three summands, $\mathfrak{u} = V_1 \oplus V_2 \oplus V_3$, but this time V_1 is a GL_{l-1}-module $\Lambda^2\mathbb{K}^{l-1}$ and its $\mathbb{K}^*$-weight is 0, V_2 is a GL_{l-1}-module $\mathbb{K}^{l-1}$ of weight 1, V_3 is a GL_{l-1}-module $(\mathbb{K}^{l-1})^*$ of weight 1. We denote the components of an element $u \in \mathfrak{u}$ by u_1, u_2, u_3.

The group L in this case is isomorphic to the one in the previous case, and the modules V_1 and V_2 are also the same as in the previous case. This enables us to keep the notation X_w that we have introduced for even l.

Again we may suppose that $\mathrm{rk}\, u_1 = l-2$, so it can be brought to $u_1' = e_1 \wedge e_2 + \ldots + e_{l-3} \wedge e_{l-2}$. We also suppose that $u_2 \notin X_{u_1}$, so it can be eventually moved to $u_2'' = e_{l-1}$ as in the previous case. We denote the image of u_3 under this action by u_3''. And again the subgroup of L of the elements of the form $(\mathrm{diag}(A, \lambda), \lambda^{-1})$ (where $A \in Sp_{l-2}, \lambda \neq 0$) preserves u_1' and u_2''.

This time there exists a GL_{l-1}-invariant pairing between V_2 and V_3 (denote it by $\langle\cdot,\cdot\rangle$), and we impose the condition $\langle u_2, u_3\rangle \neq 0$. It guarantees that the last coordinate of u_3'' is non-zero. We also impose the L-invariant condition $u_3 \notin \operatorname{Ker} u_1$, which guarantees that the first $l-2$ coordinates of u_3'' cannot vanish simultaneously. By the appropriate choice of λ the last coordinate of u_3'' can be brought to 1, and by the appropriate choice of A we can finally move it to $e_1^* + e_{l-1}^*$. So we have moved u_1 to $e_1 \wedge e_2 + e_3 \wedge e_4 + \ldots + e_{l-3} \wedge e_{l-2}$, u_2 to e_{l-1}, and u_3 to $e_1^* + e_{l-1}^*$. These elements satisfy the open conditions we have introduced, hence they define an open L-orbit in $\mathfrak{u}$.

Let l be odd. Proposition 2.1 does not apply, and we have to find $\operatorname{gtd}(G : G/P)$ directly. We use the following well-known fact about generically transitive actions:

Lemma 3.1. Let an algebraic group G act on irreducible algebraic varieties W and Y, and let $f\colon W \to Y$ be a G-equivariant map.

1. The following properties are equivalent:
   (a) The action of G on W is generically transitive and $f(W)$ is dense in Y.
   (b) The action of G on Y is generically transitive and if H is the G-stabilizer of a general point $y \in Y$, then there exists a dense H-orbit on $f^{-1}(y)$.
2. Assume that 1a and 1b hold. If $y \in Y$ and $w \in f^{-1}(y)$ are points such that the orbits $G \cdot y$ and $G_y \cdot w$ are dense in Y and $f^{-1}(y)$ respectively, then the orbit $G \cdot w$ is open in W.

□

By Theorem 1.1, it is sufficient to check that G acts on $G/P \times G/P \times G/P$ with an open orbit. First, G acts on G/P transitively, and it follows from Bruhat decomposition that P acts on G/P with finitely many orbits. The open orbit is the orbit of the point wP where w is a representative of the longest element of the Weyl group of G in $N_G(T)$. By Lemma 3.1 (2), the orbit $G \cdot (P, wP) \subseteq G/P \times G/P$ is open in $G/P \times G/P$. The stabilizer of the point (P, wP) equals $P \cap wPw^{-1}$. It is known that (in the case under consideration, i. e. $G = SO_{2l}$, l odd) $wP_1w^{-1} = P_1^-$ and $wP_lw^{-1} = P_{l-1}^-$, so $G_{(P,wP)} = P_{1,l} \cap P_{1,l-1}^-$. So, by Lemma 3.1 (1) it is sufficient to check that there is a dense $(P_{1,l} \cap P_{1,l-1}^-)$-orbit on G/P. Denote $H = P_{1,l} \cap P_{1,l-1}^-$.

Consider the tautological SO_{2l}-module $\mathbb{K}^{2l}$, and let $e_1, \ldots, e_{2l}$ be its standard basis. Then H is the group of all elements of G that preserve the following subspaces: $\langle e_1\rangle, \langle e_1, \ldots, e_{2l-1}\rangle$ (for P_1), $\langle e_1, \ldots, e_l\rangle$ (for P_l), $\langle e_{2l}\rangle, \langle e_2, \ldots, e_{2l}\rangle$ (for P_1^-), $\langle e_l, e_{l+2}, e_{l+3}, \ldots, e_{2l}\rangle$ (for P_{l-1}^-).

Let $X' \subset \operatorname{Gr}(l, 2l)$ be the set of all isotropic subspaces of dimension l in $\mathbb{K}^{2l}$. One easily checks that X' is a disjoint union of two SO_{2l}-orbits, and the group O_{2l} interchanges them. If two subspaces belong to the same SO_{2l}-orbit, then their intersection is non-zero. Denote the orbit $SO_{2l}\langle e_1, \ldots, e_l\rangle \subset X'$ by X. Then X is an irreducible subvariety in $\operatorname{Gr}(l, 2l)$. Denote the set of all isotropic lines in $\mathbf{P}^{2l-1}$ by Y. For each $s \in X'$ let $Y_s \subset \mathbf{P}^{2l-1}$ be the set of all lines contained in s. Clearly, $W = \cup_{s\in X}(s \times Y_s)$ is a closed G-invariant subset in $\operatorname{Gr}(l, 2l) \times \mathbf{P}^{2l-1}$. One easily checks that $G/P = W$.

We are going to use Lemma 3.1 (1) again for the map $\psi\colon W \to Y$ (the projection to the first coordinate).

First, let us find an open H-orbit $Z \subseteq Y$ and an element $b \in Z$. We define Z by introducing open conditions step by step as in previous cases. Let $a \in Y$. Choose a vector $f = \sum_{i=1}^{2l} f_i e_i$ $(f_i \in \mathbb{K})$ such that $\langle f \rangle = a$. Suppose that $a \not\subseteq \langle e_2, \dots, e_{2l-1}\rangle$. This subspace is H-invariant since it is the intersection of two H-invariant subspaces $\langle e_1, \dots, e_{2l-1}\rangle$ and $\langle e_2, \dots, e_{2l}\rangle$. Then $f_1 \neq 0$, $f_{2l} \neq 0$. There exists a diagonal matrix of the form $\mathrm{diag}(\lambda, \mathrm{id}_{2l-2}, \lambda^{-1}) \in H$ that maps f to a vector $f' = \sum f'_i e_i$ where $f'_1 = f'_{2l} \neq 0$. Without loss of generality, $f'_1 = f'_{2l} = 1$ (we may multiply f by a constant). Now suppose that $a \not\subseteq \langle e_1, \dots, e_l, e_{l+2}, e_{l+3}, \dots, e_{2l}\rangle = \langle e_1, \dots, e_l\rangle + \langle e_l, e_{l+2}, e_{l+3}, \dots, e_{2l}\rangle$. Then $f'_{l+1} \neq 0$. There exists a matrix A acting on $\langle e_2, \dots, e_{2l-1}\rangle$ (see Appendix A for an exact formula for A) such that $\mathrm{diag}(1, A, 1)$ is an element of H and maps f' to $f'' = e_1 + e_l - e_{l+1} + e_{2l}$. So we have moved every line $a \in Z$ to $b = \langle e_1 + e_l - e_{l+1} + e_{2l}\rangle \in Z$.

Now consider the action of H_b on $\psi^{-1}(b)$. In fact, the action of H_b° will be enough for our purposes. First, we need a simpler description for this action, so we use the following lemma. The proof of the lemma is technical and will be given in Appendix A. To formulate the lemma, notice first that the bilinear form on $\mathbb{K}^{2l}$ induces a non-degenerate bilinear form on $b^\perp/b$. H_b preserves $b^\perp$ (and b), so it acts on $b^\perp/b$.

Lemma 3.2. H_b is isomorphic to $GL_{l-2} \times \{1, -1\}$. There exists a basis $v_1, \dots v_{2l-2}$ of $b^\perp/b$ such that $\psi^{-1}(b)$ is (H_b-equivariantly) isomorphic to $SO(b^\perp/b)\langle v_1, \dots, v_{l-1}\rangle$ and that the matrix of the bilinear form in this basis is Q. In terms of the isomorphism between H_b° and GL_{l-2} and the basis of $b^\perp/b$ mentioned above, H_b° acts on $b^\perp/b$ by matrices of the form $\mathrm{diag}(1, (D^*)^{-1}, D, 1)$, $D \in GL(\langle v_l, \dots, v_{2l-3}\rangle)$. □

The group H_b° being written as the group of all matrices of the form $\mathrm{diag}(1, (D^*)^{-1}, D, 1)$ (where $D \in GL_{l-2}$) is a subgroup of $L' = P'_{l-1} \cap (P'_{l-1})^-$, where we denote by P'_{l-1} the parabolic subgroup of $SO(b^\perp/b)$ corresponding to the $(l-1)$-th simple root to distinguish it from parabolic subgroups of G. Notice that $SO(b^\perp/b)/P'_{l-1}$ is $SO(b^\perp/b)$-isomorphic to $SO(b^\perp/b)\langle v_1, \dots, v_{l-1}\rangle$. Denote the unipotent radical of $(P'_{l-1})^-$ by U'^-. Now we are going to use one more fact from [8] (see also [1]):

Proposition 3.3. [8, Proposition 2 (i)], see also [1, 14.21] Let P be a parabolic subgroup of a connected reductive group G. Let P^- be an opposite parabolic subgroup (i. e. $P \cap P^-$ is a Levi subgroup in P). Let U^- be the unipotent radical of P^-, $\mathfrak{u}^- = \mathrm{Lie}\, U^-$. Denote by $p \in G/P$ the image of P under the canonical map $G \to G/P$.

Then the orbit $U^- \cdot p$ is open in G/P, is L-stable and is L-isomorphic to $\mathfrak{u}^-$. □

In the case under consideration this means that $U'^- \langle v_1, \dots, v_{l-1}\rangle$ is open in $SO(b^\perp/b)/P'_{l-1}$ and is L'-isomorphic (and therefore H_b°-isomorphic) to $\mathfrak{u}'^- = \mathrm{Lie}\, U'^-$. It suffices to prove that H_b° acts on $\mathfrak{u}'^-$ with an open orbit. A direct calculation shows that $\mathfrak{u}'^-$ as $H_b^\circ = GL_{l-2}$ module (in terms of the isomorphism above) is isomorphic to $\Lambda^2\mathbb{K}^{l-2} \oplus \mathbb{K}^{l-2}$. Denote the components of an element $u \in \mathfrak{u}'^-$ by u_1 and u_2.

Now the situation is similar to the previous cases since $l-2$ is odd. Namely, every element of $\Lambda^2\mathbb{K}^{l-2}$ defines a skew-symmetric bilinear form on $(\mathbb{K}^{l-2})^*$, and the forms of rank $l-3$ form an open subset. If $\mathrm{rk}\, u_1 = l-3$,

(u_1, u_2) can be brought by the action of GL_{l-2} to (u_1', u_2'), where $u_1' = e_1 \wedge e_2 + \ldots + e_{l-4} \wedge e_{l-3}$. The pairs (u_1, u_2) where $\operatorname{rk} u_1 = l-3$ and the natural pairing between (any non-zero) element of $\ker u_1 \subset (\mathbb{K}^{l-2})^*$ and u_2 is non-zero form an open GL_{l-2}-invariant subset. If (u_1, u_2) is in this subset and u_1 is already moved to u_1', then u_2' can be moved to e_{l-2} by a matrix of the form $(*)$, which preserves u_1'.

Therefore, H_b° acts on $\mathfrak{u}'^-$ with an open orbit. By Proposition 3.3, H_b° also acts on (an irreducible variety) $SO(b^\perp/b)\langle v_1, \ldots, v_{l-1}\rangle$ with a dense orbit. By Lemma 3.1 (1), H acts on G/P with an open orbit, hence the action of G on $G/P \times G/P \times G/P$ is generically transitive.

3.3.3 $P = P_{1,l-1,l}$

The subgroups P and P^- are conjugate for all l. It is sufficient to find $\operatorname{gtd}(L : \mathfrak{u})$, where $L = (\mathbb{K}^*)^2 \times GL_{l-2}$ and the L-module $\mathfrak{u}$ is isomorphic to the direct sum of 7 simple modules, which we denote by $V_1, \ldots, V_7$. Namely, V_1 is a GL_{l-2}-module $(\mathbb{K}^{l-2})^*$ and its $(\mathbb{K}^*)^2$-weight is $(1,0)$, V_2 is a trivial GL_{l-2}-module of weight $(1,1)$, V_3 is a GL_{l-2}-module $\mathbb{K}^{l-2}$ of weight $(0,1)$, V_4 is a trivial GL_{l-2}-module of weight $(1,-1)$, V_5 is a GL_{l-2}-module $\mathbb{K}^{l-2}$ of weight $(0,-1)$, V_6 is a GL_{l-2}-module $\mathbb{K}^{l-2}$ of weight $(1,0)$, V_7 is a GL_{l-2}-module $\Lambda^2\mathbb{K}^{l-2}$ of weight $(0,0)$. Denote the components of $u \in \mathfrak{u}^-$ by $u_1, \ldots, u_7$.

There exists a GL_{l-2}-invariant pairing between V_1 and V_3 whose $(\mathbb{K}^*)^2$-weight is $(1,1)$. The following function is a rational L-invariant:

$$\frac{(u_1, u_3)}{u_2}.$$

Thus, the G-action on G/P is not generically 3-transitive.

3.4 **Groups of type E_6**

The only parabolic subgroup to consider is $P = P_{1,6}$. The set $\{1,6\}$ of Dynkin diagram vertices is invariant under all automorphisms of the Dynkin diagram. Hence the Weyl group element of the maximal length interchanges P and P^-. We have to find $\operatorname{gtd}(L : \mathfrak{u}^-)$.

The Levi subgroup L is locally isomorphic to $(\mathbb{K}^*)^2 \times Spin_8$, and the L-module $\mathfrak{u}^-$ is isomorphic to $V_1 \oplus V_2 \oplus V_3$. Here, as subspaces in $\mathfrak{g}$, V_1 (resp. V_2, V_3) is the direct sum of all $\mathfrak{g}_\alpha$ for which the decomposition of $\alpha \in \Phi^-$ into the sum of simple roots contains both α_1 and α_6 with coefficients -1 (resp. α_1 with coefficient -1, α_6 with coefficient 0 for V_2, α_1 with coefficient 0, α_6 with coefficient -1 for V_3). Consider the following embedding of Dynkin diagrams $D_4 \to E_6$: the vertex 1 (resp. 3, 4) is mapped to the vertex 2 (resp. 3, 5). Then, as L-modules, V_1 is a $Spin_8$-module with the lowest weight $-\pi_1$, i. e. it is a tautological SO_8-module ($Spin_8$ acts on V_1 with a two-element kernel, and the quotient group is SO_8), V_2 is a $Spin_8$-module with the lowest weight $-\pi_3$, V_3 is a $Spin_8$-module with the lowest weight $-\pi_4$. Denote the components of $u \in \mathfrak{u}^-$ by u_1, u_2, u_3.

Since V_1 is a tautological SO_8-module, there exists an SO_8-invariant (and hence $Spin_8$-invariant) quadratic form on it, which we denote by (u_1, u_1). There exist diagram automorphisms of $Spin_8$ that transform the tautological SO_8-module to $Spin_8$-modules isomorphic to V_2 and V_3. So there exist a quadratic $Spin_8$-invariant on V_2, which we denote by (u_2, u_2), and a quadratic $Spin_8$-invariant on V_3, which we denote by (u_3, u_3). From the description of V_1, V_2 and V_3 in terms of root subspaces one can deduce that the $(\mathbb{K}^*)^2$-weight of V_1 is the sum of the $(\mathbb{K}^*)^2$-weights of V_2 and V_3. Hence, the following function is a non-trivial rational L-invariant:

$$\frac{(u_2, u_2)(u_3, u_3)}{(u_1, u_1)},$$

and the G-action on $G/P \times G/P \times G/P$ is not generically transitive.

4 Finite Number of Orbits

Proposition 4.1. Let G be a simple algebraic group and P be a proper parabolic subgroup. If $n \geq 4$, the number of G-orbits on $(G/P)^n$ is infinite. □

Proof. Let $P = P_{i_1,\ldots,i_s}$. Consider the dominant weight $\lambda = \pi_{i_1} + \ldots + \pi_{i_s}$. Then G/P is isomorphic to the projectivization of the orbit of the highest weight vector $v_\lambda \in V(\lambda)$. In the sequel we shortly write $i = i_1$. It is easy to check that $y_i^2 v_\lambda = 0$. Denote the unipotent subgroup $\exp(ty_i)$ by U_i. We see that $U_i v_\lambda$ is an affine line not containing zero. The closure of its image in the projectivization $\mathbf{P}(V(\lambda))$ is a projective line $\mathbf{P}^1 \subseteq G/P \subseteq \mathbf{P}(V(\lambda))$. Choose $n \geq 4$ points $(x_1, \ldots, x_n) \in \mathbf{P}^1 \times \ldots \times \mathbf{P}^1 \subseteq G/P \times \ldots \times G/P$. The double ratio of the first four of these points does not change under G-action. Hence, two n-tuples with different double ratios cannot belong to the same orbit, and the number of orbits is infinite. ■

Now we prove that in the cases $G = SO_{2l}, P = P_{l-1,l}$ and $G = SO_{2l}, P = P_{1,l-1}$ the number of orbits on $G/P \times G/P \times G/P$ is infinite (again, the case $G = SO_{2l}, P = P_{1,l}$ can be reduced to the latter one).

Let $\mathbb{K}^{2l}$ be the tautological SO_{2l}-module and let $e_1, \ldots, e_{2l}$ be the standard basis. Let $X' \subset \mathrm{Gr}(l, 2l)$ be the set of all isotropic subspaces of dimension l in $\mathbb{K}^{2l}$. It is known that G/P_{l-1} and G/P_l are isomorphic to the two connected components of X' (respectively). In the sequel we suppose that $G/P_{l-1} = X \subseteq X'$. For each $s \in X$ let $Y_s \subset \mathbf{P}^{2l-1}$ be the set of all lines contained in s. One easily checks that the closed subset $Y = \cup_{s\in X}(s \times Y_s) \subset \mathrm{Gr}(l, 2l) \times \mathbf{P}^{2l-1}$ is isomorphic to $SO_{2l}/P_{1,l-1}$.

Similarly, if $s \in X$, denote by $Z_s \subset \mathrm{Gr}(l-1, 2l)$ the set of all subspaces of dimension $l-1$ in s. Let Z be the closed subset $\cup_{s\in X}(s \times Z_s) \subset \mathrm{Gr}(l, 2l) \times \mathrm{Gr}(l-1, 2l)$. One easily checks that it is isomorphic to $SO_{2l}/P_{l-1,l}$.

First, let $l = 3$. Consider the following isotropic subspaces: $S_1 = \langle e_1, e_2, e_4\rangle$, $S_2 = \langle e_2, e_3, e_6\rangle$, $S_3 = \langle e_1, e_3, e_5\rangle$. They belong to the same SO_6-orbit, and it follows from the choice of the group P_l in the beginning of Section 3.3 that $S_1, S_2, S_3 \in X$. Choose a line $T_1 \subset S_1$ such that $T_1 \subset \langle e_1, e_2\rangle$. Also choose lines $T_2 \subset S_2$ and $T_3 \subset S_3$ such that $T_2 \subset \langle e_2, e_3\rangle$ and $T_3 \subset \langle e_1, e_3\rangle$. Impose one more restriction, namely, the sum $T_2 + T_3$ should be direct and should not be equal to $\langle e_1, e_2\rangle$. Consider the point $((S_1, T_1), (S_2, T_2), (S_3, T_3)) \in Y \times Y \times Y$. There are four subspaces of $\langle e_1, e_2\rangle$: $\langle e_1\rangle = S_1 \cap S_3$, $\langle e_2\rangle = S_2 \cap S_3$, T_1 and $T_4 = (T_2 \oplus T_3) \cap \langle e_1, e_2\rangle$. Thus, we have defined four lines in $\mathbb{K}^6$ in terms of intersections and sums of S_i and T_i. If we apply an element $g \in G$ to these four lines, we will obtain four lines defined in the same way using gS_i and gT_i instead of S_i and T_i. The double ratio of these four lines in their sum of dimension two is not changed under G-action. Since T_1 is chosen arbitrarily, this double ratio can be any number and the number of orbits is infinite.

Consider the same subspaces S_i and T_i and set $U_1 = T_1 \oplus \langle e_4\rangle$, $U_2 = T_2 \oplus \langle e_6\rangle$ and $U_3 = T_3 \oplus \langle e_5\rangle$. The point $((S_1, U_1), (S_2, U_2), (S_3, U_3))$ belongs to $Z \times Z \times Z$. Note that $\langle e_1, e_2, e_3\rangle = (S_1 \cap S_2) \oplus (S_2 \cap S_3) \oplus (S_1 \cap S_3)$ and $T_i = U_i \cap \langle e_1, e_2, e_3\rangle$. Again we have a subspace of dimension two and four lines in it defined in terms of intersections and sums of S_i and U_i. The existence of SO_6-invariant double ratio in this case yields that the number of orbits is infinite.

Let $l > 3$. Construct the subspaces S_i, T_i and U_i as above, using the last three basis vectors instead of e_4, e_5, e_6. Let $S'_i = S_i \oplus \langle e_4, \ldots, e_l\rangle$ and $U'_i = U_i \oplus \langle e_4, \ldots, e_l\rangle$. The points $((S'_1, T_1), (S'_2, T_2), (S'_3, T_3))$ and $((S'_1, U'_1), (S'_2, U'_2), (S'_3, U'_3))$ belong to $Y \times Y \times Y$ and $Z \times Z \times Z$, respectively. Consider also the subspace $V = (S'_1 \cap S'_2 \cap S'_3)^{\perp}$. The restriction of the bilinear form to this subspace is degenerate, its kernel is $S'_1 \cap S'_2 \cap S'_3 = \langle e_4, \ldots, e_l\rangle$. The quotient is a space of dimension 6 with a bilinear form. The quotient morphism restricted to $\langle e_1, e_2, e_3, e_{2l-2}, e_{2l-1}, e_{2l}\rangle$ is an isomorphism, so we have subspaces S_i, T_i, U_i in the 6-dimensional space. This is exactly the same situation as we had above for the group SO_6, and it enables us to define double ratios for the points of $G/P_{1,l-1} \times G/P_{1,l-1} \times G/P_{1,l-1}$ and $G/P_{l-1,l} \times G/P_{l-1,l} \times G/P_{l-1,l}$ under consideration, which are preserved under the G-action. Therefore the number of SO_{2l}-orbits on these multiple flag varieties is infinite. This finishes the proof of Theorem 1.3.

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A Appendix: Technical details for the proof in Section 3.3.2 (odd case)

We use the notation introduced in Section 3.3.2.

A.1 **Matrix A**

A matrix with the desired properties can be written as follows:

$$A = \begin{pmatrix} & & & 0 & -f'_2/f'_{l+1} & & & \\ & \mathrm{id}_{l-2} & & \vdots & \vdots & & 0 & \\ & & & 0 & -f'_{l-1}/f'_{l+1} & & & \\ -f'_{2l-1} & \cdots & -f'_{l+2} & -f'_{l+1} & -f'_l - 1/f'_{l+1} & -f'_{l-1} & \cdots & -f'_2 \\ 0 & \cdots & 0 & 0 & -1/f'_{l+1} & 0 & \cdots & 0 \\ & & & 0 & -f'_{l+2}/f'_{l+1} & & & \\ & 0 & & \vdots & \vdots & & \mathrm{id}_{l-2} & \\ & & & 0 & -f'_{2l-1}/f'_{l+1} & & & \end{pmatrix}.$$

After a suitable permutation of basis vectors this matrix becomes upper-triangular, so $\det A = (-f'_{l+1})(-1/f'_{l+1}) = 1$. One can check directly that $\mathrm{diag}(1, A, 1) \in SO_{2l}$, the only non-trivial part of the computation is that $\mathrm{diag}(1, A, 1)e_{l+1}$ is an isotropic vector. Its square equals

$$2((-f'_2/f'_{l+1})(-f'_{2l-1}/f'_{l+1}) + \ldots + (-f'_{l-1}/f'_{l+1})(-f'_{l+2}/f'_{l+1})) + 2(-f'_l - 1/f'_{l+1})(-1/f'_{l+1}) =$$
$$2(f'_2 f'_{2l-1} + \ldots + f'_l f'_{l+1} + 1)/(f'^2_{l+1}) =$$
$$2(f'_2 f'_{2l-1} + \ldots + f'_l f'_{l+1} + f'_1 f'_{2l})/(f'^2_{l+1}) = 0$$

since f' is isotropic.

Now we have to check that $\mathrm{diag}(1, A, 1)f' = e_1 + e_l - e_{l+1} + e_{2l}$. Denote $\mathrm{diag}(1, A, 1)f' = f'' = \sum f''_i e_i$. It is clear directly from the description of A that all f''_i's except f''_l have the desired values. But then f''_l must equal 1 since this is the only possibility to make f'' isotropic.

A.2 **Proof of Lemma 3.2**

If $D \in GL(\langle e_{l+1}, \ldots, e_{2l-1}\rangle)$ and $\lambda = \pm 1$, then $\mathrm{diag}(\lambda, (D^*)^{-1}, \lambda, \lambda, D, \lambda) \in H_b$. Suppose that $g \in H_b$. Then g preserves the subspaces $\langle e_1\rangle$, $\langle e_{2l}\rangle$, $\langle e_l\rangle$ and b, so it preserves their sum $\langle e_1, e_l, e_{l+1}, e_{2l}\rangle$. Since $H\langle e_2, \ldots, e_{2l-1}\rangle = \langle e_2, \ldots, e_{2l-1}\rangle$, g preserves $\langle e_1, e_l, e_{l+1}, e_{2l}\rangle \cap \langle e_2, \ldots, e_{2l-1}\rangle = \langle e_l, e_{l+1}\rangle$. In particular, $ge_{l+1} = \lambda e_{l+1} + \mu e_l$ for some λ and μ, moreover, $\lambda \neq 0$, otherwise $gb \subset \langle e_1, e_l, e_{2l}\rangle$. But then $\mu = 0$, otherwise ge_{l+1} is not isotropic. So g preserves $\langle e_{l+1}\rangle$ and, since $gb = b$, g multiplies e_1, e_l, e_{l+1} and e_{2l} by the same constant that can equal ± 1 if $g \in SO_{2l}$. Finally, $g\langle e_1, e_l, e_{l+1}, e_{2l}\rangle = \langle e_1, e_l, e_{l+1}, e_{2l}\rangle$, so g preserves $\langle e_1, e_l, e_{l+1}, e_{2l}\rangle^\perp = \langle e_2, \ldots, e_{l-1}, e_{l+2}, \ldots, e_{2l-1}\rangle$ and the intersection of the latter subspace with H-invariant subspaces $\langle e_1, \ldots, e_l\rangle$ and $\langle e_l, e_{l+2}, \ldots, e_{2l}\rangle$, i. e. $\langle e_2, \ldots, e_{l-1}\rangle$ and $\langle e_{l+2}, \ldots, e_{2l-1}\rangle$, respectively. Therefore, $g = \mathrm{diag}(\lambda, (D^*)^{-1}, \lambda, \lambda, D, \lambda)$ for some $D \in GL(\langle e_{l+1}, \ldots, e_{2l-1}\rangle)$, $\lambda = \pm 1$, and $H_b = GL_{l-2} \times \{1, -1\}$.

The set $\psi^{-1}(b) \subseteq W$ is (H_b-equivariantly) isomorphic to the set $S \subseteq X$ consisting of all $s \in X$ that contain b. Consider the operator that maps e_1 to $e_1 - e_{l+1}$, e_l to $e_l + e_{2l}$ and does not move all other basis vectors. One checks directly that it is an element of SO_{2l}. It brings the subspace $\langle e_1, \dots, e_l \rangle$ to the subspace $M = \langle e_1 - e_{l+1}, e_2, \dots, e_{l-1}, e_l + e_{2l} \rangle$ containing b, so $M \in S$.

Now, consider the variety S' ($S \subseteq S' \subseteq X'$) consisting of all isotropic subspaces r' of dimension l containing b. Then $S = S' \cap X$. Consider the operator B that interchanges e_2 and e_{2l-1} and does not move all other basis vectors. $B \in O_{2l} \setminus SO_{2l}$, so B interchanges X and $X' \setminus X$. Notice that $Bb = b$, so $BS' = S'$, and B interchanges S and $S' \setminus S$. Since X is a connected component of X' and S' is a closed subset of X', S (resp. $S' \setminus S$) contains at least the connected component of M (resp. BM) in S'. Clearly, if $r' \in S'$, then $r' \subset b^{\perp}$. The canonical projection $\pi\colon b^{\perp} \to b^{\perp}/b$ establishes an isomorphism φ between S' and the set of $(l-1)$-dimensional isotropic subspaces of $b^{\perp}/b$. The latter variety consists of two connected components, which are $SO(b^{\perp}/b)$-orbits, and $\varphi(S)$ (resp. $\varphi(S' \setminus S)$) contains at least the connected component of the point $\pi(M)$ (resp. $\pi(BM)$). Therefore, S is connected and is exactly isomorphic to $SO(b^{\perp}/b)\pi(M)$.

Consider the following basis of $b^{\perp}/b$: $v_1 = \pi(e_1 - e_l - e_{l+1} - e_{2l})/2$, $v_2 = \pi(e_2)$, $\dots$, $v_{l-1} = \pi(e_{l-1})$, $v_l = \pi(e_{l+2})$, $\dots$, $v_{2l-3} = \pi(e_{2l-1})$, $v_{2l-2} = \pi(e_1 + e_l + e_{l+1} - e_{2l})/2$. One checks directly that the matrix of the bilinear form in this basis is Q, that $\pi(M) = \langle v_1, \dots, v_{l-1} \rangle$. Finally, it is clear from the description of H_b above and the definition of $\{v_i\}$ that H_b° acts by the matrices of the desired form. ■

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