On Factor Complexity of Morphic Sequences

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Abstract

The paper is devoted to an object well known in combinatorcs of words, namely to so-called morphic sequences. The main goal of the paper is to solve (at least partially) the following question raised by J.-J. Pansiot in 1985: what can the *factor complexity function* of an arbitrary morphic sequence be?

We study structure of pure morphic and morphic sequences and prove the following result: the factor complexity of an arbitrary morphic sequence is either $\Theta(n^{1+1/k})$ for some $k \in \mathbb{N}$, or is $O(n \log n)$.

1 Introduction

Combinatorics of words is an area of combinatorics, which studies various properties of finite and infinte words over a finite alphabet. In this setup, an *alphabet* is an arbitrary finite set, its elements are called *letters*. A *word* is just an arbitrary sequence (finite, or infinte to the right, or infinite to the right, or infinite to both sides) of letters. Infinite words are usually called *infinite sequences* rather than infinite words.

Morphisms and morphic sequences are well known and well studied in combinatorics on words (e. g., see [1]). Morphic sequences and so-called D0L-systems, which are more general objects built by a very similar procedure, appear in different areas of mathematics, for example, in dynamical systems [2, Chapter 5] and mathematical applications to biology [3].

Let Σ be a finite alphabet. A mapping $\varphi\colon \Sigma^* \to \Sigma^*$ is called a *morphism* if $\varphi(\gamma\delta) = \varphi(\gamma)\varphi(\delta)$ for all $\gamma, \delta \in \Sigma^*$. In other terms, this is an endomophism of the free semigroup generated by Σ. A morphism is determined by its values on single-letter words. A morphism is called *nonerasing* if $|\varphi(a)| \geq 1$ for each $a \in \Sigma$, and is called *coding* if $|\varphi(a)| = 1$ for each $a \in \Sigma$. Let $|\varphi|$ denote $\max_{a\in\Sigma} |\varphi(a)|$. Morphisms can be applied to infinite sequences as well, the image of an infinite sequence is naturally defined as the concatenation of the images of all letters.

Let $\varphi(a) = a\gamma$ for some $a \in \Sigma$, $\gamma \in \Sigma^*$, and suppose $\forall n\ \varphi^n(\gamma)$ is not empty. Then an infinite sequence $\varphi^\infty(a) = \lim_{n\to\infty} \varphi^n(a)$ is well-defined and is called *pure morphic*. Pure morphic sequences are fixed points of φ in the set of words infinite to the right. A natural generalization of the notion of a pure morphic sequence is the notion of a morphic sequence, namely, sequences of the form $\psi(\varphi^\infty(a))$ with coding ψ are called *morphic*. We will study properties of an arbitrary single morphic sequence, so we fix until the end of the paper the notation for an alphabet Σ, for a morphism $\varphi\colon \Sigma \to \Sigma$, for a letter $a \in \Sigma$ such that $\varphi(a)$ begins with a and that $|\varphi^n(a)|$ strictly grows for $n \to \infty$. We will also denote the resulting pure morphic sequence by $\alpha = \varphi^\infty(a)$. Finally, we fix a coding $\psi\colon \Sigma^* \to \Sigma^*$ and the resulting morphic sequence $\beta = \psi(\alpha)$. By Theorem 7.7.1 from [1] every morphic sequence can be generated by a nonerasing morphism, so further we assume that φ is nonerasing.

Morphic sequences can also be defined as infinte images of pure morphic sequences under arbitrary morphisms, and the union of classes of all morphic sequneces and finite words is the closure of the class of pure morphic sequence under the operation of applying a morphism. Morphic sequences arise in various areas of combinatorics of words, for example, they are source of soultions of so-called *pattern avoidability problems*, see [4, Chapter 3] for definitions and formulations. There are two kinds of pattern avoidability problems: the one where the cardinality of the alphabet is allowed to be arbitrarily large, and the one where it is fixed (or is required to be as small as possible). The first kind of pattern avoidability problem is completely solved (see, for example, [4, Section 3.2]), and a general source of solutions is the class of pure morphic sequences. The second kind of pattern avoidability problem is open in some cases, but sometimes a solution is provided by a morphic sequence (and here it is essential that the sequence is morphic, not just pure morphic, because the alphabet contains less letters), see [4, Section 3.3].

In this paper we study one of standard and natural complexity measures of sequences, namely the so-called factor complexity. The *factor complexity* of a sequence γ is a function $p_\gamma\colon \mathbb{N} \to \mathbb{N}$ where $p_\gamma(n)$ is the number of all different n-length factors occurring in β. The factor complexity functions of different words was probably first introduced in [5] (where their values were called *permutation indices*). There are many interesting facts known about factor complexity. For example, if an infinite sequence γ containing k different letters is not eventually periodic, its factor complexity satisfies $p_\gamma(n) \geq n + k - 1 \geq n + 1$, see, for example, [4, Theorem 1.3.13]. (Clearly, if an infinite sequnece is eventually periodic, its factor complexity is bounded.) Also, if the factor complexity p_γ of a sequence γ grows linearly, then all differences between its consequent values (i. e. of the form $p_\gamma(n+1) - p_\gamma(n)$) are bounded by a single constant, see [6]. Sturmian words, which are well known in combinatorics of words and have many interesting properties (see, for example, [4, Chapter 2]) and many equivalent definitions, can be defined as words in a two-letter alphabet with factor complexity $p(n) = n+1$. Factor complexity was also studied in [7], and in [8]. In particular, the paper [8] studies factor complexity functions of HD0L systems, which generalize factor complexity functions of morphic sequences. Some results about factors of sequences (including morphic sequences) or languages (including D0L-systems) were recently established in [9] and in [10]. For a survey on factor complexity, see, e. g., [11] or [12].

Pansiot showed [13] that the factor complexity of an arbitrary pure morphic sequence adopts one of the five following asymptotic behaviors: $O(1)$, $\Theta(n)$, $\Theta(n \log\log n)$, $\Theta(n \log n)$, or $\Theta(n^2)$. Since codings can only decrease factor complexity, the factor complexity of every morphic sequence is $O(n^2)$. In [14], it was shown that one can construct a morphic sequence with factor complexity asymptotically equal to $\Theta(n^{1+1/k})$, and the question about all possible asymptotic behaviors of the factor complexity of an arbitrary morphic sequence was raised. We formulate the following main result.

Theorem 1.1. *The factor complexity p_β of a morphic sequence β is either $p_\beta(n) = \Theta(n^{1+1/k})$ for some $k \in \mathbb{N}$, or $p_\beta(n) = O(n \log n)$.*

As we will see in Section 2.2, in some cases (which will be described explicitly) Theorem 1.1 follows directly from the results of the paper [13] (more precisely, in these cases the factor complexity is $O(n \log n)$). However, the proof in the present paper works independently of the results of [13].

We give an example of a morphic sequence β with $p_\beta = \Theta(n^{3/2})$ in Section 8.

To prove Theorem 1.1, we will have to develop some "structure theory" of pure morphic and morphic sequences. First, in Section 2 we will do some preparations to simplify the definitions and the proofs we give. Namely, we will introduce the notion of a letter of order k (Definition 2.6). Also we will replace φ by its power, and possibly add some more letters to Σ (these new letters will never occur in α, and α will stay unchanged), so that φ will be, as we will call it (see Definition 2.14), strongly 1-periodic morphism with long images. Then, in Sections 3–6, we will actually study the structure of pure morphic and morphic sequences, namely, we will introduce and study the notions of a letter of order k, of a k-block, of a k-multiblock, of a stable k-(multi)block, of an evolution, and of a continuously periodic evolution. Actually, these notions will be defined correctly only after we modify φ and Σ appropriately, so that φ becomes a strongly 1-periodic morphism with long images. These studies of the structure of pure morphic and morphic sequences may be of independent interest. More precisely, in Section 3 we will study the notion of a k-block, which is a finite occurrence in α consisting of letters of order at most k (see Definition 3.1). Then, in Section 4, we will study the structure of 1-blocks (which is simpler than the structure of k-blocks for a general k) in more details. Section 5 is devoted to stable k-blocks (Definition 5.1), which are, roughly speaking, k-blocks that are long enough to have a 'developed' structure, which we will be able to understand. And in Section 6 we define continiously periodic evolutions (Definition 6.19), which are sequences of k-blocks satisfying some very strond periodicity properties. The definition is based on the complicated structure of k-blocks studied in the previous sections, so it is also quite complicated. This notion will be out main tool for the proof Theorem 1.1. Then we establish criteria for continious periodicity of evolutions. Finally, in Section 7 we will prove Theorem 1.1 using the notions introduced before.

Using the notions mentioned in the previous paragraph, we can formulate the following two propositions, which the proof of Theorem 1.1 is based on. The proofs of these propositions will be given in Section 7.

Proposition 1.2. *Suppose that φ is a strongly 1-periodic morphism with long images. Let $k \in \mathbb{N}$. If $a \in \Sigma$ is a letter such that $\varphi(a) = a\gamma$ for some $\gamma \in \Sigma^*$, and there are evolutions of k-blocks arising in $\alpha = \varphi^\infty(a)$ that are not continuously periodic, then the factor complexity of $\beta = \psi(\alpha)$ is $\Omega(n^{1+1/(k-1)})$.*

Proposition 1.3. *Suppose that φ is a strongly 1-periodic morphism with long images. Let $k \in \mathbb{N}$. If $a \in \Sigma$ is a letter of order at least $k+2$ such that $\varphi(a) = a\gamma$ for some $\gamma \in \Sigma^*$, and all evolutions of k-blocks arising in $\alpha = \varphi^\infty(a)$ are continuously periodic, then the factor complexity of $\beta = \psi(\alpha)$ is $O(n^{1+1/k})$.*

However, these two propositions do not cover all cases needed to prove Theorem 1.1. This is not clear right now, before we give the definitions, but, for example, if a is a letter of order $k+2$, where $k \in \mathbb{N}$, such that $\varphi(a) = a\gamma$ for some $\gamma \in \Sigma^*$, and evolutions of $(k+1)$-blocks that are not continuously periodic do not exist (as we will see later, in this case evolutions of $(k+1)$-blocks do not exist at all), then Proposition 1.2 does not give us any upper estimate, and we cannot use Proposition 1.3 either, because if we want to use it for $(k+1)$-blocks, a has to be a letter of order at least $k+3$. Also, Propositions 1.2 and 1.3 do not say anything about complexity $O(n \log n)$. The following three propositions will help us to prove Theorem 1.1 in these cases. The proofs of these propositions will be also given in Section 7.

Proposition 1.4. *Let $k \in \mathbb{N}$. Suppose that φ is a strongly 1-periodic morphism with long images and $a \in \Sigma$ is a letter of order $k+2$ such that $\varphi(a) = a\gamma$ for some $\gamma \in \Sigma^*$. Suppose that all evolutions of k-blocks arising in $\alpha = \varphi^\infty(a)$ are continuously periodic.*

Let α_i be the rightmost letter of order $k+1$ in $\varphi^{3k+1}(a)$, and let α_j be the rightmost letter of order $k+1$ in $\varphi^{3k+2}(a)$.

If there exists a final period λ such that $\psi(\alpha_{i+1\dots j})$ is a completely $|\lambda|$-periodic word with period λ, then the factor complexity of $\beta = \psi(\alpha)$ is $O(1)$, otherwise it is $\Theta(n^{1+1/k})$.

Proposition 1.5. *Suppose that φ is a strongly 1-periodic morphism with long images. If $a \in \Sigma$ is a letter of order 2 such that $\varphi(a) = a\gamma$ for some $\gamma \in \Sigma^*$, then the factor complexity of $\beta = \psi(\varphi^\infty(a))$ is $O(1)$.*

Proposition 1.6. *Suppose that φ is a strongly 1-periodic morphism with long images. Let $k \in \mathbb{N}$. Let $a \in \Sigma$ be a letter of order ∞ such that $\varphi(a) = a\gamma$ for some $\gamma \in \Sigma^*$, and let $\alpha = \varphi^\infty(a)$. Suppose that if $b \in \Sigma$ is a letter of finite order k' and b occurs in α, then $k' < k$. Suppose that all evolutions of k-blocks arising in α are continuously periodic.*

Then the factor complexity of $\beta = \psi(\alpha)$ is $O(n \log n)$.

Main results of the present paper were previously announced in [15] (without proofs and with slightly different intermediate definitions).

2 Preliminaries

When we speak about finite words or about words infinite to the right, their letters are enumerated by nonnegative integer indices (starting from 0). The length of a finite word γ is denoted by $|\gamma|$.

We will speak about occurrences in $\alpha = \alpha_0\alpha_1\alpha_2\dots\alpha_i\dots$. Strictly speaking, we call a pair of a word γ and a location i in α *an occurrence* if the factor of α that starts from position i in α and is of length $|\gamma|$

is γ. This occurrence is denoted by $\alpha_{i\dots j}$ if j is the index of the last letter that belongs to the occurrence. In particular, $\alpha_{i\dots i}$ denotes a single-letter occurrence, and $\alpha_{i\dots i-1}$ denotes an occurrence of the empty word between the $(i-1)$-th and the i-th letters. Since $\alpha = \alpha_0\alpha_1\alpha_2\dots = \varphi(\alpha) = \varphi(\alpha_0)\varphi(\alpha_1)\varphi(\alpha_2)\dots$, φ might be considered either as a morphism on words (which we call abstract words sometimes), or as a mapping on the set of occurrences in α. Usually we speak of the latter, unless stated otherwise. Sometimes we write φ^0 for the identity morphism.

A finite word δ is called a *prefix* of a (finite or infinite to the right) word γ if $\gamma_{0\dots|\delta|-1} = \delta$. A finite word δ is called a *suffix* of a finite word γ if $\gamma_{|\gamma|-|\delta|\dots|\gamma|-1} = \delta$.

2.1 Periodicity of words

We call a finite word γ *weakly p-periodic* with a *left* (resp. *right*) period δ (where $p \in \mathbb{N}$) if $|\delta| = p$ and $\gamma = \delta\delta\dots\delta\delta_{0\dots r-1}$ (resp. $\gamma = \delta_{p-r\dots p-1}\delta\dots\delta$), where r is the remainder of $|\gamma|$ modulo p, $r = 0$ is allowed here. We shortly say "a weakly left (resp. right) δ-periodic word" instead of "a weakly $|\delta|$-periodic word with left (resp. right) period δ". δ will be always considered as an abstract word. The factor $\gamma_{|\gamma|-r\dots|\gamma|-1}$ (resp. $\gamma_{0\dots r-1}$) is called the *incomplete occurrence*. All the same is with sequences of symbols or numbers. If $r = 0$, then γ is called *completely p-periodic* with period δ (which is both left period and right period in this case, so we sometimes call it a *complete* period). Again, we shortly say "a completely δ-periodic word" instead of "a completely $|\delta|$-periodic word with period δ".

Clearly, a weakly p-periodic word with some left period always is also weakly p-periodic with some right period, and these periods are cyclic shifts of each other. So, we introduce some notation for cyclic shifts. If δ is a finite word and $0 \le r < |\delta|$, we denote the cyclic shift of δ that begins with the last $|\delta| - r$ letters of δ and ends with the first r letters of δ by $\mathrm{Cyc}_r(\delta)$. In other words, $\mathrm{Cyc}_r(\delta) = \delta_{r\dots|\delta|-1}\delta_{0\dots r-1}$. If $n \in \mathbb{Z}$ and r is the residue of n modulo $|\delta|$, we denote $\mathrm{Cyc}_n(\delta) = \mathrm{Cyc}_r(\delta)$. In particular, if $0 < n < |\delta|$, then $\mathrm{Cyc}_{-n}(\delta) = \mathrm{Cyc}_{|\delta|-n}(\delta) = \delta_{|\delta|-n\dots|\delta|-1}\delta_{0\dots|\delta|-n-1}$, in other words, $\mathrm{Cyc}_{-n}(\delta)$ is the cyclic shift of δ that begins with the last n letters of δ and ends with the first $|\delta| - n$ letters of δ.

We widely use the following easy properties of periods and cyclic shifts:

Remark 2.1. *1. If $n, m \in \mathbb{Z}$, then $\mathrm{Cyc}_{n+m}(\delta) = \mathrm{Cyc}_n(\mathrm{Cyc}_m(\delta))$.*

2. *If a finite word γ is weakly p-periodic with left period δ, where δ is a word of length p, then γ is also weakly p-periodic with right period $\delta' = \mathrm{Cyc}_{|\gamma|}(\delta)$.*

3. *If a finite word γ is weakly p-periodic with right period δ, where δ is a word of length p, then γ is also weakly p-periodic with left period $\delta' = \mathrm{Cyc}_{-|\gamma|}(\delta)$.*

4. *If δ is a word of length p, two finite words γ and γ' are weakly p-periodic, and γ (resp. γ') is weakly p-periodic with right (resp. left) period δ, then the concatenation $\gamma\gamma'$ is weakly p-periodic with left period $\delta' = \mathrm{Cyc}_{-|\gamma|}(\delta)$ and is also weakly p-periodic with right period $\delta'' = \mathrm{Cyc}_{|\gamma'|}(\delta)$.*

We will use the following well known Fine-Wilf theorem and some of its corollaries.

Theorem 2.2 (Fine, Wilf)**.** *Let γ be a finite word. Suppose that γ is weakly p_1-periodic with a left period δ and is weakly p_2-periodic with a left period σ at the same time. Suppose also that $|\gamma| \ge p_1 + p_2 - \gcd(p_1, p_2)$. Then there exists a finite word λ such that δ is λ repeated k times and σ is λ repeated l times for some $k, l \in \mathbb{N}$.*

Clearly, the same theorem for right periods instead of left ones is also true (for example, one can "read the words in the opposite direction"). After we have this theorem, it is reasonable to give the following definition. A finite word λ is called the *minimal left (resp. right) period* of a finite word γ if $2|\lambda| \le |\gamma|$, γ is weakly left (resp. right) λ-periodic and γ is not weakly p-periodic if $p < |\lambda|$. The following corollary provides more properties of the minimal periods if they exist.

Corollary 2.3. *Let γ be a finite word. If there exists $p \in \mathbb{N}$ such that γ is weakly p-periodic and $2p \le |\gamma|$, then there exist minimal left and right periods of γ.*

If λ is the minimal left (resp. right) period of γ, and γ is weakly p-periodic with left (resp. right) period δ, where $2p \le |\gamma|$, then p is divisible by $|\lambda|$ and δ is λ repeated $p/|\lambda|$ times. ☐

A similar statement in the case of complete p-periodicity follows directly since a word is completely p-periodic exactly if it is weakly p-periodic and its length is divisible by p. A finite word λ is called the *minimal complete period* of a finite word λ if $2|\lambda| \le |\gamma|$, γ is completely λ-periodic, and γ is not weakly p-periodic if $p < |\lambda|$.

Corollary 2.4. *Let γ be a finite word. If there exists $p \in \mathbb{N}$ such that γ is completely p-periodic and $2p \le |\gamma|$, then there exist a minimal complete period of γ.*

If λ is the complete period of γ, and γ is weakly p-periodic with left (resp. right) period δ, where $2p \le |\gamma|$, then p is divisible by $|\lambda|$ and δ is λ repeated $p/|\lambda|$ times. ☐

Corollary 2.5. *Let γ be a finite word, let $\gamma_{i\dots j}$ and $\gamma_{i'\dots j'}$ be two occurrences in γ. Suppose that $\gamma_{i\dots j}$ is weakly p-periodic, and $\gamma_{i'\dots j'}$ is weakly p'-periodic. Suppose also that these two occurrences overlap, and their intersection (denote it by $\gamma_{s\dots t}$) has length at least $2\max(p,p')$. In other words, $s = \max(i,i')$, $t = \min(j,j')$, and $t - s + 1 \ge 2\max(p,p')$.*

Then the union of these two occurrences (i. e. the occurrence $\gamma_{s'\dots t'}$, where $s' = \min(i,i')$ and $t' = \max(j,j')$) is a weakly $\gcd(p,p')$-periodic word.

Proof. Without loss of generality, $i \le i'$. Then $s = i'$ and $s' = i$. Let δ be the left period of $\gamma_{i\dots j}$ (so that $|\delta| = p$), and let δ' be the left period of $\gamma_{i'\dots j'}$ (so that $|\delta'| = p'$). Denote the residue of $i' - i$ modulo p by r. Then, if we write $\gamma_{i\dots j}$ as δ repeated several times, $\gamma_{i'}$ will be δ_r. Moreover, $\gamma_{s\dots t}$ becomes a weakly p-periodic word with left period $\delta'' = \delta_{r\dots|\delta|-1}\delta_{0\dots r-1} = \mathrm{Cyc}_r(\delta)$. Since $\gamma_{s\dots t}$ is a prefix of $\gamma_{i'\dots j'}$, $\gamma_{s\dots t}$ is also a weakly p'-periodic word with left period δ'. Now, by Theorem 2.2, there exists a word λ of length $\gcd(p,p')$ such that δ' is λ repeated $p'/\gcd(p,p')$ times and δ'' is λ repeated $p/\gcd(p,p')$ times. But then δ can also be written as a cyclic shift of λ repeated $p/\gcd(p,p')$ times.

Now, since $\gamma_{i\dots j}$ is weakly p-periodic with left period δ, it is also weakly $\gcd(p,p')$-periodic. Since $\gamma_{i'\dots j'}$ is weakly p'-periodic with left period δ', it is also weakly $\gcd(p,p')$-periodic. In other words, if k and $k + \gcd(p,p')$ are two indices such that $i \le k$ and $k + \gcd(p,p') \le j$, then $\gamma_k = \gamma_{k+\gcd(p,p')}$ as an abstract letter. Also, if k and $k + \gcd(p,p')$ are two indices such that $i' \le k$ and $k + \gcd(p,p') \le j'$, then again $\gamma_k = \gamma_{k+\gcd(p,p')}$ as an abstract letter.

If $j \ge j'$, we are done. Otherwise $t = j$ and $t' = j'$, and we have $j - i' + 1 \ge 2\max(p,p') \ge 2\gcd(p,p')$. So, if $k + \gcd(p,p') \le t' = j'$, but $k + \gcd(p,p') > j$, then $k + \gcd(p,p') \ge j + 1$, $k \ge j + 1 - \gcd(p,p') \ge i'$, and $\gamma_k = \gamma_{k+\gcd(p,p')}$ anyway. Hence, $\gamma_{s'\dots t'} = \gamma_{i\dots j'}$ is weakly $\gcd(p,p')$-periodic. ☐

An infinite word $\gamma = \gamma_0\gamma_1\dots\gamma_i\dots$ (where $\gamma_i \in \Sigma$) is called periodic with a period δ (where $\delta = \delta_0\dots\delta_{p-1}$, $\delta_i \in \Sigma$) if $\gamma = \delta\delta\delta\dots$, in other words, if $\gamma_{ip+j} = \delta_j$ for all $i = 0,1,2,\dots$, $j = 0,1,\dots,p-1$. An infinite word $\gamma = \gamma_0\gamma_1\dots\gamma_i\dots$ (where $\gamma_i \in \Sigma$) is called eventually periodic with a period δ (where $\delta = \delta_0\dots\delta_{p-1}$, $\delta_i \in \Sigma$) and a preperiod δ' (where $\delta' = \delta'_0\dots\delta'_{p'-1}$, $\delta_i \in \Sigma$) if $\gamma = \delta'\delta\delta\delta\dots$, in other words, if $\gamma_i = \delta'_i$ for $i = 0,1,\dots,p'-1$ and $\gamma_{p'+ip+j} = \delta_j$ for all $i = 0,1,2,\dots$, $j = 0,1,\dots,p-1$.

Sometimes we will also speak about words infinite to the left. We enumerate indices in such words by nonpositive indices, i. e. such a word can be written as $\gamma = \dots\gamma_{-i}\dots\gamma_{-1}\gamma_0$ (where $\gamma_{-i} \in \Sigma$, $i \in \mathbb{Z}_{\ge 0}$). Such a word is called periodic with a period δ (where $\delta = \delta_0\dots\delta_{p-1}$, $\delta_i \in \Sigma$) if $\gamma = \dots\delta\delta\delta$, in other words, if $\gamma_{-ip+j+1} = \delta_j$ for all $i = 1,2,\dots$, $j = 0,1,\dots,p-1$.

2.2 Orders and periodicity of letters

For each letter $b \in \Sigma$, the function $r_b\colon \mathbb{N} \to \mathbb{N}$, $r_b(n) = |\varphi^n(b)|$ is called *the growth rate* of b. Let us define *orders of letters* with respect to φ.

Definition 2.6. We say that $b \in \Sigma$ has *order* k if $r_b(n) = \Theta(n^{k-1})$, and has *order* ∞ if $r_b(n) = \Omega(q^n)$ for some $q > 1$ ($q \in \mathbb{R}$).

Consider a directed graph G defined as follows. Vertices of G are letters of Σ. For every $b, c \in \Sigma$, for each occurrence of c in $\varphi(b)$, construct an edge $b \to c$. For instance, if $\varphi(b) = bccbc$, we construct two edges $b \to b$ and three edges $b \to c$. Fig. 1 shows an example of graph G.

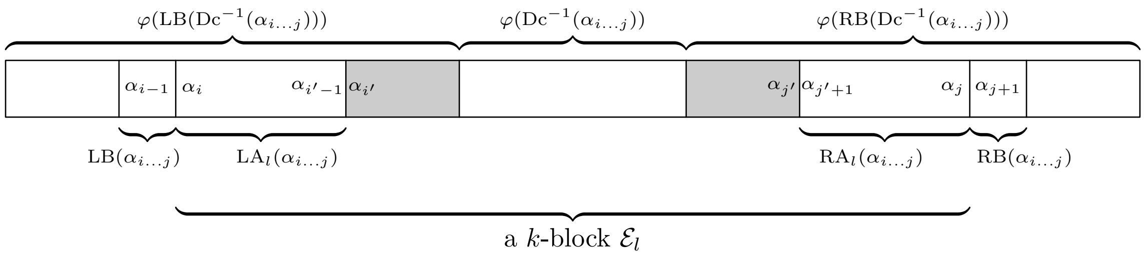

Figure 1: An example of graph G for the following morphism φ: $\varphi(a) = aab$, $\varphi(b) = c$, $\varphi(c) = cde$, $\varphi(d) = e$, $\varphi(e) = d$. Here a is a letter of order ∞, b is a preperiodic letter of order 2, c is a periodic letter of order 2, d and e are periodic letters of order 1.

Using the graph G, let us prove the following lemma.

Lemma 2.7. *For every $b \in \Sigma$, either b has some order $k < \infty$, or has order ∞. If b is a letter of order k, then $\varphi(b)$ contains at least one letter of order k. For every b of order $k < \infty$, either b never appears in $\varphi^n(b)$ (and then b is called* preperiodic*), or for each n a unique letter c_n of order k occurs in $\varphi^n(b)$, and the sequence $(c_n)_{n \in \mathbb{Z}_{\geq 0}}$ is periodic (then b is called* periodic*).*

If b is a letter of order ∞, then $\varphi(b)$ contains at least one letter of order ∞, and $\varphi^n(b)$ contains at least two letters of order ∞ if n is large enough.

If b is a periodic letter of order $k > 1$ and b occurs in $\varphi^n(b)$, then at least one letter of order $k-1$ occurs in $\varphi^n(b)$.

Proof. Consider also the following graph $\overline{G}$. Vertices of $\overline{G}$ are strongly connected components of G. There is an edge from $v \in \overline{G}$ to $u \in \overline{G}$ iff there is an edge from some of v vertices (in G) to some of u vertices. Fig. 2 shows an example of the corresponding graph $\overline{G}$.

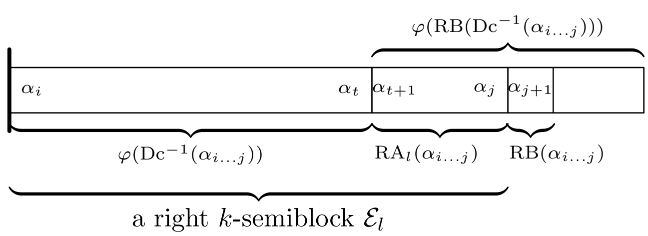

Figure 2: An example of the graph $\overline{G}$ corresponding to the graph G from the previous example.

Let $\overline{G}'$ be the subgraph of $\overline{G}$ induced by vertices $v \in \overline{G}$ such that for all vertices $a \in v \subset G$ there is at most one edge outgoing from b to a vertex $c \in v$. Let $\overline{G}''$ be the subgraph of $\overline{G}'$ induced by vertices $v \in \overline{G}'$ such that for all vertices $b \in v \subset G$ there are no edges outgoing from b to a vertex $c \in v$. In Fig. 2 and 1 the vertices of $\overline{G} \setminus \overline{G}'$ (resp. the corresponding vertices of G) are black, the vertices of $\overline{G}' \setminus \overline{G}''$ (resp. the corresponding vertices of G) are gray, and the vertices of $\overline{G}''$ (resp. the corresponding vertices of G) are white. We will now assign orders (natural numbers or infinity) to the vertices of $\overline{G}$ (hence, to the vertices of G too).

A vertex $v \in \overline{G}'$ is called a vertex of order one if it does not have outgoing edges (in $\overline{G}'$, not in G). Then assign order one to the vertices (if any) of graph $\overline{G}''$ that have outgoing edges to the vertices that are already of order one only. Repeat this operation until there are no new vertices of order one.

Suppose some vertices already are of order k (and we don't want to assign order k to any other vertex of $\overline{G}$). Then a vertex $v \in \overline{G}'$ is called vertex of order $k+1$ if all the edges outgoing from it lead to vertices of order k or less. Then, consider a vertex $w \in \overline{G}''$ that has not been currently assigned to be of some order. If all its outgoing edges lead to vertices of orders $\leq k+1$, assign w to be of order $k+1$. Repeat this operation until there are no new vertices of order $k+1$.

All vertices that currently have no order assigned (after completing the above procedure for each k), are called vertices of order ∞.

We have assigned orders to the vertices of $\overline{G}$, hence also to the vertices of G (that are the letters of Σ). It follows directly from the definition of the order of a vertex that if $b \in G$ is a vertex of order k, then there is an edge going from a to (possibly another) vertex of order k. One can prove by induction on k that

Any letter of finite order k has the rate of growth $\Theta(n^{k-1})$. Any letter of infinite order has the rate of growth $\Omega(\gamma^n)$ for some $\gamma > 1$.

Thus, two definitions of the order of a letter are equivalent.

Vertices v of $\overline{G}$ of order ∞ are exactly the vertices of $\overline{G}$ such that there exists a path from v to a vertex $w \in \overline{G} \setminus \overline{G}'$. It is already clear that if b is a letter of order ∞, then $\varphi(b)$ contains a letter of order ∞. To prove that if n is large enough, then $\varphi^n(b)$ contains at least two letters of order ∞, we may assume without loss of generality that b already belongs to a strongly connected component v of G such that $v \notin \overline{G}'$. Then there exists a vertex $c \in v$ such that there are at least two edges leading from c to vertices of G in $v \in \overline{G}$. This means that $\varphi(c)$ contains at least two letters of order ∞, and $\varphi^{n_0}(b)$ contains c for some n_0. Then $\varphi^n(b)$ contains at least two letters of order ∞ if $n > n_0$.

A vertex of $\overline{G}'$ of finite order is called *preperiodic* if it actually belongs to $\overline{G}''$, otherwise it is called *periodic*. A vertex of G (i. e. a letter) is called *periodic* (resp. *preperiodic*) iff the corresponding vertex of $\overline{G}$ is periodic (resp. preperiodic). If $b \in G$ is a periodic vertex of order $k < \infty$, it has exactly one outgoing edge to a vertex of order k. These two vertices correspond to the same vertex $v \in \overline{G}$, and all vertices of G that correspond to v (i. e. that belong to the strongly connected component v) actually form a directed loop. Unlike that, any edge that starts in a preperiodic vertex $b \in G$ of order k, leads to a vertex that had been assigned to be of some order $\leq k$ before a. Hence, this definition of a periodic letter and the definition from the lemma statement are equivalent.

To prove the last claim, observe that if $v \in \overline{G}'$ is a periodic letter of order k, then there must be an edge going from v to a vertex of order $k-1$, otherwise we would have assigned v to be a vertex of order $k-1$ or less. Therefore, there is a vertex $b \in G$ corresponding to v such such that there is an edge going from b to a vertex of G of order $k-1$. In other words, $\varphi(b)$ contains a letter of order $k-1$. Let c be (possibly another) vertex of G corresponding to $v \in \overline{G}'$. Then we already know that b and c are contained in a directed loop in G. If c occurs in $\varphi^n(c)$, then n is divisible by the length of this loop, hence n is greater than or equal to the length of this loop, and there exists m ($0 \leq m < n$) such that $\varphi^m(c)$ contains b. Then $\varphi(\varphi^m(c))$ contains a letter d of order $k-1$, and $\varphi^n(c)$ contains $\varphi^{n-m-1}(d)$. The image of a letter of order $k-1$ always contains a letter of order $k-1$, so a letter of order $k-1$ occurs in $\varphi^{n-m-1}(d)$ and hence in $\varphi^n(c)$. □

In the example of a graph G in Fig. 1, d and e are vertices of order one. We cannot assign any other vertex to be of order one, so we assign then c to be of order two. It is a periodic vertex. Then we can see that b has a single outgoing edge, and it leads to c. Thus, b should be a preperiodic vertex of order two. The remaining vertex a cannot be of finite order since it does not belong to $\overline{G}'$. It is a vertex of order ∞.

In general, it is possible that all letters in Σ have order ∞. In this case it follows from [13, Theorem 3.4] that the factor complexity of α (and therefore the factor complexity of β) is $O(n \log n)$. Therefore Theorem 1.1 is true if all letters have order ∞. However, it is possible to prove all results of the present paper (for example, Propositions 1.2–1.6) for words α and β obtained from such morphisms φ as well. But to make our defintions (in particular, the definitions used in the statements of these propositions) and our arguments simpler, it will be convenient for us to suppose that at least one periodic letter of order 1 and at least one periodic letter of order 2 exists.

So, first, if periodic letters of order 1 do not exist in Σ (then it follows from the construction above that all letters in Σ have order ∞), we add one more letter (that we temporarily denote by b) to Σ and set $\varphi(b) = b$, $\psi(b) = b$ (without varying φ and ψ on other letters). Then b is a periodic letter of order 1. From now on, we suppose that periodic letters of order 1 exist in Σ.

Second, suppose that periodic letters of order 1 exist in Σ, but periodic letters of order 2 do not exist (it follows from the above construction that in this case all letters in Σ have either order 1, or order ∞). Let $b \in \Sigma$ be a periodic letter of order 1. We add one more letter to Σ (denote it temporarily by c) and set $\varphi(c) = bc$, $\psi(c) = c$ (again, we do not change φ and ψ on other letters). Then c is a periodic letter of order 2. From now on, we suppose that periodic letters of order 2 exist in Σ.

2.3 Periodicity properties of morphisms

Now we are going to replace φ by φ^n for some $n \in \mathbb{N}$ to get a morphism satisfying better properties. Namely, first let us call a nonerasing morphism φ' *weakly 1-periodic* if:

1. If b is a preperiodic letter of order k, then all letters of order k in $\varphi'(b)$ are periodic.
2. If b is a periodic letter of order k, then the letter of order k contained in $\varphi'(b)$ is b.

We would like to choose n so that φ^n is a weakly 1-periodic morphism. Note first that the orders of letters with respect to φ^n are the same as their orders with respect to φ. Periodic and preperiodic letters with respect to φ remain periodic and preperiodic (respectively) with respect to φ^n. If the first letter in $\varphi(a)$ is a for some $a \in \Sigma$, then $\varphi^n(a)$ begins with a as well, and $(\varphi^n)^\infty(a) = \varphi^\infty(a)$.

Lemma 2.8. *There exists $n \in \mathbb{N}$ such that φ^n is a weakly 1-periodic morphism.*

Proof. If b is a preperiodic letter of order k, then $\varphi^n(b)$ does not contain b for any $n \geq 1$. Therefore, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then all letters of order k in $\varphi^n(b)$ are periodic. Take any $n \in \mathbb{N}$ such that $n \geq n_0$ for all these numbers n_0 for all preperiodic letters $b \in \Sigma$ of finite order. (Clearly, $n = |\Sigma|$ is sufficient, in the example above we can take $n = 1$.) Set $\varphi'' = \varphi^n$. If b is a preperiodic letter of order k, all letters of order k in $\varphi''(b)$ are periodic, and all letters of order k in $\varphi''^m(b)$ are also periodic. So, now it is sufficient to choose m so that if b is a periodic letter of order k, then the letter of order k occurring in $\varphi''^m(b)$ is b again. By the definition of a periodic letter, for each individual periodic letter b there exists $m_0 \in \mathbb{N}$ such that if m is divisible by m_0, then the letter of order k contained in $\varphi''^m(b)$ is b. Now let us take $m \in \mathbb{N}$ divisible by all numbers m_0 for all periodic letters b. (E. g., we can always take $m = |\Sigma|!$, and in the example above we can take $m = 2$.) Then $\varphi' = \varphi''^m$ is a weakly 1-periodic morphism. ☐

From now on, we replace φ by φ' from the proof and assume that φ is a weakly 1-periodic morphism.

Actually, we want to improve φ more. For each $k \in \mathbb{N}$ and for each letter $b \in \Sigma$ of order $> k$, the leftmost and rightmost letters of order $> k$ in $\varphi(b)$ will be important for us. If $k \in \mathbb{N}$ and γ is a finite word in Σ containing at least one letter of order $> k$, denote the leftmost (resp. rightmost) letter of order $> k$ in γ by $\mathrm{LL}_k(\gamma)$ (resp. by $\mathrm{RL}_k(\gamma)$). Observe that if $b \in \Sigma$, then $\mathrm{LL}_k(\varphi^n(b)) = \mathrm{LL}_k(\varphi(\mathrm{LL}_k(\varphi^{n-1}(b))))$ since if c is a letter of order k or less, then $\varphi(c)$ consists of letters of order k or less only. Hence, $b, \mathrm{LL}_k(\varphi(b)), \mathrm{LL}_k(\varphi^2(b)), \ldots, \mathrm{LL}_k(\varphi^n(b)), \ldots$ is an eventually periodic sequence. Similarly, $b, \mathrm{RL}_k(\varphi(b)), \mathrm{RL}_k(\varphi^2(b)), \ldots, \mathrm{RL}_k(\varphi^n(b)), \ldots$ is also an eventually periodic sequence. We want to make these sequence as simple as possible, so we call a morphism φ *strongly 1-periodic* if for each $k \in \mathbb{N}$ and for each letter $b \in \Sigma$ of order $> k$, one has $\mathrm{LL}_k(\varphi(b)) = \mathrm{LL}_k(\varphi^2(b)) = \ldots = \mathrm{LL}_k(\varphi^n(b)) = \mathrm{LL}_k(\varphi^{n+1}(b)) = \ldots$ and $\mathrm{RL}_k(\varphi(b)) = \mathrm{RL}_k(\varphi^2(b)) = \ldots = \mathrm{RL}_k(\varphi^n(b)) = \mathrm{RL}_k(\varphi^{n+1}(b)) = \ldots$, in other words, the sequences $b, \mathrm{LL}_k(\varphi(b)), \mathrm{LL}_k(\varphi^2(b)), \ldots, \mathrm{LL}_k(\varphi^n(b)), \ldots$ and $b, \mathrm{RL}_k(\varphi(b)), \mathrm{RL}_k(\varphi^2(b)), \ldots, \mathrm{RL}_k(\varphi^n(b)), \ldots$ are both eventually periodic with periods of length ***one*** and preperiods of length 1.

Observe that the definition of a weakly 1-periodic morphism guarantees that if b is a letter of ***finite*** order, then these sequences have periods of length 1, but we cannot say anything about the length of the preperiods. Also, we cannot say anything about the length of the period if all letters $b, \mathrm{LL}_k(\varphi(b)), \mathrm{LL}_k(\varphi^2(b)), \ldots, \mathrm{LL}_k(\varphi^n(b)), \ldots$ have order ∞.

Lemma 2.9. *There exists $n \in \mathbb{N}$ such that φ^n is a strongly 1-periodic morphism.*

Proof. The proof is similar to the proof of the previous lemma. Namely, if $n \in \mathbb{N}$ is large enough, then for $\varphi'' = \varphi^n$ sequences $b, \mathrm{LL}_k(\varphi''(b)), \mathrm{LL}_k(\varphi''^2(b)), \ldots, \mathrm{LL}_k(\varphi''^l(b)), \ldots$ and $b, \mathrm{RL}_k(\varphi''(b)), \mathrm{RL}_k(\varphi''^2(b)), \ldots, \mathrm{RL}_k(\varphi''^l(b)), \ldots$ are eventually periodic with preperiods of length 1 for all $k \in \mathbb{N}$ and for all letters $a \in \Sigma$ of order $> k$. Again, $n = |\Sigma|$ is sufficient for this purpose.

Now, if we take a large enough $m \in \mathbb{N}$ and set $\varphi' = \varphi''^m$, then the sequences $b, \mathrm{LL}_k(\varphi'(b)), \mathrm{LL}_k(\varphi'^2(b)), \ldots$ and $b, \mathrm{RL}_k(\varphi'(b)), \mathrm{RL}_k(\varphi'^2(b)), \ldots$ will become eventually periodic with periods of length 1 for all $k \in \mathbb{N}$ and for all letters $a \in \Sigma$ of order $> k$. This time, $m = |\Sigma|!$ is sufficient. Clearly, the preperiods of length 1 will remain the same. □

From now on, we replace φ by φ' from the proof and assume that φ is strongly 1-periodic.

Our final improvement of the morphism φ will guarantee that the image of each letter is "sufficiently long". Namely, first we are going to define the set of *final periods*. Let $b \in \Sigma$ be a letter such that $\mathrm{LL}_1(\varphi(b)) = b$. Then the prefix of $\varphi(b)$ to the left of the leftmost occurrence of b in $\varphi(b)$ consists of letters of order 1 only, denote it by γ. That is, if $\varphi(b)_i = b$ and $\varphi(b)_j \neq b$ for $0 \leq j < i$, then $\gamma = \varphi(b)_{0\ldots i-1}$. Suppose that γ is nonempty. Then $\varphi(\gamma)$ consists of periodic letters of order 1 only. Recall that to construct a morphic sequence, we use φ and also a coding ψ. Consider the word $\psi(\varphi(\gamma))\psi(\varphi(\gamma))$. Since we have repeated $\psi(\varphi(\gamma))$ twice, we can apply Corollary 2.4 and conclude that there exists the minimal complete period of $\psi(\varphi(\gamma))\psi(\varphi(\gamma))$, denote it by λ. We call λ, as well as all its cyclic shifts, *final periods*. Similarly, we can define a final period using a letter b such that $\mathrm{RL}_1(\varphi(b)) = b$ and considering the suffix of $\varphi(b)$ to the right of the rightmost occurrence of b. These are all words we call final periods, i. e. a *final period* is a word obtained from a letter $b \in \Sigma$ such that $\mathrm{LL}_1(\varphi(b)) = b$ and $\varphi(b)_0 \neq b$ by the procedure described above or a word obtained from a letter $b \in \Sigma$ such that $\mathrm{RL}_1(\varphi(b)) = b$ and $\varphi(b)$ does not end with b by a similar procedure.

Lemma 2.10. *If λ is a final period, then λ cannot be written as a finite word repeated more than once.*

Proof. Since λ is a final period, there exists a finite word λ' (which is also a final period) and a finite word γ such that λ' is the minimal complete period of $\psi(\varphi(\gamma))\psi(\varphi(\gamma))$ and $\lambda = \mathrm{Cyc}_r(\lambda')$ for some r $(0 \leq r < |\lambda'|)$.

Suppose that λ can be written as a finite word μ repeated more than once, in other words, that λ is a completely μ-periodic word and $|\mu| < |\lambda|$. But then $\lambda' = \mathrm{Cyc}_{-r}(\lambda)$ is a completely μ'-periodic word, where $\mu' = \mathrm{Cyc}_{-r}(\mu)$. Then the word $\psi(\varphi(\gamma))\psi(\varphi(\gamma))$, which is λ' repeated several times, is also a completely μ'-periodic word. But $|\mu'| = |\mu| < |\lambda| = |\lambda'|$, and this is a contradiction with the fact that λ' is the minimal complete period of $\psi(\varphi(\gamma))\psi(\varphi(\gamma))$. □

The following two corollaries follow directly from Lemma 2.10 and Theorem 2.2, and we will use them several times exactly in this form.

Corollary 2.11. *Let λ be a final period, and Let γ be a finite weakly left (resp. right) λ-periodic word of length at least $2|\lambda|$. Then λ is the minimal left (resp. right) preiod of γ.* □

Corollary 2.12. *Let γ be a finite word of length at least $2\mathbf{L}$. Suppose that γ is both weakly left (resp. right) λ-periodic and weakly left (resp. right) λ'-periodic for some final periods λ and λ'. Then $\lambda = \lambda'$.* □

Note that final periods always exist if φ is a strongly 1-periodic morphism and there is a periodic letter of order 2 in Σ (we have already assumed that this is true). Indeed, if b is a periodic letter of order 2, then $\varphi(b)$ contains exactly one occurrence of order 2, which is b, and at least one letter of order 1. In other words,

$\varphi(b)$ can be written as $\gamma b\gamma'$, where the words γ and γ' consist of letters of order 1 only, and at least one of these words is nonempty. We can use this nonempty word to construct a final period.

Clearly, the amount of final periods is finite and their lengths are bounded. Denote the maximal length of a final period by $\mathbf{L}$.

Lemma 2.13. *Let $k \in \mathbb{N}$. Then the sets of final periods for φ and for $\varphi' = \varphi^k$ are the same.*

If γb (resp. $b\gamma$) is a prefix (resp. suffix) of $\varphi(b)$, where $b \in \Sigma$ and γ is a finite word consisting of letters of order 1 only, then $\varphi(\gamma)\dots\varphi(\gamma)\gamma b$ (resp. $b\gamma\varphi(\gamma)\dots\varphi(\gamma)$), where $\varphi(\gamma)$ is repeated $k-1$ times, is a prefix (resp. suffix) of $\varphi^k(b)$.

Proof. Choose a letter $b \in \Sigma$ such that $\mathrm{LL}_1(\varphi(b)) = b$. (The case $\mathrm{RL}_1(\varphi(b)) = b$ is completely symmetric.) Suppose that $\varphi(b)_0 \neq b$ and denote the prefix of $\varphi(b)$ to the left of the leftmost occurrence of b by γ. Then γb is a prefix of $\varphi(b)$ and γ consists of letters of order 1 only. Let us prove by induction on $k \in \mathbb{N}$ that $\varphi(\gamma)\dots\varphi(\gamma)\gamma b$, where $\varphi(\gamma)$ is repeated $k-1$ times, is a prefix of $\varphi^k(b)$. For $k = 1$ we already know this. If $\varphi(\gamma)\dots\varphi(\gamma)\gamma b$, where $\varphi(\gamma)$ is repeated $k-1$ times, is a prefix of $\varphi^k(b)$, then $\varphi^2(\gamma)\dots\varphi^2(\gamma)\varphi(\gamma)\varphi(b)$, where $\varphi^2(\gamma)$ is repeated $k-1$ times, is a prefix of $\varphi^{k+1}(b)$. Recall that γb is a prefix of $\varphi(b)$, so $\varphi^2(\gamma)\dots\varphi^2(\gamma)\varphi(\gamma)\gamma b$, where $\varphi^2(\gamma)$ is repeated $k-1$ times, is also a prefix of $\varphi^{k+1}(b)$. Finally, recall that φ is (in particular) weakly 1-periodic, so $\varphi(\gamma)$ consists of periodic letters of order 1 only, and $\varphi^2(\gamma) = \varphi(\gamma)$. Therefore, $\varphi(\gamma)\dots\varphi(\gamma)\gamma b$, where $\varphi(\gamma)$ is repeated k times, is a prefix of $\varphi^{k+1}(b)$.

So, the prefix of $\varphi'(b) = \varphi^k(b)$ to the left of the leftmost occurrence of b is $\gamma' = \varphi(\gamma)\dots\varphi(\gamma)\gamma$, where $\varphi(\gamma)$ is repeated $k-1$ times. If we apply φ' to this prefix, we will get $\varphi(\gamma)$ repeated k times (here we again use the fact that $\varphi^2(\gamma) = \varphi(\gamma)$). Finally, $\psi(\varphi'(\gamma'))\psi(\varphi'(\gamma'))$ is $\psi(\varphi(\gamma))$ repeated $2k$ times, and, by Corollary 2.4, $\psi(\varphi'(\gamma'))\psi(\varphi'(\gamma'))$ has a minimal complete period, and it coincides with the minimal complete period of $\psi(\varphi(\gamma))\psi(\varphi(\gamma))$. □

After we have this lemma, it is reasonable to give the following definition:

Definition 2.14. A strongly 1-periodic morphism is called *a strongly 1-periodic morphism with long images* if the following holds:

1. For each letter $b \in \Sigma$ such that $\mathrm{LL}_1(\varphi(b)) = b$ and the prefix γ of $\varphi(b)$ to the left of the leftmost occurrence of b is nonempty, we have $|\gamma| \geq 2\mathbf{L}$.

2. For each letter $b \in \Sigma$ such that $\mathrm{RL}_1(\varphi(b)) = b$ and the suffix γ of $\varphi(b)$ to the right of the rightmost occurrence of b is nonempty, we have $|\gamma| \geq 2\mathbf{L}$.

Lemma 2.15. *Let φ be a strongly 1-periodic morphism with long images. Then for each letter $b \in \Sigma$ such that $\mathrm{LL}_1(\varphi(b)) = b$ (resp. $\mathrm{RL}_1(\varphi(b)) = b$) and the prefix (resp. suffix) γ of $\varphi(b)$ to the left (resp. to the right) of the leftmost (resp. rightmost) occurrence of b is nonempty, there exists a minimal complete period of $\psi(\varphi(\gamma))$, and it is a final period.*

Proof. The claim for $\psi(\varphi(\gamma))\psi(\varphi(\gamma))$ instead of $\psi(\varphi(\gamma))$ is just the definition of a final period. Let λ be the minimal complete period of $\psi(\varphi(\gamma))\psi(\varphi(\gamma))$. By Corollary 2.4, $\psi(\varphi(\gamma))$ is λ repeated $|\psi(\varphi(\gamma))|/|\lambda|$ times, i. e. $\psi(\varphi(\gamma))$ is completely λ-periodic. Since $|\gamma| \geq 2\mathbf{L}$, we also have $|\psi(\varphi(\gamma))| \geq 2\mathbf{L} \geq 2|\lambda|$, and by Corollary 2.4 again, there exists a minimal complete period λ' of $\psi(\varphi(\gamma))$ and λ is λ' repeated several times. But then $\psi(\varphi(\gamma))\psi(\varphi(\gamma))$ is also completely λ'-periodic, but λ was the minimal complete period of $\psi(\varphi(\gamma))\psi(\varphi(\gamma))$, so $\lambda' = \lambda$. □

Again, let us prove that we can make a strongly 1-periodic morphism with long images out of φ by replacing it with φ^n.

Lemma 2.16. *There exists $n \in \mathbb{N}$ such that φ^n is a strongly 1-periodic morphism with long images.*

Proof. Observe first that if φ is strongly 1-periodic, then φ^n is also strongly 1-periodic.

Choose a letter $b \in \Sigma$ such that $\mathrm{LL}_1(\varphi(b)) = b$. (The case $\mathrm{RL}_1(\varphi(b)) = b$ is completely symmetric.) Suppose that $\varphi(b)_0 \neq b$. Then, by the second statement of Lemma 2.13, if n is large enough, then the length of the prefix of $\varphi^n(b)$ to the left of the leftmost occurrence of b is at least $2\mathbf{L}$ (here we also use the fact that φ is nonerasing, so in the statement of Lemma 2.13 we have $|\varphi(\gamma)| \geq |\gamma|$).

Let n_0 be the maximum of all these numbers n for all letters $a \in \Sigma$ and for the left and the right side. ($n_0 = 2\mathbf{L}$ is sufficient for this purpose, but a smaller n_0 can also work.) Then, by Lemma 2.13, $\varphi' = \varphi^n$ is a strongly 1-periodic morphism with long images. □

From now on, we replace φ by φ' from the proof and assume that φ is a strongly 1-periodic morphism with long images.

3 Blocks

Definition 3.1. A (possibly empty) finite occurrence $\alpha_{i\ldots j}$ is *a k-block* if it consists of letters of order $\leq k$, $i > 0$, and the letters α_{i-1} and α_{j+1} both have order $> k$.

The occurrence of a single letter α_{i-1} is called the *left border* of this block and is denoted by $\mathrm{LB}(\alpha_{i\ldots j})$. The occurrence of a single letter α_{j+1} is called the *right border* of this block and is denoted by $\mathrm{RB}(\alpha_{i\ldots j})$. Observe that if we have constructed the whole morphic sequence starting with a letter $a \in \Sigma$ (i. e. $\alpha = \varphi^\infty(a)$), and a is a letter of a finite order k, then all letters in α are of order $\leq k$. So it makes no sense to define "k-blocks" of the form $\alpha_{0\ldots j}$ since it is not possible that all letters $\alpha_0 = a, \alpha_1, \ldots, \alpha_j$ have orders $\leq k$, and α_{j+1} has order $> k$.

Note that even if there are no letters of order k in Σ, k-blocks still may exist, then all letters in k-blocks will be of order $< k$ (or a k-block can also be empty), and the borders of such a k-block will be letters of order $> k$ (in fact, as one can deduce from the assignment of orders to letters in the previous section, these letters must have order ∞). A problem that can arise is that letters of order $< k$ (or $\leq k$) may form an infinite sequence, then they do not form a k-block by definition. Later we will see that this can really happen if all letters in α have finite orders, we will discuss this in Lemma 3.5.

The image under φ of a letter of order $\leq k$ cannot contain letters of order $> k$. Let $\alpha_{i\ldots j}$ be a k-block. Then $\varphi(\alpha_{i\ldots j})$ is a suboccurrence of some k-block which is called the *descendant* of $\alpha_{i\ldots j}$ and is denoted by $\mathrm{Dc}_k(\alpha_{i\ldots j})$. (We use the subscript k here to underline that the same occurrence $\alpha_{i\ldots j}$ can be a k-block and an m-block for some $m \neq k$ at the same time, for example, if $\mathrm{LB}(\alpha_{i\ldots j})$ and $\mathrm{RB}(\alpha_{i\ldots j})$ are both of order $> k+1$, then $\alpha_{i\ldots j}$ is a $(k+1)$-block as well. In this case, $\mathrm{Dc}_k(\alpha_{i\ldots j})$ and $\mathrm{Dc}_{k+1}(\alpha_{i\ldots j})$ could be different occurrences in α.) The l-th superdescendant (denoted by $\mathrm{Dc}_k^l(\alpha_{i\ldots j})$) is the descendant of ... of the descendant of $\alpha_{i\ldots j}$ (l times).

Let $\alpha_{s\ldots t}$ be a k-block in α. Then if there exists a k-block $\alpha_{i\ldots j}$ such that $\mathrm{Dc}_k(\alpha_{i\ldots j}) = \alpha_{s\ldots t}$, it is unique. Indeed, otherwise there would be a letter of order $> k$ between those two k-blocks, and its image would contain a letter of order $> k$ again. But this letter would belong to $\alpha_{s\ldots t}$. If the k-block $\alpha_{i\ldots j}$ exists, is called the *ancestor* of $\alpha_{s\ldots t}$ and is denoted $\mathrm{Dc}_k^{-1}(\alpha_{s\ldots t})$. The l-th superancestor (denoted by $\mathrm{Dc}_k^{-l}(\alpha_{s\ldots t})$) is the ancestor of ... of the ancestor of $\alpha_{s\ldots t}$ (l times). If $\mathrm{Dc}_k^{-1}(\alpha_{s\ldots t})$ does not exist (this can happen only if α_{s-1} and α_{t+1} belong to the image of the same letter), then $\alpha_{s\ldots t}$ is called an *origin*.

Definition 3.2. A sequence $\mathscr{E}$ of k-blocks, $\mathscr{E}_0 = \alpha_{i\ldots j}, \mathscr{E}_1 = \mathrm{Dc}_k(\alpha_{i\ldots j})$, $\mathscr{E}_2 = \mathrm{Dc}^2(\alpha_{i\ldots j}), \ldots, \mathscr{E}_l = \mathrm{Dc}_k^l(\alpha_{i\ldots j}), \ldots$, where $\alpha_{i\ldots j}$ is an origin, is called an *evolution*.

The number l is called the *evolutional sequence number* of a k-block $\mathscr{E}_l$.

Let $\mathscr{E}$ be an evolution of k-blocks. The letter $\mathrm{LB}(\mathscr{E}_{l+1})$ is the rightmost letter of order $> k$ in $\varphi(\mathrm{LB}(\mathscr{E}_l))$, i. e. $\mathrm{LB}(\mathscr{E}_{l+1}) = \mathrm{RL}_k(\varphi(\mathrm{LB}(\mathscr{E}_l)))$. Since φ is a strongly 1-periodic morphism, this means that $\mathrm{LB}(\mathscr{E}_l)$ does not depend on l if $l \geq 1$. Similarly, $\mathrm{RB}(\mathscr{E}_l)$ does not depend on l if $l \geq 1$. We call the abstract letter $\mathrm{LB}(\mathscr{E}_l)$ (resp. $\mathrm{RB}(\mathscr{E}_l)$) for any $l \geq 1$ the *left (resp right.) border of $\mathscr{E}$* and denote it by $\mathrm{LB}(\mathscr{E})$ (resp. by $\mathrm{RB}(\mathscr{E})$).

Lemma 3.3. *The set of all abstract words that can be origins in α, is finite.*

Proof. Each origin is a factor of $\varphi(b)$ where b is a single letter. Moreover, this occurrence inside $\varphi(b)$ cannot be a prefix or a suffix. □

Corollary 3.4. *The set of all possible evolutions in α (considered as sequences of abstract words rather than sequences of occurrences in α), is finite.*

Proof. Let $\mathscr{E}_0$ be an origin, which is a suboccurrence of $\varphi(\alpha_i)$. Here α_i is a letter of order $> k$. Then $\varphi(\alpha_i)$ also contains $\mathrm{LB}(\mathscr{E}_0)$ and $\mathrm{RB}(\mathscr{E}_0)$. $\mathrm{LB}(\mathscr{E}_{l+1})$, $\mathrm{RB}(\mathscr{E}_{l+1})$ and $\mathscr{E}_{l+1}$ itself depend on *abstract words* $\mathrm{LB}(\mathscr{E}_l)$, $\mathrm{RB}(\mathscr{E}_l)$ and $\mathscr{E}_l$ only. Thus, all these words became known after we had selected an abstract letter $b = \alpha_i$ and a suboccurrence inside $\varphi(b)$. □

Lemma 3.5. *Let $\alpha = \varphi^\infty(a)$, where $a \in \Sigma$. Then:*

1. *If a is a letter of a finite order K, then a is the only letter of order $\geq K$ in α, and it only occurs once, as α_0. For each $k < K - 1$, $k \in \mathbb{N}$, α splits into a concatenation of k-blocks and letters of order $> k$.*

2. *If a is a letter of order ∞, then for each $k \in \mathbb{N}$, α splits into a concatenation of k-blocks and letters of order $> k$.*

Proof. First assume that a is a letter of finite order K. Then a is a periodic letter of order K since $\varphi(a)$ begins with a. Then each word $\varphi^l(a)$ ($l \in \mathbb{N}$) contains only one letter of order $\geq K$ by a property of periodic letters. To prove the claim in this case, it suffices to prove that α contains infinitely many letters of order $K-1$. Let γ be the finite word such that $\varphi(a) = a\gamma$. Then $\alpha = a\gamma\varphi(\gamma)\varphi^2(\gamma)\ldots\varphi^l(\gamma)\ldots$. Since a is a letter of order k, $\varphi(a)$ contains at least one letter of order $k-1$. But then $\varphi^l(\gamma)$ contains at least one letter of order $k-1$ for each l.

Now let us consider the case when a is a letter of order ∞. Then it is sufficient to prove that α contains infinitely many letters of order ∞. By Lemma 2.7, $\varphi^l(a)$ contains at least two letters of order ∞ if l is large enough. Again write $\varphi(a) = a\gamma$, then $\varphi^l(a) = a\gamma\varphi(\gamma)\varphi^2(\gamma)\ldots\varphi^{l-1}(\gamma)$ and $\alpha = a\gamma\varphi(\gamma)\varphi^2(\gamma)\ldots\varphi^l(\gamma)\ldots$. If $\varphi^{l_0}(a)$ contains at least two letters of order ∞, then at least one of the words $\gamma, \varphi(\gamma), \ldots, \varphi^{l_0-1}(\gamma)$ contains a letter of order ∞. But then, by Lemma 2.7 again, all words $\varphi^l(\gamma)$ for $l \geq l_0$ also contain a letter of order ∞, and α contains infinitely many letters of order ∞. □

Now, when we know that α can be split into a concatenation of alternating letters of order $> k$ and k-blocks (at least for some $k \in \mathbb{N}$), it is convenient to consider concatenations of finitely many k-blocks and letters of order $> k$ between them. However, it is not very convenient to consider them as just occurrences in α, because k-blocks can be empty occurrences, and we want to distinguish clearly whether we include a k-block of the form $\alpha_{i+1\ldots i}$ (as it was pointed out above, this notation denotes the occurrence of the empty word between α_i and α_{i+1}) into a concatenation of the form $\alpha_{i+1\ldots j}$ or no. Also, we will need to consider possibly empty concatenations of k-blocks, and their exact locations will be important for us, in particular, if $\alpha_{i+1\ldots i}$ is an empty k-block, we want to distinguish "the empty concatenation located directly to the left of $\alpha_{i+1\ldots i}$" from "the empty concatenation located directly to the right of $\alpha_{i+1\ldots i}$". So we start with the following definition:

A pair of occurrences $(\alpha_{i\ldots j}, \alpha_{j+1\ldots s})$ is called a k-*delimiter* ($k \in \mathbb{N}$) in α in one of the two cases:

1. if exactly one of these two occurrences is a (possibly empty) k-block, and the other one is a single letter of order $> k$, or

2. if $\alpha_{i\ldots j} = \alpha_{0\ldots -1}$, the occurrence of the empty word before the actual beginning of α, and $\alpha_{j+1\ldots s} = \alpha_{0\ldots 0}$, a letter of order $> k$ (it follows from Lemma 3.5 that α_0 cannot be contained in a k-block since k-blocks are finite by definition).

Here $\alpha_{i\dots j}$ is called *the left part* of the k-delimiter and $\alpha_{j+1\dots s}$ is called *the right part* of the k-delimiter. Split α into a concatenation of k-blocks and letters of order $> k$. Write all these occurrences in α in an infinite sequence, mentioning each empty k-block explicitly. For example, if $\Sigma = \{a, b, c\}$, $\varphi(a) = abb$, $\varphi(b) = bcc$, $\varphi(c) = c$, then the orders of letters a, b, c are $3, 2, 1$, respectively, $\alpha = abbbccbccbccccbcccc\dots$, and this sequence of occurrences is: $\alpha_{0\dots 0}, \alpha_{1\dots 0}, \alpha_{1\dots 1}, \alpha_{2\dots 1}, \alpha_{2\dots 2}, \alpha_{3\dots 2}, \alpha_{3\dots 3}, \alpha_{4\dots 5}, \alpha_{6\dots 6}, \alpha_{7\dots 8}, \dots$. As abstract words, the nonempty words in this sequence are: $\alpha_{0\dots 0} = a, \alpha_{1\dots 1} = b, \alpha_{2\dots 2} = b, \alpha_{3\dots 3} = b, \alpha_{4\dots 5} = cc, \alpha_{6\dots 6} = b, \alpha_{7\dots 8} = cc, \dots$. The occurrences $\alpha_{1\dots 0}$, $\alpha_{2\dots 1}$, and $\alpha_{3\dots 2}$ here are empty 1-blocks. Informally speaking, a k-delimiter is the "empty space" between two members of this sequence (the left and the right parts of the k-delimiter) or the "empty space" to the left of the whole sequence. We say that a k-block or a single letter of order $> k$ $\alpha_{i\dots j}$ is located *strictly to the left* from a k-block or a single letter of order $> k$ $\alpha_{s\dots t}$ if $\alpha_{i\dots j}$ is written in this sequence before $\alpha_{s\dots t}$. In terms of indices this means that either $i < s$ ("the position where $\alpha_{i\dots j}$ starts is before the position where $\alpha_{s\dots t}$ starts") or $i = s$ and $j < t$ ("the positions where $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$ start coincide, but $\alpha_{i\dots j}$ ends before $\alpha_{s\dots t}$ ends"), this is possible only if $\alpha_{i\dots j}$ is an occurrence of the empty word ($j = i - 1$) since k-blocks and letters of order $> k$ do not overlap. We also say that a k-block or a single letter of order $> k$ $\alpha_{i\dots j}$ is located *strictly to the right* from a k-block or a single letter of order $> k$ $\alpha_{s\dots t}$ if $\alpha_{s\dots t}$ is located strictly to the left from $\alpha_{i\dots j}$. Clearly, if $\alpha_{i\dots j}$ is a k-block or a letter of order $> k$ and $\alpha_{s\dots t}$ is also a k-block or a letter of order $> k$, then either $\alpha_{i\dots j}$ is located strictly to the left from $\alpha_{s\dots t}$, or $\alpha_{i\dots j} = \alpha_{s\dots t}$, or $\alpha_{i\dots j}$ is located strictly to the right from $\alpha_{s\dots t}$.

Now let $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ be a k-delimiter, and let $\alpha_{i'\dots j'}$ be a k-block or a single letter of order $> k$. Then we want to define when $\alpha_{i'\dots j'}$ is *located at the right-hand side* of $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$. If $(\alpha_{i\dots j}, \alpha_{j+1\dots s}) = (\alpha_{0\dots -1}, \alpha_{0\dots 0})$, then we always say that $\alpha_{i'\dots j'}$ is located at the right-hand side of $(\alpha_{0\dots -1}, \alpha_{0\dots 0})$. Otherwise we say that $\alpha_{i'\dots j'}$ is located at the right-hand side of $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ if either $\alpha_{i'\dots j'} = \alpha_{j+1\dots s}$ as occurrences in α, or $\alpha_{i'\dots j'}$ is located strictly to the right from $\alpha_{j+1\dots s}$. In terms of indices this means that either $j+1 = i'$ and $s = j'$, or $j + 1 < i'$, or $j + 1 = i'$ and $s < j'$. This can be rewritten shorter as follows: either $j + 1 = i'$ and $s \le j'$, or $s < j'$. Similarly, if $(\alpha_{i\dots j}, \alpha_{j+1\dots s}) = (\alpha_{0\dots -1}, \alpha_{0\dots 0})$, then we never say that $\alpha_{i'\dots j'}$ is *located at the left-hand side of* $(\alpha_{0\dots -1}, \alpha_{0\dots 0})$. If $(\alpha_{i\dots j}, \alpha_{j+1\dots s}) \ne (\alpha_{0\dots -1}, \alpha_{0\dots 0})$, then we say that $\alpha_{i'\dots j'}$ is located at the left-hand side of $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ if either $\alpha_{i'\dots j'} = \alpha_{i\dots j}$ as occurrences in α, or $\alpha_{i'\dots j'}$ is located strictly to the left from $\alpha_{i\dots j}$. In terms of indices this means that either $i' = i$ and $j' = j$, or $i' < i$, or $i' = i$ and $j' < j$. This can be rewritten shorter as follows: either $i' = i$ and $j' \le j$, or $i' < i$. Again, if $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ is a k-delimiter, and $\alpha_{i'\dots j'}$ is a k-block or a single letter of order $> k$, then either $\alpha_{i'\dots j'}$ is located at the left-hand side of $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$, or $\alpha_{i'\dots j'}$ is located at the right-hand side of $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$.

If $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ and $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$ are k-delimiters, we say that $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$ is *located at the right-hand side of* $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ if $\alpha_{i'\dots j'}$ is located at the right-hand side of $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$. And $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ is said to be *located at the left-hand side of* $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$ if $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$ is located at the right-hand side of $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$. And again, if we have two k-delimiters, then either they coincide, or one of them is located at the left-hand side of the other one, or one of them is located at the right-hand side of the other one. Finally, we say that a k-block or a letter of order $> k$ is *located between* one k-delimiter and another k-delimiter if this k-block or this letter of order $> k$ is located at the right-hand side of the first k-delimiter and at the left-hand side of the second k-delimiter.

Now we are ready to define k-multiblocks.

Definition 3.6. We say that a *k-multiblock* is defined by the following data:

1. Two k-delimiters $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ and $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$, where $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ either coincides with $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$, or is located at the left-hand side of $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$. Here $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ (resp. $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$) is called the *left* (resp. *right*) k-delimiter of the k-multiblock,

2. The set of all k-blocks and letters of order $> k$ located between $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ and $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$.

Two k-multiblocks are called *consecutive* if the right k-delimiter of first k-multiblock coincides with the left k-delimiter of the second k-multiblock. The k-multiblock whose left (resp. right) k-delimiter is the left (resp. right) k-delimiter of the first (resp. second) k-multiblock is called their *concatenation.* A k-multiblock is called *empty* if the left and the right k-delimiters coincide, in other words, if the set of k-blocks and letters of order k is empty. A k-multiblock consisting of a single empty k-block is not called an empty k-multiblock.

We need to introduce some convenient notation for k-multiblocks. First, a k-multiblock is determined by two k-delimiters $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ and $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$, so we can denote it by $[(\alpha_{i\dots j}, \alpha_{j+1\dots s}), (\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})]$. Second, each k-delimiter is determined by its left or right part, so we can denote the same k-multiblock by $[\alpha_{j+1\dots s}, \alpha_{i'\dots j'}]$ (and this notation agrees with the fact that if $(\alpha_{i\dots j}, \alpha_{j+1\dots s})$ and $(\alpha_{i'\dots j'}, \alpha_{j'+1\dots s'})$ are two different k-delimiters, then the set of k-blocks and letters of order $> k$ in this k-multiblock is the subsequence of the sequence of all k-blocks and letters of order $> k$ in α that starts at $\alpha_{j+1\dots s}$ and ends at $\alpha_{i'\dots j'}$, inclusively). Moreover, if $\alpha_{j+1\dots j}$ is not a k-block, then the occurrence of the form $\alpha_{j+1\dots s}$, which is a k-block or a letter of order $> k$, is determined uniquely by the index $j+1$. However, if $\alpha_{j+1\dots j}$ is a k-block, then there are two occurrences of the form $\alpha_{j+1\dots s}$ that we can use as a right part of a k-delimiter: the empty k-block $\alpha_{j+1\dots j}$ and also $\alpha_{j+1\dots j+1}$, which must be a letter of order $> k$ in this case. In this case we denote the k-delimiter whose right part is $\alpha_{j+1\dots j}$ (i. e. the ***leftmost*** k-delimiter whose right part is of the form $\alpha_{j+1\dots s}$) by $<, j+1$, and the k-delimiter whose right part is $\alpha_{j+1\dots j+1}$ (i. e. the ***rightmost*** k-delimiter whose right part is of the form $\alpha_{j+1\dots s}$) by $>, j+1$. If $\alpha_{j+1\dots j}$ is not a k-block, we say that $<, j+1$ and $>, j+1$ denote the same k-delimiter, namely, the unique k-delimiter whose right part is of the form $\alpha_{j+1\dots s}$. Similarly, if $\alpha_{j'+1\dots j'}$ is a k-block, then there are two occurrences of the form $\alpha_{i'\dots j'}$ that are k-blocks or letters of order $> k$: the empty k-block $\alpha_{j'+1\dots j'}$ and a letter $\alpha_{j'\dots j'}$ of order $> k$. And we denote the k-delimiter whose left part is $\alpha_{j'+1\dots j'}$ (i. e. the rightmost k-delimiter whose left part is of the form $\alpha_{i'\dots j'}$) by $j', >$, and the k-delimiter whose left part is $\alpha_{j'\dots j'}$ (i. e. the leftmost k-delimiter whose left part is of the form $\alpha_{i'\dots j'}$) by $j', <$. If $\alpha_{j'+1\dots j'}$ is not a k-block, then there exists at most one occurrence of the form $\alpha_{i'\dots j'}$ that can be the left part of a k-delimiter, and if it exists, we denote the k-delimiter with this left part by both $j', <$ and $j', >$. Now we denote the same k-multiblock as before by $\alpha[\mathfrak{x}\dots\mathfrak{y}]_k$, where $\mathfrak{x}$ (resp. $\mathfrak{y}$) is a notation for a k-delimiter of the form $<, j+1$ or $>, j+1$ (resp. $j', <$ or $j', >$).

For example, if $\alpha_{i\dots j}$ is a non-empty k-block, then the k-multiblock whose set of k-blocks and letters of order $> k$ between the k-delimiters consists of $\alpha_{i\dots j}$ only, is denoted by $\alpha[<, i\dots j, >]_k$ or by $\alpha[>, i\dots j, <]_k$ (and two more possibilities). If α_i is a letter of order $> k$, then the k-multiblock that consists of this letter itself if denoted by $\alpha[>, i\dots i, <]_k$ (and here the signs $>$ and $<$ are important if α_{i-1} or α_{i+1} is also a letter of order $> k$). Let us consider the example of an empty k-block $\alpha_{i+1\dots i}$. In this case, $\alpha[<, i+1\dots i, >]_k$ is the k-multiblock that consists of the empty k-block $\alpha_{i+1\dots i}$ only (the k-block is located between the two k-delimiters), $\alpha[<, i+1\dots i, <]_k$ is the empty k-multiblock "located directly at the left" of the k-block $\alpha_{i+1\dots i}$ (the two k-delimiters coincide and are located at the left-hand side of $\alpha_{i+1\dots i}$), $\alpha[>, i+1\dots i, >]_k$ is the empty k-multiblock "located directly at the right" of the empty k-block, and $\alpha[>, i+1\dots i, <]_k$ denotes nothing since the two k-delimiters do not coincide and are not in the correct order. The k-multiblocks $\alpha[<, i+1\dots i, <]_k$ and $\alpha[<, i+1\dots i, >]_k$ are consecutive (and their concatenation is $\alpha[<, i+1\dots i, >]_k$ again), and $\alpha[<, i+1\dots i, <]_k$ and $\alpha[>, i+1\dots i, >]_k$ are not.

More generally, if we know that there exists an (empty or nonempty) k-block of the form $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$ is a k-block or letter of order $> k$ that coincides with $\alpha_{i\dots j}$ or located strictly to the right from $\alpha_{i\dots j}$, then $\alpha[<, i\dots t, ?]_k$, where ? is one of the signs $<$ and $>$, always denotes a k-multiblock that includes $\alpha_{i\dots j}$. And if we know that α_i is a letter of order $> k$, and and $\alpha_{s\dots t}$ is a k-block or letter of order $> k$ that coincides with α_i or located strictly to the right from α_i, then then $\alpha[>, i\dots t, ?]_k$, where ? is one of the signs $<$ and $>$ always denotes a k-multiblock that begins with α_i as a set of consecutive k-blocks and letters of order $> k$.

For each k-multiblock one can consider the concatenation of all k-blocks and letters of order $> k$ between the two k-delimiters, this is an occurrence in α. As we noted before, if the right part of the first k-delimiter is of the form $\alpha_{j+1\dots s}$, and the left part of the second k-delimiter is of the form $\alpha_{i'\dots j'}$, then this concatenation is $\alpha_{j+1\dots j'}$. Therefore, if $\alpha[?, i\dots j, ?]_k$ is a k-multiblock, where each question mark denotes one of the signs $<$ or $>$, then this occurrence in α is $\alpha_{i\dots j}$. We call it the *forgetful occurrence* of the k-multiblock and denote it by $\mathrm{Fg}(\alpha[?, i\dots j, ?]_k)$.

We did not define (and we are not going to define) any 0-blocks and 0-delimiters, however, it is convenient to have uniform notation and terminology for 0-multiblocks. We say that a 0-*multiblock* is just a (possibly empty) finite occurrence in α. We denote an occurrence $\alpha_{i\dots j}$ by $\alpha[?, i\dots j, ?]_0$, where each question mark is one of the signs $<$ or $>$ (these signs do not play any role here). The notions of consecutiveness and concatenation here are the usual notions of consecutiveness and concatenation for occurrences in α. A 0-multiblock is called empty if it is an occurrence of the empty word.

Now we are ready to define descendants of k-multiblocks. First, let α_i be a letter of order $> k$ ($k \in \mathbb{N}\cup 0$). Then the occurrence $\varphi(\alpha_i)$ contains at least one letter of order $> k$. Let α_j (resp. $\alpha_{j'}$) be the leftmost (resp. the rightmost) occurrence of a letter of order $> k$ in $\varphi(\alpha_i)$. Then $\alpha[>, j\dots j', <]_k$ is a k-multiblock that begins with α_j and ends with $\alpha_{j'}$ (and does not contain k-blocks of the form $\alpha_{j\dots j-1}$ or $\alpha_{j'+1\dots j'}$ even if these empty occurrences are k-blocks). We call $\alpha[>, j\dots j', <]_k$ the descendant of the k-multiblock $\alpha[>, i\dots i, <]_k$ (which consists of a single letter α_i) and denote $\alpha[>, j\dots j', <]_k$ by $\mathrm{Dc}_k(\alpha[>, i\dots i, <]_k)$.

Remark 3.7. *If α_i is a periodic letter of order $k+1$, then $\mathrm{Dc}_k(\alpha[>, i\dots i, <]_k)$ consists of a single letter of order $k+1$, namely, the unique letter of order $k+1$ in $\varphi(\alpha_i)$. Moreover, since φ is (in particular) weakly 1-periodic, this letter coincides with α_i as an abstract letter.*

Second, as we have already noted, if $\alpha_{i\dots j}$ is a k-block ($k \in \mathbb{N}$), then $\alpha[<, i\dots j, >]_k$ is always the k-multiblock that consists of $\alpha_{i\dots j}$ only, independently of whether $\alpha_{i\dots j}$ is empty or no. If $\mathrm{Dc}_k(\alpha_{i\dots j}) = \alpha_{s\dots t}$, then we say that $\mathrm{Dc}_k(\alpha[<, i\dots j, >]_k) = \alpha[<, s\dots t, >]_k$ (the k-block that consists of $\alpha_{s\dots t}$ only).

Third, let us define the descendants of empty k-multiblocks ($k > 0$). An empty k-multiblock is determined by a delimiter $(\alpha_{i\dots j}, \alpha_{s\dots t})$ repeated twice, both as the left and as the right delimiter of the k-multiblock. If $\alpha_{i\dots j} = \alpha_{0\dots -1}$, we say that the descendant of this k-multiblock is this k-multiblock itself. Otherwise either $\alpha_{i\dots j}$ or $\alpha_{s\dots t}$ is a k-block. If $\alpha_{i\dots j}$ is a k-block, then $\alpha_{s\dots t}$ is the right border of $\alpha_{i\dots j}$, and we say that the descendant of $[(\alpha_{i\dots j}, \alpha_{s\dots t}), (\alpha_{i\dots j}, \alpha_{s\dots t})]$ is

$$[(\mathrm{Dc}_k(\alpha_{i\dots j}), \mathrm{RB}(\mathrm{Dc}_k(\alpha_{i\dots j}))), (\mathrm{Dc}_k(\alpha_{i\dots j}), \mathrm{RB}(\mathrm{Dc}_k(\alpha_{i\dots j})))].$$

Similarly, if $\alpha_{s\dots t}$ is a k-block, then $\alpha_{i\dots j}$ is its left border, and we say that the descendant of $[(\alpha_{i\dots j}, \alpha_{s\dots t}), (\alpha_{i\dots j}, \alpha_{s\dots t})]$ is

$$[(\mathrm{LB}(\mathrm{Dc}_k(\alpha_{s\dots t})), \mathrm{Dc}_k(\alpha_{s\dots t})), (\mathrm{LB}(\mathrm{Dc}_k(\alpha_{s\dots t})), \mathrm{Dc}_k(\alpha_{s\dots t}))].$$

Finally, let $k \in \mathbb{N}\cup 0$, and let $\alpha[\mathfrak{x}, i\dots j, \mathfrak{y}]_k$, where $\mathfrak{x}, \mathfrak{y} \in \{<, >\}$, be a non-empty k-multiblock. It consists of consecutive letters of order $> k$ and (if $k > 0$) k-blocks, and their descendants according to the definitions above are also consecutive k-multiblocks. We call the concatenation of these k-multiblocks the *descendant* of $\alpha[\mathfrak{x}, i\dots j, \mathfrak{y}]_k$. Denote it by $\mathrm{Dc}_k(\alpha[\mathfrak{x}, i\dots j, \mathfrak{y}]_k)$. One checks easily using the particular cases of the definition of the descendant of a k-multiblock above that $\mathrm{Dc}_k(\alpha[\mathfrak{x}, i\dots j, \mathfrak{y}]_k)$ can be written as $\alpha[\mathfrak{x}, s\dots t, \mathfrak{y}]_k$, where the indices s and t may differ from i and j, but the signs $\mathfrak{x}$ and $\mathfrak{y}$ stay the same. If $l \in \mathbb{N}$, we also write $\mathrm{Dc}_k^l(\alpha[\mathfrak{x}, i\dots j, \mathfrak{y}]_k) = \mathrm{Dc}_k(\dots\mathrm{Dc}_k(\alpha[\mathfrak{x}, i\dots j, \mathfrak{y}]_k)\dots)$, where Dc_k is repeated l times. We call $\mathrm{Dc}_k^l(\alpha[\mathfrak{x}, i\dots j, \mathfrak{y}]_k)$ the *l-th superdescendant* of $\alpha[\mathfrak{x}, i\dots j, \mathfrak{y}]_k$.

Observe that if $k = 0$, then $\mathrm{Dc}_0(\alpha[\mathfrak{x}, i\dots j, \mathfrak{y}]_0)$ is just $\varphi(\alpha[\mathfrak{x}, i\dots j, \mathfrak{y}]_0)$, but it will be useful to have Dc_k as a uniform notation later, for example, when we will define atoms inside blocks.

Remark 3.8. *The descendants of two consecutive k-multiblocks ($k \in \mathbb{N}\cup 0$) are always consecutive, even if they contain several k-blocks and letters of order $> k$ or one or two of them is empty.*

Consider the following example: let $\Sigma = \{a, b, c, d\}$, $\varphi(a) = ab$, $\varphi(b) = cdcdd$, $\varphi(c) = cdd$, $\varphi(d) = d$. The orders of letters a, b, c, d are $3, 2, 2, 1$, respectively, and b is a preperiodic letter, all other letters are periodic. This morphism φ is strongly 1-periodic, $\mathbf{L} = 1$, so φ is also a strongly 1-periodic morphism with long images. Consider the corresponding pure morphic sequence $\alpha = \varphi^\infty(a) = a\,b\,cdcdd\;cdddcdddd\;cddddddcdddddd\dots$ and a 1-multiblock $\alpha[>, 1\dots 1, <]_1$ consisting of a single letter b of order $2 > 1$. Here $\alpha_{1\dots 0}$ is an empty 1-block, and $\alpha_{2\dots 1}$ is also an empty 1-block, but we do not include them into the 1-multiblock. We have

$\mathrm{Dc}_1(\alpha[>, 1 \dots 1, <]_1) = \alpha[>, 2 \dots 4, <]_1$ ($\alpha_{2\dots4} = cdc$) and $\mathrm{Dc}_1^2(\alpha[>, 1 \dots 1, <]_1) = \mathrm{Dc}_1(\alpha[>, 2 \dots 4, <]_1) = \alpha[>, 7 \dots 11, <]_1$ ($\alpha_{7\dots11} = cdddc$). If we include both $\alpha_{1\dots0}$ and $\alpha_{2\dots1}$ into the 1-multiblock and consider a 1-multiblock $\alpha[<, 1 \dots 1, >]_1$, we will get $\mathrm{Dc}_1(\alpha[<, 1 \dots 1, >]_1) = \alpha[<, 2 \dots 6, >]_1$ ($\alpha_{2\dots6} = cdcdd$) and $\mathrm{Dc}_1^2(\alpha[<, 1 \dots 1, >]_1) = \mathrm{Dc}_1(\alpha[<, 2 \dots 6, >]_1) = \alpha[<, 5 \dots 15, >]_1$ ($\alpha_{5\dots15} = ddcdddcdddd$).

Lemma 3.9. *If $k \in \mathbb{N}$, $\alpha[\mathfrak{x}, s \dots t, \mathfrak{y}]_{k-1}$ is a $(k-1)$-multiblock consisting of a single letter of order $\geq k$ or (if $k > 1$) a single $(k-1)$-block, $\alpha_{s\dots t}$ is a suboccurrence of a k-block $\alpha_{i\dots j}$, and $\mathrm{Dc}_{k-1}(\alpha[\mathfrak{x}, s \dots t, \mathfrak{y}]_{k-1}) = \alpha[\mathfrak{x}, s' \dots t', \mathfrak{y}]_{k-1}$, then $\alpha_{s'\dots t'}$ is a suboccurrence of $\mathrm{Dc}_k(\alpha_{i\dots j})$.*

Proof. The claim follows directly from the definitions of the descendant of a k-block and of a $(k-1)$-multiblock consisting of a single $(k-1)$-block or of a single letter of order $> (k-1)$. ☐

Corollary 3.10. *If $k \in \mathbb{N}$, $\alpha[\mathfrak{x}, s \dots t, \mathfrak{y}]_{k-1}$ is a $(k-1)$-multiblock, $\alpha_{s\dots t}$ is a suboccurrence of a k-block $\alpha_{i\dots j}$, and $\mathrm{Dc}_{k-1}(\alpha[\mathfrak{x}, s \dots t, \mathfrak{y}]_{k-1}) = \alpha[\mathfrak{x}, s' \dots t', \mathfrak{y}]_{k-1}$, then $\alpha_{s'\dots t'}$ is a suboccurrence of $\mathrm{Dc}_k(\alpha_{i\dots j})$.* ☐

Now we define *atoms* inside k-blocks ($k \in \mathbb{N}$). Let $\mathscr{E} = \mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2, \dots$ be an evolution of k-blocks. The lth left and right atoms exist in a k-block $\mathscr{E}_m$ iff $m \geq l > 0$. We will also define the zeroth atom, but there will be only one zeroth atom in each k-block $\mathscr{E}_m$, it will not be left or right. First, define the l-th atoms inside the k-block $\mathscr{E}_l$ ($l > 0$). Let $\mathscr{E}_l = \alpha_{i\dots j}$. Its ancestor $\alpha_{s\dots t} = \mathrm{Dc}_k^{-1}(\alpha_{i\dots j})$ is a k-block, so it is a concatenation of letters of order k and (if $k > 1$) $(k-1)$-blocks, and we can consider a $(k-1)$-multiblock $\alpha[<, s \dots t, >]_{k-1}$. If $\alpha_{s\dots s-1}$ or $\alpha_{t+1\dots t}$ is a $(k-1)$-block, we include it into the $(k-1)$-multiblock, so we are considering a $(k-1)$-multiblock, which starts with a $(k-1)$-block and ends with a $(k-1)$-block. Now consider a $(k-1)$-block $\mathrm{Dc}_{k-1}(\alpha[<, s \dots t, >]_{k-1})$ and denote it by $\alpha[<, i' \dots j', >]_{k-1}$.

By Corollary 3.10, $\alpha_{i'\dots j'}$ is a suboccurrence of $\alpha_{i\dots j}$. It also follows from the definition of the descendant of a $(k-1)$-block that $\varphi(\alpha_{s\dots t})$ is a suboccurrence of $\alpha_{i'\dots j'}$ and that $\alpha_{i'-1}$ and $\alpha_{j'+1}$ are letters of order $> (k-1)$, more precisely, $\alpha_{i'-1}$ (resp. $\alpha_{j'+1}$) is the rightmost (resp. the leftmost) letter of order $> (k-1)$ in $\varphi(\alpha_{s-1})$ (resp. in $\varphi(\alpha_{t+1})$). The $(k-1)$-multiblock $\alpha[<, i \dots i'-1, <]_{k-1}$ that comes from the image of the left border of the ancestor, is called *the lth left atom* of the block and is denoted by $\mathrm{LA}_{k,l}(\alpha_{i\dots j})$.

Remark 3.11. *If $k > 1$, then this $(k-1)$-multiblock is either empty (does not contain any letters of order $> (k-1)$ or k-blocks, even empty ones) if $i = i'$, or it begins with a (possibly empty) $(k-1)$-block of the form $\alpha_{i\dots i''}$ and ends with a single letter $\alpha_{i'-1}$ of order k if $i < i'$. If $k = 1$, then $\alpha_{i'-1}$ is the rightmost letter in $\varphi(\alpha_{s-1})$, and $\mathrm{LA}_{k,l}(\alpha_{i\dots j}) = \alpha_{i\dots i'-1}$ is an empty occurrence in α if and only if $i = i'$ if and only if the rightmost letter in $\varphi(\alpha_{s-1})$ is of order > 1.*

Similarly, the $(k-1)$-multiblock $\alpha[>, j'+1 \dots j, >]_{k-1} = \mathrm{RA}_{k,l}(\alpha_{i\dots j})$ is called *the l-th right atom* of the k-block $\alpha_{i\dots j}$.

Remark 3.12. *If $k > 1$, then it is either an empty $(k-1)$-multiblock if $j' = j$, or it begins with a single letter $\alpha_{j'+1}$ of order k and ends with a (possibly empty) $(k-1)$-block of the form $\alpha_{j''\dots j}$ if $j' < j$. If $k = 1$, then $\alpha_{j'+1}$ is the leftmost letter in $\varphi(\alpha_{t+1})$, and $\mathrm{LA}_{k,l}(\alpha_{i\dots j}) = \alpha_{j'+1\dots j}$ is an empty occurrence in α if and only if $j' = j$ if and only if the leftmost letter in $\varphi(\alpha_{t+1})$ is of order > 1.*

Fig. 3 illustrates this construction.

Then, if $l < m$, the lth left and right atoms of $\mathscr{E}_m$ are defined as follows: $\mathrm{LA}_{k,l}(\mathscr{E}_m) = \mathrm{Dc}_{k-1}^{m-l}(\mathrm{LA}_{k,l}(\mathscr{E}_l))$, $\mathrm{RA}_{k,l}(\mathscr{E}_m) = \mathrm{Dc}_{k-1}^{m-l}(\mathrm{RA}_{k,l}(\mathscr{E}_l))$. Then, using Remarks 3.11 and 3.12 and the definitions of the descendant of a $(k-1)$-block or of a $(k-1)$-multiblock that consists of a single letter of order $> (k-1)$, we note the following:

Remark 3.13. *If $k > 1$, then each left (resp. right) atom in any k-block is either an empty $(k-1)$-multiblock (it does not contain any letters of order $> (k-1)$ or k-blocks, even empty ones), or it begins with a (possibly empty) $(k-1)$-block (resp. with a single letter of order k) and ends with a single letter of order k (resp. with a (possibly empty) $(k-1)$-block).*

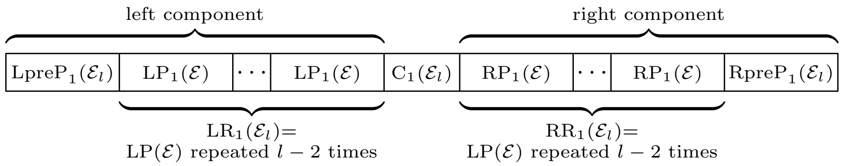

Figure 3: Structure of a k-block: $\alpha_{i'-1}$ and $\alpha_{j'+1}$ are letters of order k, all letters in the grayed areas are of order $\leq k-1$.

Finally, if $\mathscr{E}_0 = \alpha_{i\ldots j}$, then the zeroth atom of $\mathscr{E}_0$ is $\mathrm{A}_{k,0}(\mathscr{E}_0) = \alpha[<, i\ldots j, >]_{k-1}$, i. e. it is the largest (including all possible empty $(k-1)$-blocks if $k > 1$) $(k-1)$-multiblock whose forgetful occurrence is $\mathscr{E}_0$. The zeroth atoms of other blocks in the evolution are defined by $\mathrm{A}_{k,0}(\mathscr{E}_m) = \mathrm{Dc}^m_{k-1}(\mathrm{A}_{k,0}(\mathscr{E}_0))$.

Remark 3.14. *If $k > 1$, then each zeroth atom begins with a (possibly empty) $(k-1)$-block and ends with a (possibly empty) $(k-1)$-block. However, these two $(k-1)$-blocks may be the same, i. e. the zeroth atom can consist of a single $(k-1)$-block. The zeroth atom is never empty as a $(k-1)$-multiblock, i. e. it contains at least one (maybe, empty) $(k-1)$-block, but the forgetful occurrence of the zeroth atom may be empty.*

Therefore, if $\mathscr{E}_m = \alpha_{i\ldots j}$, then $\alpha[<, i\ldots j, >]_{k-1}$ (the largest $(k-1)$-multiblock whose forgetful occurrence is $\mathscr{E}_m$) splits into the concatenation of all atoms in $\mathscr{E}_m$:

$$\alpha[<, i\ldots j, >]_{k-1} = \mathrm{LA}_{k,m}(\mathscr{E}_m)\,\mathrm{LA}_{k,m-1}(\mathscr{E}_m)\ldots\mathrm{LA}_{k,1}(\mathscr{E}_m)\,\mathrm{A}_{k,0}(\mathscr{E}_m)\,\mathrm{RA}_{k,1}(\mathscr{E}_m)\ldots\mathrm{RA}_{k,m}(\mathscr{E}_m).$$

Lemma 3.15. *Let $l \geq 0$ and $m \geq 1$. Consider an occurrence $\varphi^m(\mathrm{LB}(\mathscr{E}_l))$ in α. Let α_{i-1} be the rightmost occurrence of a letter of order $> k$ in $\varphi^m(\mathrm{LB}(\mathscr{E}_l))$ and let α_j be the rightmost occurrence of a letter of order $\geq k$ in $\varphi^m(\mathrm{LB}(\mathscr{E}_l))$.*

Then $\alpha_{i-1} = \mathrm{LB}(\mathscr{E}_{l+m})$ and $\alpha_{i\ldots j} = \mathrm{Fg}(\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\ldots\mathrm{LA}_{k,l+1}(\mathscr{E}_{l+m}))$ as occurrences in α.

Proof. The first equality is proved directly by induction on m using the definition of the descendant of a k-block and the fact that the image of a letter of order $\leq k$ consists of letters of order $\leq k$ only.

The second equality for $m = 1$ it follows directly from the definitions of the $(l+1)$th atom and of the descendant of a $(k-1)$-block (see Remark 3.11). The second equality in general will follow from the first one and the fact that either $\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\ldots\mathrm{LA}_{k,l+1}(\mathscr{E}_{l+m})$ is an empty $(k-1)$-multiblock and $j = i-1$, or $\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\ldots\mathrm{LA}_{k,l+1}(\mathscr{E}_{l+m})$ is not empty, and α_j is the rightmost occurrence of a letter of order k in its forgetful occurrence. We already know this for $m = 1$, to prove this in general, we use induction on m. By the definition of a descendant of a single letter α_s of order $> (k-1)$, the rightmost letter in the forgetful occurrence of $\mathrm{Dc}_{k-1}(\alpha[>, s\ldots s, <]_{k-1})$ is the rightmost letter of order $> (k-1)$ in $\varphi(\alpha_s)$. Therefore, if $\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\ldots\mathrm{LA}_{k,l+1}(\mathscr{E}_{l+m})$ is not an empty $(k-1)$-multiblock and the rightmost letter of its forgetful occurrence is α_j, a letter of order k, then $\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m+1})\ldots\mathrm{LA}_{k,l+1}(\mathscr{E}_{l+m+1}) = \mathrm{Dc}_{k-1}(\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\ldots\mathrm{LA}_{k,l+1}(\mathscr{E}_{l+m}))$ is also a nonempty k-multiblock, and the rightmost letter in its forgetful occurrence is the rightmost letter of order $> (k-1)$ in $\varphi(\alpha_j)$. By the induction hypothesis, α_j is the rightmost occurrence of a letter of order $\geq k$ in $\varphi^m(\mathrm{LB}(\mathscr{E}_l))$, so, since images of letters of order $\leq (k-1)$ consist of letters of order $\leq (k-1)$ only if $k > 1$, we get that the rightmost occurrence of a letter of order $> (k-1)$ in $\varphi(\alpha_j)$ and rightmost occurrence of a letter of order $> (k-1)$ in $\varphi^{m+1}(\mathrm{LB}(\mathscr{E}_l))$ coincide. If $\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\ldots\mathrm{LA}_{k,l+1}(\mathscr{E}_{l+m})$ is an empty k-multiblock, then $\mathrm{LA}_{k,l+m+1}(\mathscr{E}_{l+m+1})\,\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m+1})\ldots\mathrm{LA}_{k,l+1}(\mathscr{E}_{l+m+1}) = \mathrm{LA}_{k,l+m+1}(\mathscr{E}_{l+m+1})$, and (by the induction hypothesis) the rightmost occurrence of a letter of order $\geq k$ in $\varphi^m(\mathrm{LB}(\mathscr{E}_l))$ is $\mathrm{LB}(\mathscr{E}_{l+m})$. Now it suffices to use the claim for $l+m$ instead of l and 1 instead of m, but we have already considered this case before. $\square$

Lemma 3.16. *Let $l \geq 0$ and $m \geq 1$. Consider an occurrence $\varphi^m(\mathrm{RB}(\mathscr{E}_l))$ in α. Let α_{i+1} be the leftmost occurrence of a letter of order $> k$ in $\varphi^m(\mathrm{RB}(\mathscr{E}_l))$ and let α_j be the leftmost occurrence of a letter of order $\geq k$ in $\varphi^m(\mathrm{RB}(\mathscr{E}_l))$.*

Then $\alpha_{i+1} = \mathrm{RB}(\mathscr{E}_{l+m})$ and $\alpha_{j\dots i} = \mathrm{Fg}(\mathrm{RA}_{k,l+1}(\mathscr{E}_{l+m})\dots\mathrm{RA}_{k,l+m}(\mathscr{E}_{l+m}))$ as occurrences in α.

Proof. The proof is completely symmetric to the proof of the previous lemma. □

Corollary 3.17. *If $l \geq 2$ and $m, n \geq 0$, then*

$$\mathrm{Fg}(\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\dots\mathrm{LA}_{k,l}(\mathscr{E}_{l+m}))$$

is the same abstract word as

$$\mathrm{Fg}(\mathrm{LA}_{k,l+m+n}(\mathscr{E}_{l+m+n})\dots\mathrm{LA}_{k,l+n}(\mathscr{E}_{l+m+n}))$$

and

$$\mathrm{Fg}(\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})\dots\mathrm{RA}_{k,l+m}(\mathscr{E}_{l+m}))$$

is the same abstract word as

$$\mathrm{Fg}(\mathrm{RA}_{k,l+n}(\mathscr{E}_{l+m+n})\dots\mathrm{RA}_{k,l+m+n}(\mathscr{E}_{l+m+n})).$$

In other words, if $l \geq 2$ and $m \geq 0$, then the abstract words

$$\mathrm{Fg}(\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\dots\mathrm{LA}_{k,l}(\mathscr{E}_{l+m}))$$

and

$$\mathrm{Fg}(\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})\dots\mathrm{RA}_{k,l+m}(\mathscr{E}_{l+m}))$$

do not depend on l.

Proof. Since $l \geq 2$ and φ is a strongly 1-periodic morphism, $\mathrm{LB}(\mathscr{E}_{l-1}) = \mathrm{LB}(\mathscr{E}_{l-1+n})$ as abstract letters. Denote this abstract letter by b. Denote $\varphi^{m+1}(b) = \gamma$, this is a finite abstract word. Let γ_{i-1} (resp. γ_j) be the rightmost occurrence of a letter of order $> k$ (resp. $\geq k$) in γ. By the previous lemma, $\gamma_{i\dots j} = \mathrm{Fg}(\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\dots\mathrm{LA}_{k,l}(\mathscr{E}_{l+m}))$ as abstract words and $\gamma_{i\dots j} = \mathrm{Fg}(\mathrm{LA}_{k,l+m+n}(\mathscr{E}_{l+m+n})\dots\mathrm{LA}_{k,l+n}(\mathscr{E}_{l+m+n}))$ as abstract words. The proof for right atoms is analogous. □

Corollary 3.18. *If $l \geq 2$ and $m, n \geq 0$, then $\mathrm{Fg}(\mathrm{LA}_{k,l}(\mathscr{E}_{l+m}))$ is the same abstract word as $\mathrm{Fg}(\mathrm{LA}_{k,l+n}(\mathscr{E}_{l+m+n}))$. Moreover, if $\mathscr{E}_{l+m} = \alpha_{i\dots j}$, $\mathrm{Fg}(\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})) = \alpha_{s\dots t}$, $\mathscr{E}_{l+m+n} = \alpha_{i'\dots j'}$, and $\mathrm{Fg}(\mathrm{LA}_{k,l+n}(\mathscr{E}_{l+m+n})) = \alpha_{s'\dots t'}$, then $s - i = s' - i'$. In other words, if $l \geq 2$ and $m \geq 0$, then $\mathrm{Fg}(\mathrm{LA}_{k,l}(\mathscr{E}_{l+m}))$ does not depend on l, as an abstract word, and the numbers of letters in α between $\mathrm{LB}(\mathscr{E}_l)$ and $\mathrm{Fg}(\mathrm{LA}_{k,l}(\mathscr{E}_{l+m}))$ also does not depend on l.*

Proof. If $m = 0$, then this is just the previous corollary. If $m > 0$, then by the previous corollary,

$$\mathrm{Fg}(\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\dots\mathrm{LA}_{k,l}(\mathscr{E}_{l+m}))$$

is the same abstract word as

$$\mathrm{Fg}(\mathrm{LA}_{k,l+m+n}(\mathscr{E}_{l+m+n})\dots\mathrm{LA}_{k,l+n}(\mathscr{E}_{l+m+n})),$$

denote this abstract word by γ, and

$$\mathrm{Fg}(\mathrm{LA}_{k,l+m}(\mathscr{E}_{l+m})\dots\mathrm{LA}_{k,l+1}(\mathscr{E}_{l+m}))$$

is the same abstract word as

$$\mathrm{Fg}(\mathrm{LA}_{k,l+m+n}(\mathscr{E}_{l+m+n})\dots\mathrm{LA}_{k,l+n+1}(\mathscr{E}_{l+m+n})),$$

denote this abstract word by δ. Clearly, δ is a prefix of γ, so write $\gamma = \delta\delta'$ for some finite abstract word δ'. But then $\mathrm{Fg}(\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})) = \delta'$ as abstract words, $\mathrm{Fg}(\mathrm{LA}_{k,l}(\mathscr{E}_{l+m+n})) = \delta'$ as abstract words, $s - i = |\delta|$ and $s' - i' = |\delta|$. □

Corollary 3.19. *If $l \geq 2$ and $m, n \geq 0$, then* $\mathrm{Fg}(\mathrm{RA}_{k,l}(\mathscr{E}_{l+m}))$ *is the same abstract word as* $\mathrm{Fg}(\mathrm{RA}_{k,l+n}(\mathscr{E}_{l+m+n}))$. *Moreover, if $\mathscr{E}_{l+m} = \alpha_{i\dots j}$,* $\mathrm{Fg}(\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})) = \alpha_{s\dots t}$, *$\mathscr{E}_{l+m+n} = \alpha_{i'\dots j'}$, and* $\mathrm{Fg}(\mathrm{RA}_{k,l+n}(\mathscr{E}_{l+m+n})) = \alpha_{s'\dots t'}$, *then $j - t = j' - t'$. In other words, if $l \geq 2$ and $m \geq 0$, then* $\mathrm{Fg}(\mathrm{RA}_{k,l}(\mathscr{E}_{l+m}))$ *does not depend on l, as an abstract word, and the numbers of letters in α between* $\mathrm{Fg}(\mathrm{RA}_{k,l}(\mathscr{E}_{l+m}))$ *and* $\mathrm{LB}(\mathscr{E}_l)$ *also does not depend on l.*

Proof. The proof is completely symmetric to the proof of the previous corollary. □

Observe that the condition $l \geq 2$ cannot be omitted since in the proof of Corollary 3.17 we used the fact that $\mathrm{LB}(\mathscr{E}_{l-1}) = \mathrm{LB}(\mathscr{E}_{l-1+n})$ as abstract letters. Moreover, $\mathrm{LA}_{k,1}(\mathscr{E}_1)$ is a $(k-1)$-multiblock contained in the image of $\mathrm{LB}(\mathscr{E}_0)$, and $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ for $l > 1$ is contained in the image of $\mathrm{LB}(\mathscr{E}_l)$, a letter which does not have to be equal to $\mathrm{LB}(\mathscr{E}_0)$, so the letters of order k and (if $k > 1$) $(k-1)$-blocks in $\mathrm{LA}_{k,1}(\mathscr{E}_1)$ and $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ may be different. And the first left atoms of other k-blocks in the evolution are superdescendants of $\mathrm{LA}_{k,1}(\mathscr{E}_1)$, while the lth atoms of other k-blocks in the evolution are superdescendants of $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ for $l > 1$. So, the $(k-1)$-blocks in $\mathrm{LA}_{k,1}(\mathscr{E}_m)$ may belong to totally different evolutions than $(k-1)$-blocks in $\mathrm{LA}_{k,l}(\mathscr{E}_n)$ for $l > 1$ belong to, while the $(k-1)$-blocks in $\mathrm{LA}_{k,l}(\mathscr{E}_m)$ and in $(k-1)$-blocks in $\mathrm{LA}_{k,l'}(\mathscr{E}_n)$ by Corollary 3.18 belong to the same evolutions if evolutions are understood as sequences of abstract words (as in Lemma 3.4).

These observations and these corollaries justify the following definitions. If $l \geq 1$, we call the concatenation of the $(k-1)$-multiblocks

$$\mathrm{LA}_{k,1}(\mathscr{E}_l)\,\mathrm{A}_{k,0}(\mathscr{E}_l)\,\mathrm{RA}_{k,1}(\mathscr{E}_l)$$

the *core* of $\mathscr{E}_l$. The core of $\mathscr{E}_l$ is denoted by $\mathrm{C}_k(\mathscr{E}_l)$. If $l \geq 2$, the concatenation of the $(k-1)$-multiblocks

$$\mathrm{LA}_{k,l}(\mathscr{E}_l)\,\mathrm{LA}_{k,l-1}(\mathscr{E}_l)\dots\mathrm{LA}_{k,2}(\mathscr{E}_l)$$

$$(\text{resp. } \mathrm{RA}_{k,2}(\mathscr{E}_l)\dots\mathrm{RA}_{k,l-1}(\mathscr{E}_l)\,\mathrm{RA}_{k,l}(\mathscr{E}_l))$$

is called the *left (resp. right) component*.

By Remark 3.11, $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ is either an empty $(k-1)$-multiblock, or it contains (actually, the rightmost letter of its forgetful occurrence is) a letter of order k. By Corollary 3.18, either for all $l > 1$ $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ is an empty $(k-1)$-multiblock, or for all $l > 1$ $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ contains a letter of order k. So, if each atom $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ for $l > 1$ contains a letter of order k, we say that *Case I holds for $\mathscr{E}$ at the left.* If all atoms $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ for $l > 1$ are empty $(k-1)$-blocks, we say that *Case II holds for $\mathscr{E}$ at the left.* Similarly, cases I and II are defined for right atoms. These cases happen independently at right and at left, in any combination.

Remark 3.20. *The left (resp. right) component is empty if and only if Case II holds at the left (resp. at the right). If Case II holds both at the left and at the right for an evolution $\mathscr{E}$ of k-blocks and $l \geq 1$, then* $\mathscr{E}_l = \mathrm{Fg}(\mathrm{C}_k(\mathscr{E}_l))$.

Note that if $k \in \mathbb{N}$, then k-blocks may exist by definition even if all letters in α have either order $< k$, or order ∞ (see also Lemma 3.5). In this situation, Case II holds for all evolutions of k-blocks both at the left and at the right.

4 1-Blocks

Now we will consider 1-blocks more accurately. The fact that φ is a strongly 1-periodic morphism makes the structure of a 1-block quite easy. During this section, it will be useful to keep in mind that 0-multiblocks are just occurrences in α and their descendants are just their images under φ.

Lemma 4.1. *Let $\mathscr{E}$ be an evolution of 1-blocks. Then:*

If $l > 1$, then $\mathrm{C}_1(\mathscr{E}_l)$ does not depend on l as an abstract word and consists of periodic letters only.

If $l > 1$, then $\mathrm{LA}_{1,l}(\mathscr{E}_l)$ and $\mathrm{RA}_{1,l}(\mathscr{E}_l)$ do not depend on l as abstract words.

If $l > 1$ and $m \geq 1$, then $\mathrm{LA}_{1,l}(\mathscr{E}_{l+m})$ and $\mathrm{RA}_{1,l}(\mathscr{E}_{l+m})$ as abstract words depend neither on l nor on m. They equal $\varphi(\mathrm{LA}_{1,l}(\mathscr{E}_l))$ and $\varphi(\mathrm{RA}_{1,l}(\mathscr{E}_l))$ as abstract words, respectively and consist of periodic letters of order 1 only.

Proof. Since φ is (in particular) weakly 1-periodic, the image of a preperiodic letter of order 1 consists of periodic letters of oder 1 only. The image of a periodic letter of order 1 is a (single) periodic letter of order 1. We have $\mathrm{C}_1(\mathscr{E}_l) = \mathrm{LA}_{1,1}(\mathscr{E}_l)\,\mathrm{A}_{1,0}(\mathscr{E}_l)\,\mathrm{RA}_{1,1}(\mathscr{E}_l) = \mathrm{Dc}_0^{l-1}(\mathrm{LA}_{1,1}(\mathscr{E}_1)\,\mathrm{A}_{1,0}(\mathscr{E}_1)\,\mathrm{RA}_{1,1}(\mathscr{E}_1)) = \varphi^{l-1}(\mathrm{C}_1(\mathscr{E}_1))$. So, if $l > 1$, all letters in $\varphi^{l-1}(\mathrm{C}_1(\mathscr{E}_1))$ are periodic letters of order 1. By weak 1-periodicity again, $\varphi(\varphi^{l-1}(\mathrm{C}_1(\mathscr{E}_1))) = \varphi^{l-1}(\mathrm{C}_1(\mathscr{E}_1))$ as abstract words. But $\mathrm{C}_1(\mathscr{E}_l) = \varphi^{l}(\mathrm{C}_1(\mathscr{E}_1)) = \varphi(\varphi^{l-1}(\mathrm{C}_1(\mathscr{E}_1)))$, so we have the first claim.

The second claim is just a particular case of Corollaries 3.18 and 3.19. For the third claim, we write $\mathrm{LA}_{1,l}(\mathscr{E}_l + m) = \mathrm{Dc}_0^{m}(\mathrm{LA}_{1,l}(\mathscr{E}_l)) = \varphi^{m}(\mathrm{LA}_{1,l}(\mathscr{E}_l))$. Using the second claim, we see that it is sufficient to prove that $\varphi^{m}(\mathrm{LA}_{1,l}(\mathscr{E}_l))$ does not depend on m as an abstract word if $m \geq 1$ (for $m = 1$ it clearly equals $\varphi(\mathrm{LA}_{1,l}(\mathscr{E}_l)))$. Again, since φ is weakly 1-periodic, $\varphi^{m}(\mathrm{LA}_{1,l}(\mathscr{E}_l))$ consists of periodic letters of order 1 only if $m \geq 1$, and, by weak 1-periodicity again, $\varphi^{m+1}(\mathrm{LA}_{1,l}(\mathscr{E}_l)) = \varphi^{m}(\mathrm{LA}_{1,l}(\mathscr{E}_l))$ as an abstract words if $m \geq 1$. The computation for the right atoms is the same. □

After we have this lemma, we can give the following definitions:

Given an evolution $\mathscr{E}$ of 1-blocks, we call the abstract word $\mathrm{C}_1(\mathscr{E}_l)$ for any $l > 1$ the *core of* $\mathscr{E}$ and denote it by $\mathrm{C}_1(\mathscr{E})$. The abstract word $\mathrm{LA}_{1,l}(\mathscr{E}_l)$ (resp. $\mathrm{RA}_{1,l}(\mathscr{E}_l)$) for any $l > 1$ is called the *left (resp. right) preperiod of* $\mathscr{E}$ and is denoted by $\mathrm{LpreP}_1(\mathscr{E})$ (resp. by $\mathrm{RpreP}_1(\mathscr{E})$). The lth left (resp. right) atom of a particular 1-block $\mathscr{E}_l$, where $l > 1$ is called the *left (resp. right) preperiod of* $\mathscr{E}_l$ and is denoted by by $\mathrm{LpreP}_1(\mathscr{E}_l)$ (resp. by $\mathrm{RpreP}_1(\mathscr{E}_l)$. The abstract word $\mathrm{LA}_{1,l}(\mathscr{E}_{l+m})$ (resp. $\mathrm{RA}_{1,l}(\mathscr{E}_{l+m})$) for any $l > 1$ and $m \geq 1$ is called the *left (resp. right) period of* $\mathscr{E}$ and is denoted by $\mathrm{LP}_1(\mathscr{E})$ (resp. by $\mathrm{RP}_1(\mathscr{E})$). By Lemma 4.1, it equals $\varphi(\mathrm{LpreP}_1(\mathscr{E}))$ (resp. $\varphi(\mathrm{RpreP}_1(\mathscr{E})))$. If $l > 1$, the occurrence between $\mathrm{LpreP}_1(\mathscr{E}_l)$ and $\mathrm{C}_1(\mathscr{E}_l)$ (resp. between $\mathrm{C}_1(\mathscr{E}_l)$ and $\mathrm{RpreP}_1(\mathscr{E}_l)$) is called the *left (resp. right) regular part* of $\mathscr{E}_l$ and is denoted by $\mathrm{LR}_1(\mathscr{E}_l)$ (resp. by $\mathrm{RR}_1(\mathscr{E}_l)$). If $l = 2$, it is an occurrence of the empty word, and if $l \geq 3$, it is the concatenation of left atoms $\mathrm{LA}_{1,l-1}(\mathscr{E}_l)\ldots\mathrm{LA}_{1,2}(\mathscr{E}_l)$ (resp. of right atoms $\mathrm{RA}_{1,2}(\mathscr{E}_l)\ldots\mathrm{RA}_{1,l-1}(\mathscr{E}_l)$), all these atoms equal $\mathrm{LP}_1(\mathscr{E})$ (resp. $\mathrm{RP}_1(\mathscr{E})$) as abstract words.

Using this terminology, we formulate the following corollary.

Corollary 4.2. *If $\mathscr{E}$ is an evolution of 1-blocks and $l > 1$, then the 1-block $\mathscr{E}_l$ equals the following abstract word:*

$$\mathrm{LpreP}_1(\mathscr{E})\,\mathrm{LP}_1(\mathscr{E})\ldots\mathrm{LP}_1(\mathscr{E})\,\mathrm{C}_1(\mathscr{E})\,\mathrm{RP}_1(\mathscr{E})\ldots\mathrm{RP}_1(\mathscr{E})\,\mathrm{RpreP}_1(\mathscr{E}),$$

where $\mathrm{LP}_1(\mathscr{E})$ and $\mathrm{RP}_1(\mathscr{E})$ are repeated $l-2$ times each.

$\mathrm{LpreP}_1(\mathscr{E})$ *(resp.* $\mathrm{RpreP}_1(\mathscr{E})$*) is an empty word if and only if Case II holds at the left (resp. at the right) for $\mathscr{E}$.*

$\mathrm{LP}_1(\mathscr{E})$ *(resp.* $\mathrm{RP}_1(\mathscr{E})$*) is an empty word if and only if Case II holds at the left (resp. at the right) for $\mathscr{E}$.*

The left (resp. right) regular part of $\mathscr{E}_l$ consists of periodic letters of order 1 only. It is an empty word if and only if Case II holds at the left (resp. at the right) for $\mathscr{E}$ or $l = 2$. □

The terminology we introduced and the structure of a 1-block is illustrated by Fig. 4.

Definition 4.3. We call a 1-block $\mathscr{E}_l$ *stable* if $l \geq 3$, otherwise it is called *unstable*.

If a 1-block is stable, then its left and right components, preperiods and regular parts, as well as its core, are defined. The following corollary about lengths of factors inside 1-blocks follows directly from what we already know about the structure of 1-blocks and from Corollary 3.4.

Corollary 4.4. *The lengths of all unstable 1-blocks are bounded by a single constant that depends on Σ and φ only. The lengths of all cores and left and right preperiods of all stable 1-blocks are bounded by a single constant that depends on Σ and φ only.*

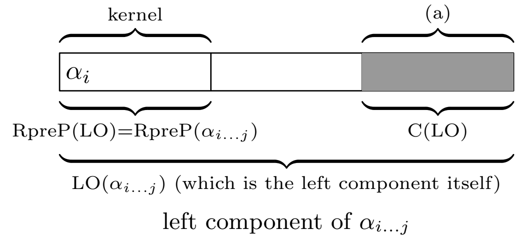

Figure 4: Detailed structure of a 1-block $\mathscr{E}_l$.

The the left (resp. right) regular part of a stable 1-block $\mathscr{E}_l$ is a nonempty word if and only if Case I holds at the left (resp. at the right). Moreover, it is completely $\mathrm{LP}_1(\mathscr{E})$-periodic (resp. $\mathrm{RP}_1(\mathscr{E})$-periodic), and the length of the left (resp. right) regular part equals $(l-2)|\,\mathrm{LP}_1(\mathscr{E})|$ (resp. $(l-2)|\,\mathrm{LP}_1(\mathscr{E})|$).

In particular, the length of the left (resp. right) regular part of a 1-block $\mathscr{E}_l$, as well as the length of the left (resp. right) component is either $\Theta(l)$ if Case I holds at the left (resp. at the right), or 0 if Case II holds at the left (resp. at the right).

The length of the whole 1-block $\mathscr{E}_l$ is always $O(l)$. It is $\Theta(l)$ if Case I holds at the left or at the right, and is $O(1)$ if Case II holds both at the left and at the right. All constants in the Θ- and O-notations in this corollary depend on Σ and φ only. $\square$

Now let us recall the definition of a strongly 1-periodic morphism with long images. Let $\mathscr{E}$ be an evolution of 1-blocks. As we already noted, $\mathrm{LB}(\mathscr{E}_{l+1}) = \mathrm{RL}_1(\varphi(\mathrm{LB}(\mathscr{E}_l)))$ for all $l \geq 0$. Moreover, suppose now that $l \geq 1$ and $\mathrm{LB}(\mathscr{E}_{l+1}) = \mathrm{RL}_1(\varphi(\mathrm{LB}(\mathscr{E}_l))) = \mathrm{LB}(\mathscr{E})$ as an abstract letter. Then $\varphi(\mathrm{LB}(\mathscr{E}_l))$ has a suffix $\mathrm{LB}(\mathscr{E}_{l+1})\,\mathrm{LpreP}_1(\mathscr{E}_{l+1})$. Hence, the word γ we used in the definition of a final period for $a = \mathrm{LB}(\mathscr{E})$ is $\mathrm{LpreP}_1(\mathscr{E})$, and $\varphi(\gamma) = \varphi(\mathrm{LpreP}_1(\mathscr{E})) = \mathrm{LP}_1(\mathscr{E})$ by Lemma 4.1 (and by the definitions of the left preperiod and the left period of an evolution). So, the following lemma follows now directly from Lemma 2.15 and from the definition of a strongly 1-periodic morphism with long images.

Lemma 4.5. *If $\mathscr{E}$ is an evolution of 1-blocks and Case I holds at the left (resp. at the right), then $\psi(\mathrm{LP}_1(\mathscr{E}))$ (resp. $\psi(\mathrm{RP}_1(\mathscr{E}))$) has a minimal complete period λ, and λ is a final period. $|\,\mathrm{LP}_1(\mathscr{E})| \geq 2\mathbf{L}$ and $|\,\mathrm{RP}_1(\mathscr{E})| \geq 2\mathbf{L}$.*

If $\mathscr{E}_l$ is a stable 1-block and Case I holds at the left (resp. at the right) for $\mathscr{E}$, then λ is the minimal complete period of $\psi(\mathrm{LR}_1(\mathscr{E}_l))$ (resp. $\psi(\mathrm{RR}_1(\mathscr{E}_l))$). $|\,\mathrm{LR}_1(\mathscr{E}_l)| \geq 2\mathbf{L}$ and $|\,\mathrm{RR}_1(\mathscr{E}_l)| \geq 2\mathbf{L}$. $\square$

The core of a stable 1-block is called its (unique) *prime central kernel*. It is also called its (unique) *composite central kernel*. If $\mathscr{E}$ is an evolution of 1-blocks and $l \geq 3$ (so that $\mathscr{E}_l$ is stable), then the prime (resp. composite) central kernel of $\mathscr{E}_{l+1}$ is called the *descendant* of the prime (resp. composite) central kernel of $\mathscr{E}_l$.

5 Stable k-Blocks

Now we are going to consider k-blocks more accurately. In this section we mostly focus on k-blocks for $k > 1$, referring to the previous section for similar results for $k = 1$. Through this section, we will give examples based on $\Sigma = \{a, \mathfrak{b}, b, \mathfrak{c}, c, \mathfrak{d}, d, \mathfrak{e}, e, \mathfrak{f}, f\}$ and on the following morphism φ: $\varphi(a) = a\mathfrak{b}\mathfrak{d}\mathfrak{b}$, $\varphi(\mathfrak{b}) = \mathfrak{cbee}$, $\varphi(b) = \mathfrak{cbee}$, $\varphi(\mathfrak{c}) = \mathfrak{eecee}$, $\varphi(c) = \mathfrak{eecee}$, $\varphi(\mathfrak{d}) = \mathfrak{ffdff}$, $\varphi(d) = \mathfrak{ffdff}$, $\varphi(\mathfrak{e}) = e$, $\varphi(e) = e$, $\varphi(\mathfrak{f}) = f$, $\varphi(f) = f$. Then $\varphi^\infty(a) = \alpha = a\,\mathfrak{b}\mathfrak{d}\mathfrak{b}\,\mathfrak{cbeeff}d\mathfrak{ff}\mathfrak{c}b\mathfrak{ee}\,\mathfrak{ee}c\mathfrak{eec}b\mathfrak{ee}e\mathfrak{e}ff\mathfrak{ff}d\mathfrak{ff}ff\mathfrak{ee}c\mathfrak{eec}b\mathfrak{eee}e\ldots$. Here a is a periodic letter of order 4, $\mathfrak{b}$ is a preperiodic letter of order 3, b is a periodic letter of order 3, $\mathfrak{c}$ and $\mathfrak{d}$ are preperiodic letters of order 2, c and d are periodic letters of order 2, $\mathfrak{e}$ and $\mathfrak{f}$ are preperiodic letters of order 1, and e and f are periodic letters of order 1. Consider an evolution $\mathscr{E}$ of 2-blocks, whose origin is $\alpha_{2\ldots 2} = \mathfrak{d}$. A 2-block $\mathscr{E}_l$ where l is large enough looks as follows:

$$\underbrace{\mathfrak{e}\mathfrak{e}ee..eff..\mathfrak{f}\mathfrak{f}\mathfrak{f}d\mathfrak{f}\mathfrak{f}ff..fee..ee\mathfrak{e}\mathfrak{e}c\mathfrak{e}\mathfrak{e}ee..ee\mathfrak{e}\mathfrak{e}}_{\text{core}}\underbrace{c\mathfrak{e}\mathfrak{e}ee..ee\mathfrak{e}\mathfrak{e}c\mathfrak{e}\mathfrak{e}ee..ee\mathfrak{e}\mathfrak{e}c\dots c\mathfrak{e}\mathfrak{e}ee..ee\mathfrak{e}\mathfrak{e}c\dots ee\mathfrak{e}\mathfrak{e}c\mathfrak{e}\mathfrak{e}ee\mathfrak{e}\mathfrak{e}c\mathfrak{e}\mathfrak{e}\mathfrak{c}}_{\text{right component}}.$$

Here Case I holds at the right and Case II holds at the left (and the left component is empty). Intervals denoted by ... may contain many intervals denoted by .. The (forgetful occurrence of) the zeroth atom is $\mathfrak{e}\mathfrak{e}ee..eff..\mathfrak{f}\mathfrak{f}\mathfrak{f}d\mathfrak{f}\mathfrak{f}ff..fee..ee\mathfrak{e}\mathfrak{e}$, the (forgetful occurrence of) the mth right atom, where $0 < m < l-1$ is of the form $c\mathfrak{e}\mathfrak{e}e..e\mathfrak{e}\mathfrak{e}$, where e is repeated $4(l-m)-2$ times, the (forgetful occurrence of) the $(l-1)$th right atom is $c\mathfrak{e}\mathfrak{e}$, and the (forgetful occurrence of) the lth right atom is $\mathfrak{c}$, the lth atom itself also includes the empty 1-block located immediately to the right of this $\mathfrak{c}$.

First, let us define *stable k-blocks*.

Definition 5.1. A k-block is called *stable* if its evolutional sequence number is at least $3k$.

(For $k = 1$ we get exactly the definition from the previous section.) Let $\mathscr{E} = \mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2, \dots$ be an evolution of k-blocks. If $\mathscr{E}_l$ is a stable k-block (i. e. if $l \geq 3k$), the concatenation of atoms $\mathrm{LA}_{k,l}(\mathscr{E}_l)\,\mathrm{LA}_{k,l-1}(\mathscr{E}_l)\dots\mathrm{LA}_{k,l-3k+3}(\mathscr{E}_l)$ (resp. $\mathrm{RA}_{k,l-3k+3}(\mathscr{E}_l)\dots\mathrm{RA}_{k,l-1}(\mathscr{E}_l)\,\mathrm{RA}_{k,l}(\mathscr{E}_l)$) is called the *left (resp. right) preperiod* of $\mathscr{E}_l$ and is denoted by $\mathrm{LpreP}_k(\mathscr{E}_l)$ (resp. by $\mathrm{RpreP}_k(\mathscr{E}_l)$). The concatenation of all atoms between the left preperiod and the core (resp. between the core and the right preperiod), i. e. the concatenation $\mathrm{LA}_{k,l-3k+2}(\mathscr{E}_l)\,\mathrm{LA}_{k,l-3k+1}(\mathscr{E}_l)\dots\mathrm{LA}_{k,2}(\mathscr{E}_l)$ (resp. $\mathrm{RA}_{k,2}(\mathscr{E}_l)\dots\mathrm{RA}_{k,l-3k+1}(\mathscr{E}_l)\,\mathrm{RA}_{k,l-3k+2}(\mathscr{E}_l)$) is called the *left (resp. right) regular part* of $\mathscr{E}_l$. It is denoted by $\mathrm{LR}_k(\mathscr{E}_l)$ (resp. by $\mathrm{RR}_k(\mathscr{E}_l)$). Again, these definitions for $k = 1$ coincide with the definition from the previous section. The following remark is a particular case of Corollary 3.17.

Remark 5.2. *If $\mathscr{E}_l$ is a stable k-block, then* $\mathrm{Fg}(\mathrm{LpreP}_k(\mathscr{E}_l))$ *and* $\mathrm{Fg}(\mathrm{RpreP}_k(\mathscr{E}_l))$ *do not depend on l as abstract words if $l \geq 3k$.*

So, we call the abstract word $\mathrm{Fg}(\mathrm{LpreP}_k(\mathscr{E}_l))$ (resp. $\mathrm{Fg}(\mathrm{RpreP}_k(\mathscr{E}_l))$) for any $l \geq 3k$ *the left (resp. right) preperiod of* $\mathscr{E}$ and denote it by $\mathrm{LpreP}_k(\mathscr{E})$ (resp. by $\mathrm{RpreP}_k(\mathscr{E})$). In the example above, $\mathrm{LpreP}_k(\mathscr{E})$ is empty since Case II holds for $\mathscr{E}$ at the left, and $\mathrm{RpreP}_k(\mathscr{E}) = c\mathfrak{e}\mathfrak{e}eeeeee\mathfrak{e}\mathfrak{e}c\mathfrak{e}\mathfrak{e}ee\mathfrak{e}\mathfrak{e}c\mathfrak{e}\mathfrak{e}\mathfrak{c}$.

Corollary 5.3. *The lengths of all left and right preperiods of all evolutions of k-blocks arising in α are bounded by a single constant that depends on Σ, φ, and k only. In particular, only finitely many abstract words can equal left and right preperiods of evolutions of k-blocks arising in α.*

Proof. By Corollary 3.4, only finitely many sequences of abstract words can be evolutions in α. Therefore, there exists a single constant x that depends on Σ, φ and k only such that if $\mathscr{E}$ is an evolution of k-blocks, then $|\mathscr{E}_{3k}| \leq x$. By Remark 5.2, $\mathrm{LpreP}_k(\mathscr{E})$ and $\mathrm{RpreP}_k(\mathscr{E})$ are factors of $\mathscr{E}_{3k}$, so $|\mathrm{LpreP}_k(\mathscr{E})| \leq x$ and $|\mathrm{RpreP}_k(\mathscr{E})| \leq x$. ☐

Note that we ***do not*** claim that if $\mathscr{E}$ and $\mathscr{E}'$ are two evolutions of k-blocks such that $\mathscr{E}_l = \mathscr{E}'_l$ as an abstract word for all $l \geq 0$, then $\mathrm{LpreP}_k(\mathscr{E}) = \mathrm{LpreP}_k(\mathscr{E}')$ and $\mathrm{RpreP}_k(\mathscr{E}) = \mathrm{RpreP}_k(\mathscr{E}')$.

Now let us prove some facts about atoms inside the regular parts of a stable k-block (or about atoms of the form $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$, where m is large enough).

Lemma 5.4. *Let $\mathscr{E}$ be an evolution of k-blocks. If $l \geq 1$ and $m \geq 1$, then all letters of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$ (resp. in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$) are periodic, and there is at least one such letter if Case I holds at the left (resp. at the right). If $m \geq 1$, then all letters of order k in $\mathrm{A}_{k,0}(\mathscr{E}_m)$ are periodic.*

Proof. For $k = 1$ we already know this by Lemma 4.1. Suppose that $k > 1$. By the definition of the descendant of a ($(k-1)$-multiblock that consists of a) single letter α_i of order $> (k-1)$, it is a $(k-1)$-multiblock that consists of $(k-1)$-blocks and letters of order $> (k-1)$ inside $\varphi(\alpha_i)$. Hence, all letters of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+1})$ and in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+1})$ ($l \geq 1$) are periodic since they are contained in the images of letters of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ and in $\mathrm{RA}_{k,l}(\mathscr{E}_l)$. If Case I holds at the left (resp. at the right), then there is at least one letter of order k in in $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ (resp. in $\mathrm{RA}_{k,l}(\mathscr{E}_l)$), and its descendant gives at least one letter of order k for $\mathrm{LA}_{k,l}(\mathscr{E}_{l+1})$ (resp. for $\mathrm{RA}_{k,l}(\mathscr{E}_{l+1})$). Also, all letters of order k in $\mathrm{A}_{k,0}(\mathscr{E}_1)$ are periodic since they are contained in the images of letters of order k in $\mathrm{A}_{k,0}(\mathscr{E}_0)$. Now the claim follows from Remark 3.7 and the definition of a left (right, zeroth) atom. ☐

Corollary 5.5. *Let $\mathscr{E}$ be an evolution of k-blocks. If $l \geq 2$ and $m \geq 1$, then the amounts of letters of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$ and in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$ do not depend on l and m. The amounts of letters of order k in $\mathrm{LA}_{k,1}(\mathscr{E}_{1+m})$, in $\mathrm{RA}_{k,1}(\mathscr{E}_{1+m})$ and in $\mathrm{A}_{k,0}(\mathscr{E}_m)$ do not depend on m (but may differ from the amounts of letters of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$ and in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$ for $l \geq 2$).*

Proof. For $k = 1$ this follows from Lemma 4.1. If $k > 1$, then the fact that these amounts do not depend on m follows now from Remark 3.7, and the fact that they do not depend on l if $l \geq 2$ follows from Corollaries 3.18 (for left atoms) and 3.19 (for right atoms). ☐

Corollary 5.6. *Let $\mathscr{E}$ be an evolution of k-blocks, where $k > 1$. If $l \geq 2$ and $m \geq 1$, then the amounts of (possibly empty) $(k-1)$-blocks in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$ and in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$ do not depend on l and m and equal the amounts of letters of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$ and in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$, respectively. The amounts of (possibly empty) $(k-1)$-blocks in $\mathrm{LA}_{k,1}(\mathscr{E}_{1+m})$ and in $\mathrm{RA}_{k,1}(\mathscr{E}_{1+m})$ do not depend on m and equal the amounts of letters of order k in $\mathrm{LA}_{k,1}(\mathscr{E}_{1+m})$ and in $\mathrm{RA}_{k,1}(\mathscr{E}_{1+m})$, respectively (but may differ from the amounts of $(k-1)$-blocks in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$ and in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$ for $l \geq 2$).*

Proof. Recall that by Remark 3.13, a left (resp. right) atom is either an empty $(k-1)$-multiblock, or it begins with a (possibly empty) $(k-1)$-block (resp. with a single letter of order k) and ends with a single letter of order k (resp. with a (possibly empty) $(k-1)$-block). It follows from the general definition of a $(k-1)$-multiblock that $(k-1)$-blocks and letters of order $> (k-1)$ always alternate inside a $(k-1)$-multiblock. Hence, the amount of letters of order k in a left (resp. right) atom is always the same as the amount of $(k-1)$-blocks in it. ☐

Corollary 5.7. *Let $\mathscr{E}$ be an evolution of k-blocks, where $k > 1$. If $m \geq 1$, then the amount of (possibly empty) $(k-1)$-blocks in $\mathrm{A}_{k,0}(\mathscr{E}_m)$ does not depend on m and equals one plus the amount of letters of order k in $\mathrm{A}_{k,0}(\mathscr{E}_m)$.*

Proof. By Remark 3.14, the zeroth atom either consists of a single (possibly empty) $(k-1)$-block, or it begins with a (possibly empty) $(k-1)$-block and ends with another (possibly empty) $(k-1)$-block. Again, it follows from the general definition of a $(k-1)$-multiblock that $(k-1)$-blocks and letters of order $> (k-1)$ always alternate inside a $(k-1)$-multiblock. Hence, the amount of $(k-1)$-blocks in the zeroth atom always equals one plus the amount of letters of order k in it. ☐

Lemma 5.8. *Let $\mathscr{E}$ be an evolution of k-blocks, where $k > 1$. Let $m \geq 1$. Let $\alpha_{i\dots j}$ be a $(k-1)$-block in a left atom $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$, in a right atom $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$, or in a zeroth atom $\mathrm{A}_{k,0}(\mathscr{E}_m)$. Then the evolutional sequence number of $\alpha_{i\dots j}$ is either m or $m-1$.*

Proof. Observe first that all $(k-1)$-blocks in $\mathrm{LA}_{k,l}(\mathscr{E}_l)$ and in $\mathrm{RA}_{k,l}(\mathscr{E}_l)$ are origins since by Lemma 3.15 they are contained in $\varphi(\mathrm{LB}(\mathscr{E}_{l-1}))$ and $\varphi(\mathrm{RB}(\mathscr{E}_{l-1}))$, respectively, and cannot be prefixes or suffixes of $\varphi(\mathrm{LB}(\mathscr{E}_{l-1}))$ and $\varphi(\mathrm{RB}(\mathscr{E}_{l-1}))$, respectively. Also, all $(k-1)$-blocks in $\mathrm{A}_{k,0}(\mathscr{E}_0)$ are origins since $\mathrm{Fg}(\mathrm{A}_{k,0}(\mathscr{E}_0)) = \mathscr{E}_0$ is an origin, so it is contained in the image of a single letter and cannot be a prefix or a suffix there.

Now, let $\alpha_{i\dots j}$ be a $(k-1)$-block in $\mathrm{LA}_{k,l+1}(\mathscr{E}_l) = \mathrm{Dc}_{k-1}(\mathrm{LA}_{k,l}(\mathscr{E}_l))$, in $\mathrm{RA}_{k,l+1}(\mathscr{E}_l) = \mathrm{Dc}_{k-1}(\mathrm{RA}_{k,l}(\mathscr{E}_l))$, or in $\mathrm{A}_{k,0}(\mathscr{E}_1) = \mathrm{Dc}_{k-1}(\mathrm{A}_{k,0}(\mathscr{E}_0))$. Then there are two possibilities for $\alpha_{i\dots j}$: The first possibility is that $\alpha_{i\dots j}$ is the descendant of a $(k-1)$-block in $\mathrm{LA}_{k,l}(\mathscr{E}_l)$, in $\mathrm{RA}_{k,l}(\mathscr{E}_l)$, or in $\mathrm{A}_{k,0}(\mathscr{E}_0)$, respectively, and then the evolutional sequence number of $\alpha_{i\dots j}$ is 1. The second possibility is that $\alpha_{i\dots j}$ is a suboccurrence of the descendant of a letter α_s of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_l)$, in $\mathrm{RA}_{k,l}(\mathscr{E}_l)$, or in $\mathrm{A}_{k,0}(\mathscr{E}_0)$, respectively. It follows from the definition of the descendant of a $(k-1)$-multiblock consisting of a single letter of order $> (k-1)$ only, that in this case $\alpha_{i\dots j}$ is a suboccurrence of $\varphi(\alpha_s)$, and it cannot be a prefix or a suffix of $\varphi(\alpha_s)$. Then $\alpha_{i\dots j}$ is an origin, and its evolutional sequence number is 0.

Finally, we do induction on m. By Lemma 5.4, all letters of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$, in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$, and in $\mathrm{A}_{k,0}(\mathscr{E}_m)$ are periodic. Then it follows from Remark 3.7 that all $(k-1)$-blocks in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m+1})$, in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m+1})$, and in $\mathrm{A}_{k,0}(\mathscr{E}_{m+1})$ are the descendants of $(k-1)$-blocks in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$, in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$, and in $\mathrm{A}_{k,0}(\mathscr{E}_m)$, respectively. So, if $\alpha_{i\dots j}$ is a $(k-1)$-block in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m+1})$, in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m+1})$, or in $\mathrm{A}_{k,0}(\mathscr{E}_{m+1})$, then $\mathrm{Dc}_{k-1}^{-1}(\alpha_{i\dots j})$ is a $(k-1)$-block in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$, in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$, or in $\mathrm{A}_{k,0}(\mathscr{E}_m)$. By induction hypothesis, the evolutional sequence number of $\mathrm{Dc}_{k-1}^{-1}(\alpha_{i\dots j})$ is m or $m-1$, so the evolutional sequence number of $\alpha_{i\dots j}$ is $m+1$ or m. □

Note that a $(k-1)$-block with evolutional sequence number $m-1$ can appear in the lth atom of $\mathscr{E}_{l+m}$ only if there is a letter b of order k in the lth atom of $\mathscr{E}_l$ whose image contains several letters of order k. In particular, b must be preperiodic. In our example of an evolution of 2-blocks, the lth atoms of blocks $\mathscr{E}_l$ contain preperiodic letters of order 1, but their images contain only one letter of order 1, so in our example, each $(k-1)$-block the lth atom of $\mathscr{E}_{l+m}$ has evolutional sequence number m, not $m-1$.

Corollary 5.9. *If $\mathscr{E}_l$ is a stable k-block, where $k > 1$, then all letters of order k in $\mathrm{LR}_k(\mathscr{E}_l)$, in $\mathrm{RR}_k(\mathscr{E}_l)$ and in $\mathrm{C}_k(\mathscr{E}_l)$ are periodic, and all $(k-1)$-blocks in $\mathrm{LR}_k(\mathscr{E}_l)$, in $\mathrm{RR}_k(\mathscr{E}_l)$ and in $\mathrm{C}_k(\mathscr{E}_l)$ are stable.* □

Lemma 5.10. *Let $\mathscr{E}$ be an evolution of k-blocks ($k > 1$) such that Case I holds at the left. Let $l \geq 2$ and $m \geq 2$. There exists a $(k-1)$-block $\alpha_{i\dots j}$ in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$ such that Case I holds at the left or at the right for the evolution of $\alpha_{i\dots j}$.*

Proof. By Lemma 5.4, there is at least one letter of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$, and all these letters of order k are periodic.

Let us first assume that there are at least two periodic letters of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$. Let α_s be the leftmost of these letters. Then by Remark 3.13, $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$ contains a $(k-1)$-block of the form $\alpha_{i\dots s-1}$ and a $(k-1)$-block of the form $\alpha_{s+1\dots j}$. We have $\mathrm{RB}(\alpha_{i\dots s-1}) = \alpha_s$ and $\mathrm{LB}(\alpha_{s+1\dots j}) = \alpha_s$. By Lemma 5.8, the evolutional sequence numbers of these blocks are at least 1, and the evolutional sequence numbers of the blocks $\mathrm{Dc}_{k-1}(\alpha_{i\dots s-1})$ and $\mathrm{Dc}_{k-1}(\alpha_{s+1\dots j})$ are at least 2. Since α_s is a periodic letter of order k, $\varphi(\alpha_s)$ contains at least one letter of order $k-1$. If it is located to the left from (the unique) letter of order k in $\varphi(\alpha_s)$, then by Remark 3.12 the nth right atom of $\mathrm{Dc}_{k-1}(\alpha_{i\dots s-1})$, where $n \geq 2$ is the evolutional sequence number of $\mathrm{Dc}_{k-1}(\alpha_{i\dots s-1})$, contains a letter of order $k-1$, and Case I holds at the right for the evolution of $\alpha_{i\dots s-1}$. Similarly, if there is a letter of order $k-1$ in $\varphi(\alpha_s)$ located to the right from the letter of order k in $\varphi(\alpha_s)$, then by Remark 3.11, Case I holds at the left for the evolution of $\alpha_{s+1\dots j}$.

Now suppose that there is exactly one periodic letter of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$. By Corollary 5.5, $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$ contains exactly one $(k-1)$-block, denote it by $\alpha_{i\dots j}$. By Lemma 5.4 and by Corollaries 5.5 and 5.6, $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m-1})$ also consists of one $(k-1)$-block and one periodic letter of order k. Moreover, the periodic letter of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m}) = \mathrm{Dc}_{k-1}(\mathrm{LA}_{k,l}(\mathscr{E}_{l+m-1}))$ is the descendant of the periodic letter of order k in $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m-1})$, so they coincide as abstract letters. Recall also that by Remark 3.13, these letters of order k are the rightmost letters in the forgetful occurrences of $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m-1})$ and of $\mathrm{LA}_{k,l}(\mathscr{E}_{l+m})$. By Corollary 3.18, the rightmost letter in $\mathrm{Fg}(\mathrm{LA}_{k,l+1}(\mathscr{E}_{l+m}))$ is the same letter of order k. Therefore, $\mathrm{LB}(\alpha_{i\dots j})$ and $\mathrm{RB}(\alpha_{i\dots j})$ coincide as abstract letters, denote this abstract letter by $b \in \Sigma$. By Lemma 5.8, the evolutional sequence number of $\alpha_{i\dots j}$ is at least 1. So, again, $\varphi(b)$ contains at least one letter of order $k-1$. If it is located to the left (resp. to the right) of the unique occurrence of b in $\varphi(b)$, then by Remark 3.12 (resp. 3.11) and by the definition of Case I, Case I holds at the right (resp. at the left) for the evolution of $\mathrm{Dc}_{k-1}(\alpha_{i\dots j})$, i. e. for the evolution of $\alpha_{i\dots j}$. □

Lemma 5.11. *Let $\mathscr{E}$ be an evolution of k-blocks ($k > 1$) such that Case I holds at the right. Let $l \geq 2$ and $m \geq 2$. There exists a $(k-1)$-block $\alpha_{i\dots j}$ in $\mathrm{RA}_{k,l}(\mathscr{E}_{l+m})$ such that Case I holds at the left or at the right for the evolution of $\alpha_{i\dots j}$.*

Proof. The proof is completely similar to the proof of the previous lemma. □

Lemma 5.12. *Let $\mathscr{E}$ be an evolution of k-blocks such that Case I holds at the left (resp. at the right). Let $\mathscr{E}_l$ be a stable k-block. Then $|\operatorname{Fg}(\mathrm{LR}_k(\mathscr{E}_l))| \geq 2\mathbf{L}$ (resp. $|\operatorname{Fg}(\mathrm{RR}_k(\mathscr{E}_l))| \geq 2\mathbf{L}$).*

Proof. For $k = 1$ we already know this by Lemma 4.5. For $k > 1$ note first that

$$\mathrm{LR}_k(\mathscr{E}_l) = \mathrm{LA}_{k,l-3k+2}(\mathscr{E}_l)\,\mathrm{LA}_{k,l-3k+1}(\mathscr{E}_l)\ldots\mathrm{LA}_{k,2}(\mathscr{E}_l)$$

$$(\text{resp. } \mathrm{RR}_k(\mathscr{E}_l) = \mathrm{RA}_{k,2}(\mathscr{E}_l)\ldots\mathrm{RA}_{k,l-3k+1}(\mathscr{E}_l)\,\mathrm{RA}_{k,l-3k+2}(\mathscr{E}_l))$$

is nonempty since $l \geq 3k$, and $l - 3k + 2 \geq 2$. Now the claim follows by induction on k from Corollary 5.9 and from Lemmas 5.10 and 5.11. □

Lemma 5.13. *Let $\mathscr{E}$ be an evolution of k-blocks such that Case I holds at the left (resp. at the right). Let $\mathscr{E}_l$ be a stable k-block. Then $|\operatorname{Fg}(\mathrm{LR}_k(\mathscr{E}_{l+1}))| > |\operatorname{Fg}(\mathrm{LR}_k(\mathscr{E}_l))|$ (resp. $|\operatorname{Fg}(\mathrm{RR}_k(\mathscr{E}_{l+1}))| > |\operatorname{Fg}(\mathrm{RR}_k(\mathscr{E}_l))|$).*

Proof. We can write

$$\mathrm{LR}_k(\mathscr{E}_l) = \mathrm{LA}_{k,l-3k+2}(\mathscr{E}_l)\,\mathrm{LA}_{k,l-3k+1}(\mathscr{E}_l)\ldots\mathrm{LA}_{k,2}(\mathscr{E}_l)$$

$$(\text{resp. } \mathrm{RR}_k(\mathscr{E}_l) = \mathrm{RA}_{k,2}(\mathscr{E}_l)\ldots\mathrm{RA}_{k,l-3k+1}(\mathscr{E}_l)\,\mathrm{RA}_{k,l-3k+2}(\mathscr{E}_l))$$

and

$$\mathrm{LR}_k(\mathscr{E}_{l+1}) = \mathrm{LA}_{k,l+1-3k+2}(\mathscr{E}_{l+1})\,\mathrm{LA}_{k,l+1-3k+1}(\mathscr{E}_{l+1})\ldots\mathrm{LA}_{k,2}(\mathscr{E}_{l+1})$$

$$(\text{resp. } \mathrm{RR}_k(\mathscr{E}_{l+1}) = \mathrm{RA}_{k,2}(\mathscr{E}_{l+1})\ldots\mathrm{RA}_{k,l+1-3k+1}(\mathscr{E}_{l+1})\,\mathrm{RA}_{k,l+1-3k+2}(\mathscr{E}_{l+1})).$$

By Corollary 3.17,

$$\operatorname{Fg}(\mathrm{LA}_{k,l-3k+2}(\mathscr{E}_l)\ldots\mathrm{LA}_{k,2}(\mathscr{E}_l)) \text{ and } \operatorname{Fg}(\mathrm{LA}_{k,l+1-3k+2}(\mathscr{E}_{l+1})\ldots\mathrm{LA}_{k,3}(\mathscr{E}_{l+1}))$$

$$(\text{resp. } \operatorname{Fg}(\mathrm{RA}_{k,2}(\mathscr{E}_l)\ldots\mathrm{RA}_{k,l-3k+2}(\mathscr{E}_l)) \text{ and } \operatorname{Fg}(\mathrm{RA}_{k,3}(\mathscr{E}_{l+1})\ldots\mathrm{RA}_{k,l+1-3k+2}(\mathscr{E}_{l+1})))$$

coincide as abstract words, and $\mathrm{LA}_{k,2}(\mathscr{E}_{l+1})$ (resp. $\mathrm{RA}_{k,2}(\mathscr{E}_{l+1})$) contains at least one letter of order k since Case I holds at the left (resp. at the right). □

Lemma 5.14. *Let $\mathscr{E}$ be an evolution of k-blocks. Then $|\mathscr{E}_l|$ is $O(l^k)$ and $|\operatorname{Fg}(\mathrm{C}_k(\mathscr{E}_l))|$ is $O(l^{k-1})$ (for $l \to \infty$), and the constants in the O-notation depend on φ, Σ, and k only, but not on $\mathscr{E}$.*

Proof. For $k = 1$ we already know this by Corollary 4.4. For $k > 1$, we do induction on k. Without loss of generality, we may consider only the values of l grater than or equal to $3k$. By Remark 5.2, the lengths of the forgetful occurrences of $\mathrm{LpreP}_{k,l}(\mathscr{E}_l)$ and of $\mathrm{RpreP}_{k,l}(\mathscr{E}_l)$ are constants (they do not depend on l), and since the total number of different evolutions present in α, understood as sequences as abstract words, is finite (Corollary 3.4), the lengths of all left and right preperiods of all k-blocks are bounded by a single constant that depends on Σ, φ and k only.

By Corollaries 5.6 and 5.7, the amount of $(k-1)$-blocks in $\mathrm{C}_k(\mathscr{E}_l)$ does not depend on l, and using Corollary 3.4 again, we conclude that all amounts of $(k-1)$-blocks in $\mathrm{C}_k(\mathscr{E}_l)$ are bounded by a single constant that depends on Σ, φ and k only. By Lemma 5.8, the evolutional sequence numbers of these $(k-1)$-blocks can be l, $l-1$ or $l-2$ (since $\mathrm{C}_k(\mathscr{E}_l)$ is the concatenation of the zeroth and the first atoms). Similarly, it follows from Corollary 5.5 and from Corollary 3.4 that the amount of letters of order k in $\mathrm{C}_k(\mathscr{E}_l)$ is bounded by a single constant that depends on Σ, φ and k only. Therefore, it follows from the induction hypothesis for $k-1$ that $|\operatorname{Fg}(\mathrm{C}_k(\mathscr{E}_l))|$ is $O(l^{k-1})$.

Let us count the total amount of $(k-1)$-blocks in $\mathrm{LR}_k(\mathscr{E}_l)$ and in $\mathrm{RR}_k(\mathscr{E}_l)$. By Corollary 5.6, the amount of $(k-1)$-blocks in the nth left atom, if this atom is inside $\mathrm{LR}_k(\mathscr{E}_l)$ (i. e. if $2 \leq n \leq l - 3k + 2$), does not depend on l and n, denote this amount by x. Similarly, denote by y the amount of $(k-1)$-blocks in the nth right atom if this atom is inside $\mathrm{RR}_k(\mathscr{E}_l)$. The total amount of $(k-1)$-blocks in $\mathrm{LR}_k(\mathscr{E}_l)$ and in $\mathrm{RR}_k(\mathscr{E}_l)$ is

$(x+y)(l-3k+1)$. By Corollary 5.6, the total amount of letters of order k in $\mathrm{LR}_k(\mathscr{E}_l)$ and in $\mathrm{RR}_k(\mathscr{E}_l)$ is also $(x+y)(l-3k+1)$. Using Corollary 3.4 again, we can write this number as $O(l)$. By Lemma 5.8, the evolutional sequence numbers of all $(k-1)$-blocks in $\mathrm{LR}_k(\mathscr{E}_l)$ and in $\mathrm{RR}_k(\mathscr{E}_l)$ are at most $l-2$.

Now observe that it follows from the definition of the descendant of a $(k-1)$-block and from the fact that φ is nonerasing that the length of a $(k-1)$-block is less than or equal to the length of its descendant. Hence, it follows from the induction hypothesis for $k-1$ that if we have a $(k-1)$-block whose evolutional sequence number is at most l, then its length is bounded by a constant (that depends on Σ, φ and k only) multiplied by l^{k-1}. Therefore, the total length of $\mathrm{Fg}(\mathrm{LR}_k(\mathscr{E}_l))$ and $\mathrm{Fg}(\mathrm{RR}_k(\mathscr{E}_l))$ is $O(l)O(l^{k-1})+O(l)=O(l^k)$. Finally, $|\mathscr{E}_l|$ is $O(1)+O(l^{k-1})+O(l^k)=O(l^k)$. □

Lemma 5.15. *Let $\mathscr{E}$ be an evolution of k-blocks. If Case I holds at the left (resp. at the right) for $\mathscr{E}$, then $|\mathrm{Fg}(\mathrm{LR}_k(\mathscr{E}_l))|$ (resp. $|\mathrm{Fg}(\mathrm{RR}_k(\mathscr{E}_l))|$) is $\Theta(l^k)$. If Case I holds for $\mathscr{E}$ at least at one side (at the left or at the right), then $|\mathscr{E}_l|$ is $\Theta(l^k)$. The constants in the Θ-notation here depend on φ, Σ, and k only, but not on $\mathscr{E}$.*

Proof. For $k=1$ this is true by Corollary 4.4. For $k>1$ we are going to prove this by induction on k. Since we already have Lemma 5.14, it is sufficient to prove that if Case I holds at the left (resp. at the right) for $\mathscr{E}$, then $|\mathrm{Fg}(\mathrm{LR}_k(\mathscr{E}_l))|$ (resp. $|\mathrm{Fg}(\mathrm{RR}_k(\mathscr{E}_l))|$) is $\Omega(l^k)$.

By the induction hypothesis for $k-1$, there exist numbers $l_0\in\mathbb{N}$ and $x\in\mathbb{R}_{>0}$ such that if the evolutional sequence number of a $(k-1)$-block is $l\ge l_0$ and Case I holds at the left or at the right for its evolution, then the length of this $(k-1)$-block is at least xl^{k-1}. Set $l_1=6k+2l_0+4$. Suppose that we are considering a k-block $\mathscr{E}_l$ such that $l\ge l_1$. Note first that $l-\lfloor l/2\rfloor\ge l/2\ge l_1/2\ge 3k$ and $\lfloor l/2\rfloor\ge\lfloor l_1/2\rfloor\ge 2$. Hence, if Case I holds at the left (resp. at the right) for $\mathscr{E}$, then the concatenation of left atoms $\mathrm{LA}_{k,\lfloor l/2\rfloor}(\mathscr{E}_l)\ldots\mathrm{LA}_{k,2}(\mathscr{E}_l)$ (resp. $\mathrm{RA}_{k,2}(\mathscr{E}_l)\ldots\mathrm{RA}_{k,\lfloor l/2\rfloor}(\mathscr{E}_l)$) is contained in the left (resp. right) regular part of $\mathscr{E}_l$. By Lemma 5.8, the smallest possible evolutional sequence number of a $(k-1)$-block contained in one of these atoms is $l-\lfloor l/2\rfloor-1\ge l/2-1\ge l_1/2-1=3k+l_0+1\ge l_0$. By Lemma 5.10 (resp. 5.11), each of these atoms contains at least one $(k-1)$-block such that Case I holds at the left or at the right for its evolution. So, we have at least $\lfloor l/2\rfloor-2+1$ such $(k-1)$-blocks in this concatenation of atoms, and by the induction hypothesis, each of these $(k-1)$-blocks has length at least $x(l-\lfloor l/2\rfloor-1)^{k-1}$. Hence, the length of the forgetful occurrence of the whole left (resp. right) regular part is at least $x(l-\lfloor l/2\rfloor-1)^{k-1}(\lfloor l/2\rfloor-1)=\Omega(l^k)$. □

These two lemmas explain why letters of order k were called letters of order k, not letter of order $k-1$. Despite the growth rates of individual letters of order k (in the sense of their repeated images under φ) are $\Theta(l^{k-1})$, the "growth rates" of k-blocks (i. e. occurrences consisting of letters of order at most k) in the sense of their superdescendants and evolutions are $O(l^k)$ and sometimes $\Theta(l^k)$.

Now we are going to define stable k-multiblocks and (prime and composite) kernels in stable k-multiblocks. Here we also allow $k=0$.

Definition 5.16. We call a k-multiblock *stable* if it consists of periodic letters of order $k+1$ and (if $k\ge 1$) stable k-blocks only.

In particular, an empty k-multiblock is always stable. Note that it is not true in general that if l is large enough, then the lth superdescendant of a k-multiblock is stable, namely, if a k-multiblock contains a letter of order $>k+1$, then its superdescendants never become stable. On the other hand, we can say the following:

Remark 5.17. *If a k-multiblock $\alpha[\mathfrak{x},i\ldots j,\mathfrak{y}]_k$, where $\mathfrak{x},\mathfrak{y}\in\{<,>\}$, is stable, then each letter of order $>k$ in $\mathrm{Dc}_k([\mathfrak{x},i\ldots j,\mathfrak{y}]_k)$ is periodic, has order $k+1$ and is the descendant of (more precisely, is the only letter of order $>k$ or k-block in the descendant of) a letter of order k in $[\mathfrak{x},i\ldots j,\mathfrak{y}]_k$. If $\alpha[\mathfrak{x},i\ldots j,\mathfrak{y}]_k$ is stable and $k\ge 1$, then each k-block in $\mathrm{Dc}_k(\alpha[\mathfrak{x},i\ldots j,\mathfrak{y}]_k)$ is stable and is the descendant of a k-block in $[\mathfrak{x},i\ldots j,\mathfrak{y}]_k$. In other words, the operation of taking the descendant of a k-block or of a letter of order $>k$ establishes a bijection between the letters of order $>k$ and (if $k\ge 1$) the k-blocks in $[\mathfrak{x},i\ldots j,\mathfrak{y}]_k$ and the letters of order $>k$ and (if $k\ge 1$) the k-blocks in $\mathrm{Dc}_k([\mathfrak{x},i\ldots j,\mathfrak{y}]_k)$. In particular, $\mathrm{Dc}_k([\mathfrak{x},i\ldots j,\mathfrak{y}]_k)$ is also a stable k-multiblock.*

Note that if $k \geq 1$, then the requirement that all letters of order $> k$ in a stable k-multiblock are of order exactly $k+1$ and are periodic is essential in the sense that if we only know that all k-blocks in a given k-multiblock are stable, then it is not true in general that all k-blocks in its descendant are also stable. The descendants of letters of order $> (k+1)$ (in the sense of the definition of the descendant of a k-multiblock) or of preperiodic letters of order $k+1$ can contain k-blocks with evolutional sequence number 0 (i. e. origins), and they are unstable.

In particular, a stable 0-multiblock is just an occurrence in α consisting of periodic letters of order 1 only.

It will also be convenient for us now to introduce the notion of an evolution of stable k-multiblocks, but we will not introduce ancestors and origins. Instead, we say the following:

Definition 5.18. A sequence of ***stable*** k-multiblocks $\mathscr{F}_0, \mathscr{F}_1, \mathscr{F}_2, \dots$ is called an *evolution* if $\mathscr{F}_{l+1} = \mathrm{Dc}_k(\mathscr{F}_l)$ for all $l \geq 0$.

An evolution containing a given stable k-multiblock always exists (for example, one can take this block as $\mathscr{F}_0$ and set $\mathscr{F}_l = \mathrm{Dc}_k^l(\mathscr{F}_0)$ for $l > 0$), but is not necessarily unique, for example, if $\mathscr{F}$ is an evolution of k-multiblocks, then a k-multiblock $\mathscr{F}_l$ with $l > 0$ is also contained in the following evolution $\mathscr{F}'$: $\mathscr{F}'_m = \mathscr{F}_{m+1}$ for all $m \geq 0$.

We call two evolutions $\mathscr{F}$ and $\mathscr{F}'$ of stable k-multiblocks *consecutive* if $\mathscr{F}_l$ and $\mathscr{F}'_l$ are consecutive k-multiblocks for all $l \geq 0$. If $\mathscr{F}$ and $\mathscr{F}'$ are two consecutive evolutions of stable k-multiblocks, we call the evolution $\mathscr{F}''$, where $\mathscr{F}''_l$ is the concatenation of $\mathscr{F}_l$ and $\mathscr{F}'_l$, the *concatenation* of $\mathscr{F}$ and $\mathscr{F}'$.

Now we define prime kernels in a stable k-multiblock ($k \geq 0$). They will be suboccurrences in the forgetful occurrence of the k-multiblock. More precisely, we are going to define them by induction on k. The only prime kernel of a stable 0-multiblock is just the 0-multiblock itself (which is already an occurrence in α.

Suppose that $k \geq 1$. Let $\mathscr{F}_l$ be a stable k-multiblock. We say that a suboccurrence $\alpha_{s\dots t}$ in its forgetful occurrence $\mathrm{Fg}(\mathscr{F}_l)$ is a prime kernel of $\mathscr{F}_l$ if one of the following conditions holds:

1. $\alpha_{s\dots t}$ is the forgetful occurrence of the left (resp. right) preperiod of a k-block in $\mathscr{F}_l$ such that Case I holds at the left (resp. at the right) for its evolution.

2. $\alpha_{s\dots t}$ is a single letter of order $k+1$.

3. There exists a k-block $\alpha_{i\dots j}$ in $\mathscr{F}_l$ such that $\alpha_{s\dots t}$ is a prime kernel of $\mathrm{C}_k(\alpha_{i\dots j})$ (recall that the core of a k-block is by definition a $(k-1)$-multiblock, so its prime kernels are already defined by the induction hypothesis).

Lemma 5.19. *Each suboccurrence of $\mathscr{F}_l$ is listed in this list at most once, and kernels of a k-multiblock do not overlap.*

Proof. For $k = 0$ the claim is trivial since there is only one kernel in each 0-multiblock.

If $k > 0$, then a stable k-multiblock consists of periodic letters of order $k+1$ and stable k-blocks, and prime kernels are either these letters of order $k+1$ themselves (and they are listed only once), or are suboccurrences of the k-blocks. Clearly, suboccurrences of the k-blocks cannot overlap with letters of order $k+1$. Suppose that an occurrence $\alpha_{s\dots t}$ is a prime kernel contained in a k-block $\alpha_{i\dots j}$. It can be listed in the list above as a left (resp. right) preperiod only if Case I holds at the left (resp. at the right) for the evolution of $\alpha_{i\dots j}$. But then the left (resp. right) regular part of $\alpha_{i\dots j}$ is nonempty by Lemmas 4.5 and 5.12, so $\alpha_{s\dots t}$ cannot overlap or coincide with the prime kernels of $\mathrm{C}_k(\alpha_{i\dots j})$. By the induction hypothesis, the prime kernels of $\mathrm{C}_k(\alpha_{i\dots j})$ also do not overlap, and each of them is mentioned in the definition exactly once. ☐

Note that we do not claim (and this is not true in general) that prime kernels are nonempty occurrences. The most trivial counterexample is an empty 0-multiblock, i. e. an empty occurrence in α, then it is its prime kernel. A bit more general example is a 1-multiblock consisting of a single 1-block whose core is empty. This can happen if both borders of the origin of an evolution of 1-blocks are, for example, preperiodic letters of order 2, and their images consist of exactly one periodic letter of order 2 each. More generally, it can happen that the core of a k-block $\alpha_{i\ldots j}$, where $k > 1$, consists of a single empty $(k-1)$-block, then using our definition of a prime kernel, we get by induction that this empty $(k-1)$-block is a prime kernel inside $\alpha_{i\ldots j}$. Moreover, if Case II holds, say, at the right for the evolution of $\alpha_{i\ldots j}$, and the k-multiblock whose prime kernels we are defining contains the right border α_{j+1} of $\alpha_{i\ldots j}$ as well as $\alpha_{i\ldots j}$, then this empty prime kernel (which is $\alpha_{j+1\ldots j}$ in this case) and $\alpha_{j+1\ldots j+1}$ are both prime kernels. Thus, it is possible that two prime kernels are consecutive occurrences in α, and one of them is an empty occurrence. However, it is not possible (and the above lemma proves that this is not possible) that this empty occurrence is called a prime kernel twice, and this is guaranteed (in particular) by the fact that we call the forgetful occurrence of the right preperiod a prime kernel only if Case I holds at the right. Otherwise in the example above, we would have called $\alpha_{j+1\ldots j}$ a prime kernel twice: first as a prime kernel of $\mathrm{C}_k(\alpha_{i\ldots j})$ and second as the forgetful occurrence of $\mathrm{RpreP}_k(\alpha_{i\ldots j})$.

One more remark we make about this definition is the following.

Remark 5.20. *An empty 0-multiblock (i. e. an empty occurrence in α) has a prime kernel, which is the 0-multiblock itself, while an empty k-multiblock for $k > 0$ (which is really empty in the k-multiblock sense, i. e. it does not contain any letters of order $> k$ and k-blocks, even empty ones) does not have any prime kernels according to this definition. However, if a k-multiblock, where $k > 0$, consists of a single empty k-block (which must be stable, otherwise prime kernels are not defined anyway) does have one prime kernel, which is this k-block itself.*

For example, let us find all prime kernels in the 2-multiblock consisting of a single stable 2-block in the example above. The core of this 2-block consists the following 1-blocks and letters of order 2:

1. A 1-block $\mathfrak{e}\mathfrak{e}ee..eff..f\mathfrak{f}\mathfrak{f}$, where the amount of letters e equals the amount of letters f and is at least 10 if the ambient 2-block is stable.
2. A letter d of order 2.
3. A 1-block $\mathfrak{f}\mathfrak{f}ff..fee..ee\mathfrak{e}\mathfrak{e}$, where the amount of letters e equals two plus the amount of letters f. There are at least 8 letters f if the ambient 2-block is stable.
4. A letter c of order 2.
5. A 1-block $\mathfrak{e}\mathfrak{e}ee..ee\mathfrak{e}\mathfrak{e}$, where e is repeated an even number of times, and at least 18 times if the 2-block is stable.

So, the prime kernels of the 2-multiblock are:

1. The left preperiod of the 1-block $\mathfrak{e}\mathfrak{e}ee..eff..f\mathfrak{f}\mathfrak{f}$, which is $\mathfrak{e}\mathfrak{e}$.
2. The core of this 1-block, which is the occurrence $eeff$ in the middle.
3. The right preperiod of this 1-block, which is $\mathfrak{f}\mathfrak{f}$.
4. The letter d of order 2.
5. The left preperiod of the 1-block $\mathfrak{f}\mathfrak{f}ff..fee..ee\mathfrak{e}\mathfrak{e}$, which is $\mathfrak{f}\mathfrak{f}$.
6. The core of this 1-block, which is the rightmost occurrence of the word ff in this 1-block.
7. The right preperiod of this 1-block, which is $\mathfrak{e}\mathfrak{e}$.
8. The letter c of order 2.

9. The left preperiod of the 1-block $\mathfrak{ee}ee..ee\mathfrak{ee}$, which is $\mathfrak{ee}$.

10. The core of this 1-block, which is the occurrence of the word ee located exactly in the middle of this 1-block (recall that the number of letters e in this 1-block is even).

11. The right preperiod of this 1-block, which is $\mathfrak{ee}$.

12. The right preperiod of the whole 2-block, which is $c\mathfrak{ee}eeeeee\mathfrak{ee}c\mathfrak{ee}ee\mathfrak{ee}c\mathfrak{eec}$.

Now we define *descendants* of prime kernels of a stable k-multiblock $\mathscr{F}_l$, where $\mathscr{F}$ is an evolution of stable k-multiblocks. In general, they are prime kernels of $\mathscr{F}_{l+1} = \mathrm{Dc}_k(\mathscr{F}_l)$. More precisely, if $k = 0$, then the only prime kernel of $\mathscr{F}_l$ is $\mathscr{F}_l$ itself, and we say that $\mathscr{F}_{l+1} = \varphi(\mathscr{F}_l)$ is the descendant of the prime kernel $\mathscr{F}_l$. For $k > 0$, we define the descendants of prime kernels by induction on k. Let $\alpha_{s\ldots t}$ be a prime kernel of $\mathscr{F}_l$. Consider three cases (we can do that since we know the statement of Lemma 5.19).

If $\alpha_{s\ldots t}$ is the forgetful occurrence of the left (resp. right) preperiod of a k-block $\alpha_{i\ldots j}$ contained in $\mathscr{F}_l$ and such that Case I holds at the left (resp. at the right) for its evolution, then we say that the descendant of the prime kernel $\alpha_{s\ldots t}$ is the forgetful occurrence of $\mathrm{LpreP}_k(\mathrm{Dc}_k(\alpha_{i\ldots j}))$ (resp. of $\mathrm{RpreP}_k(\mathrm{Dc}_k(\alpha_{i\ldots j}))$).

If $\alpha_{s\ldots t}$ is a single periodic letter of order $k+1$, then its descendant in the sense of k-multiblocks, i. e. the only periodic letter of order $k+1$ in $\varphi(\alpha_{s\ldots t})$, is called the descendant of $\alpha_{s\ldots t}$ as a prime kernel.

Finally, if $\alpha_{s\ldots t}$ is a prime kernel of the $(k-1)$-block $\mathrm{C}_k(\alpha_{i\ldots j})$, where $\alpha_{i\ldots j}$ is a k-block inside $\mathscr{F}_l$, then the descendant of $\alpha_{s\ldots t}$ as of a prime kernel of a $(k-1)$-multiblock is already defined by the induction hypothesis (and is a suboccurrence in $\mathrm{Fg}(\mathrm{Dc}_{k-1}(\mathrm{C}_k(\alpha_{i\ldots j}))) = \mathrm{Fg}(\mathrm{C}_k(\mathrm{Dc}_k(\alpha_{i\ldots j}))))$, and we say that the descendant of $\alpha_{s\ldots t}$ as of a prime kernel of $\mathscr{F}_l$ is the same suboccurrence of $\mathrm{Fg}(\mathrm{Dc}_{k-1}(\mathrm{C}_k(\alpha_{i\ldots j})))$.

The descendant of a prime kernel $\alpha_{s\ldots t}$ of a stable k-multiblock is denoted by $\mathrm{kDc}_k(\alpha_{s\ldots t})$.

Remark 5.21. *A trivial induction on k shows that the operation of taking the descendant of a prime kernel establishes a bijection between the prime kernels of a stable k-multiblock and the prime kernels of its descendant.*

Lemma 5.22. *Let $\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k$, where $\mathfrak{x}, \mathfrak{y} \in \{<, >\}$, be a stable k-multiblock. If $k > 0$, suppose also that it is nonempty in the k-multiblock sense. Then there exists a prime kernel of $\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k$ of the form $\alpha_{i\ldots i'}$.*

If $\mathrm{Dc}_k(\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k) = \alpha[\mathfrak{x}, s\ldots t, \mathfrak{y}]_k$, then $\mathrm{kDc}_k(\alpha_{i\ldots i'})$ is a prime kernel of the form $\alpha_{s\ldots s'}$.

Proof. For $k = 0$ the claim follows directly from the definition. Suppose that $k > 0$. We use induction on k.

If the leftmost k-block or letter of order $k+1$ contained in $\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k$ is actually a letter of order $k+1$, then this letter is α_i, and it is a prime kernel itself. If $\mathrm{Dc}_k(\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k) = \alpha[\mathfrak{x}, s\ldots t, \mathfrak{y}]_k$, then α_s is a periodic letter of order $k+1$, and $\mathrm{kDc}_k(\alpha_i) = \alpha_s$.

If the leftmost k-block or letter of order $k+1$ contained in $\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k$ is actually a k-block, denote it by $\alpha_{i\ldots i''}$. If Case I holds at the left for the evolution of $\alpha_{i\ldots i''}$, then the forgetful occurrence of $\mathrm{LpreP}_k(\alpha_{i\ldots i''})$ is a prime kernel, and it starts from position i in α. Again, if $\mathrm{Dc}_k(\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k) = \alpha[\mathfrak{x}, s\ldots t, \mathfrak{y}]_k$, then $\mathrm{kDc}_k(\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{i\ldots i''}))) = \mathrm{Fg}(\mathrm{LpreP}_k(\mathrm{Dc}_k(\alpha_{i\ldots i''})))$ starts from position s in α.

If Case II holds at the left, then the left component of $\alpha_{i\ldots i''}$ is empty, and the core of $\alpha_{i\ldots i''}$ is a $(k-1)$-multiblock of the form $\alpha[<, i\ldots i''', >]_{k-1}$ for some i''', and, by Remark 3.14, it is a non-empty $(k-1)$-multiblock if $k-1 > 0$. By the induction hypothesis, there exists a prime kernel of $\alpha[<, i\ldots i''', >]_{k-1}$ of the form $\alpha_{i\ldots i'}$. Then $\alpha_{i\ldots i'}$ is also a kernel of $\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k$. And again, if $\mathrm{Dc}_k(\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k) = \alpha[\mathfrak{x}, s\ldots t, \mathfrak{y}]_k$, then $\mathrm{Dc}_{k-1}(\alpha[<, i\ldots i''', >]_{k-1}) = \alpha[<, s\ldots s''', >]_{k-1}$ for some s''', and by the induction hypothesis, $\mathrm{kDc}_{k-1}(\alpha_{i\ldots i'})$ is a prime kernel of $\alpha[<, s\ldots s''', >]_{k-1}$ of the form $\alpha_{s\ldots s'}$ for some s'. By the definition of the descendant of a prime kernel in this case, we also have $\mathrm{kDc}_k(\alpha_{i\ldots i'}) = \alpha_{s\ldots s'}$. $\square$

Lemma 5.23. *Let $\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k$, where $\mathfrak{x}, \mathfrak{y} \in \{<, >\}$, be a stable k-multiblock. If $k > 0$, suppose also that it is nonempty in the k-multiblock sense. Then there exists a prime kernel of $\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k$ of the form $\alpha_{j'\ldots j}$.*

If $\mathrm{Dc}_k(\alpha[\mathfrak{x}, i\ldots j, \mathfrak{y}]_k) = \alpha[\mathfrak{x}, s\ldots t, \mathfrak{y}]_k$, then $\mathrm{kDc}_k(\alpha_{j'\ldots j})$ is a prime kernel of the form $\alpha_{t'\ldots t}$.

Proof. The proof is completely symmetric to the proof of the previous lemma. □

Lemma 5.24. *If $\alpha_{s\dots t}$ is a prime kernel of a stable k-multiblock $\mathscr{F}_l$, then $\mathrm{kDc}_k(\alpha_{s\dots t})$ coincides with $\alpha_{s\dots t}$ as an abstract word.*

Proof. For $k = 0$ this is true by the definitions of a periodic letter of order 1 and a weakly 1-periodic morphism. If $k > 0$, we again use induction on k. Consider the three cases from the definition of a prime kernel.

If there exists a k-block $\alpha_{i\dots j}$ in $\mathscr{F}_l$ such that $\alpha_{s\dots t} = \mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{i\dots j}))$ (resp. $\alpha_{s\dots t} = \mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{i\dots j}))$) and Case I holds at the left (resp. at the right) for the evolution of $\alpha_{i\dots j}$, then $\mathrm{kDc}_k(\alpha_{s\dots t}) = \mathrm{Fg}(\mathrm{LpreP}_k(\mathrm{Dc}_k(\alpha_{i\dots j})))$ (resp. $\mathrm{kDc}_k(\alpha_{s\dots t}) = \mathrm{Fg}(\mathrm{RpreP}_k(\mathrm{Dc}_k(\alpha_{i\dots j})))$). By Remark 5.2, $\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{i\dots j}))$ and $\mathrm{Fg}(\mathrm{LpreP}_k(\mathrm{Dc}_k(\alpha_{i\dots j})))$ (resp. $\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{i\dots j}))$ and $\mathrm{Fg}(\mathrm{RpreP}_k(\mathrm{Dc}_k(\alpha_{i\dots j})))$) coincide as abstract words.

If $\alpha_{s\dots t}$ is a single periodic letter of order $k+1$, then by Remark 3.7, $\mathrm{kDc}_k(\alpha_{s\dots t})$ is also a single letter, and it coincides with $\alpha_{s\dots t}$ as an abstract letter.

Finally, if there is a k-block $\alpha_{i\dots j}$ in $\mathscr{F}_l$ such that $\alpha_{s\dots t}$ is a kernel of $\mathrm{C}_k(\alpha_{i\dots j})$, then the claim follows from the induction hypothesis. □

Lemma 5.25. *Let $\mathscr{F}_l$ be a stable k-multiblock, $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$ be its two prime kernels. Suppose that $\alpha_{i\dots j}$ is located to the left from $\alpha_{s\dots t}$ and that there are no other prime kernels between $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$.*

Then they are either consecutive occurrences in α, or the occurrence $\alpha_{j+1\dots s-1}$ between them is the forgetful occurrence of the (left or right) regular part of a stable k'-block $\alpha_{u\dots v}$ ($1 \le k' \le k$) such that Case I holds for its evolution at the left or at the right, respectively.

$\alpha_{i\dots j}$ and $\alpha_{s\dots t}$ are consecutive occurrences in α if and only if $\mathrm{kDc}_k(\alpha_{i\dots j})$ and $\mathrm{kDc}_k(\alpha_{s\dots t})$ are consecutive occurrences. If $\alpha_{j+1\dots s-1} = \mathrm{Fg}(\mathrm{LR}_{k'}(\alpha_{u\dots v}))$ (resp. $\alpha_{j+1\dots s-1} = \mathrm{Fg}(\mathrm{RR}_{k'}(\alpha_{u\dots v}))$) and Case I holds at the left (resp. at the right) for the evolution of $\alpha_{u\dots v}$, then the occurrence between $\mathrm{kDc}_k(\alpha_{i\dots j})$ and $\mathrm{kDc}_k(\alpha_{s\dots t})$ is $\mathrm{Fg}(\mathrm{LR}_{k'}(\mathrm{Dc}_{k'}(\alpha_{u\dots v})))$ (resp. $\mathrm{Fg}(\mathrm{RR}_{k'}(\mathrm{Dc}_{k'}(\alpha_{u\dots v})))$).

Proof. For $k = 0$ the statement is clear since each 0-multiblock has only one prime kernel. For $k > 0$, we prove the statement by induction on k.

$\alpha_{i\dots j}$ and $\alpha_{s\dots t}$ cannot be occurrences in two different k-blocks, otherwise there would be a letter of order $k+1$ between these two k-blocks, and this letter of order $k+1$ would also be located between $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$. So, there are two possible cases: either $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$ are both occurrences in the same k-block, or one of these occurrences is located in a k-block, and the other is the left or the right border of this k-block.

Suppose that $\alpha_{i\dots j}$ is an occurrence in a k-block $\alpha_{i'\dots j'}$, and $\alpha_{s\dots t}$ is the right border of $\alpha_{i'\dots j'}$, i. e. $s = t = j'$. By Lemma 5.23, the k-multiblock consisting of the k-block $\alpha_{i'\dots j'}$ has a prime kernel of the form $\alpha_{j''\dots j'}$, i. e. $i = j''$ and $j = j'$. Then $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$ are consecutive, and $\mathrm{kDc}_k(\alpha_{j''\dots j'})$ and $\mathrm{kDc}_k(\alpha_{j'\dots j'}) = \mathrm{RB}(\mathrm{Dc}_k(\alpha_{i'\dots j'}))$ are consecutive by the second part of Lemma 5.23.

The case when $\alpha_{s\dots t}$ is an occurrence in a k-block $\alpha_{i'\dots j'}$, and $\alpha_{i\dots j}$ is the left border of $\alpha_{i'\dots j'}$ is similar to the previous one. In this case, we have $i = j = i'$, and, by Lemma 5.22, $\alpha_{s\dots t}$ starts from position i' in α, i. e. $s = i'$. Then $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$ are consecutive occurrences in α, and, by the second part of Lemma 5.22, $\mathrm{kDc}_k(\alpha_{i\dots j}) = \mathrm{RB}(\mathrm{Dc}_k(\alpha_{i'\dots j'}))$ and $\mathrm{kDc}_k(\alpha_{s\dots t})$ are also consecutive.

Suppose now that $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$ are both occurrences in a k-block $\alpha_{i'\dots j'}$. Again, there are several possibilities:

First, it is possible that Case I holds at the left for the evolution of $\alpha_{i'\dots j'}$, and $\alpha_{i\dots j} = \mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{i'\dots j'}))$. Then $\alpha_{s\dots t}$ is a prime kernel of $\mathrm{C}_k(\alpha_{i'\dots j'})$ (recall that there are no prime kernels between $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$, that $\mathrm{C}_k(\alpha_{i'\dots j'})$ is a nonempty $(k-1)$-multiblock if $k-1 > 0$ by Remark 3.14, so by Remark 5.20, $\mathrm{C}_k(\alpha_{i'\dots j'})$ has at least one prime kernel). By Lemma 5.22, then the forgetful occurrence of $\mathrm{C}_k(\alpha_{i'\dots j'})$ starts from position s in α, so $\alpha_{j+1\dots s-1} = \mathrm{Fg}(\mathrm{LR}_k(\alpha_{i'\dots j'}))$. We also have $\mathrm{kDc}_k(\alpha_{i\dots j}) = \mathrm{Fg}(\mathrm{LR}_k(\alpha_{i'\dots j'}))$, and $\mathrm{kDc}_k(\alpha_{s\dots t})$ is the leftmost prime kernel of $\mathrm{C}_k(\mathrm{Dc}_k(\alpha_{i'\dots j'}))$, so the occurrence between $\mathrm{kDc}_k(\alpha_{i\dots j})$ and $\mathrm{kDc}_k(\alpha_{s\dots t})$ is $\mathrm{Fg}(\mathrm{LR}_k(\mathrm{Dc}_k(\alpha_{i'\dots j'})))$.

Second, it is possible that Case I holds at the right for the evolution of $\alpha_{i'\dots j'}$, and $\alpha_{s\dots t} = \mathrm{Fg}(\mathrm{RR}_k(\alpha_{i'\dots j'}))$, but this case is completely symmetric to the previous one.

The remaining possibility is that both $\alpha_{i\dots j}$ and $\alpha_{s\dots t}$ are prime kernels of $\mathrm{C}_k(\alpha_{i'\dots j'})$, which is a $(k-1)$-multiblock, but then the claim follows from the induction hypothesis. □

This lemma (together with Lemma 5.12, which implies that the forgetful occurrence of the left or right regular part is nonempty if Case I holds at the left or at the right, respectively) enables us to define *composite kernels* of stable k-multiblocks as maximal (by inclusion) concatenations of consecutive prime kernels. In other words, if $\mathscr{F}_l$ is a stable k-multiblock, then an occurrence $\alpha_{s\dots t}$ is called a composite kernel, if it is a concatenation of consecutive prime kernels of $\mathscr{F}_l$, and letters α_{s-1} (if $s > 0$) and α_{t+1} do not belong to any prime kernels of $\mathscr{F}_l$. (Empty occurrences are allowed by this definition, so, if $\alpha_{s\dots s-1}$ is an empty prime kernel, and letters α_{s-1} and α_s do not belong to any prime kernel, then $\alpha_{s\dots s-1}$ is also a composite kernel). If $\mathscr{F}_l$ is a stable k-multiblock, we can write its composite kernels in a list, as they occur in $\mathrm{Fg}(\mathscr{F}_l)$ from the right to the left. Denote the number of these composite kernels by $\mathrm{nker}_k(\mathscr{F}_l)$ We refer to the elements of this list as to the first, the second, ..., the $\mathrm{nker}_k(\mathscr{F}_l)$th composite kernel of $\mathscr{F}_l$, and denote them by $\mathrm{Ker}_{k,1}(\mathscr{F}_l), \mathrm{Ker}_{k,2}(\mathscr{F}_l), \dots, \mathrm{Ker}_{k,\mathrm{nker}_k(\mathscr{F}_l)}(\mathscr{F}_l)$.

We also can define the *descendant* of a composite kernel as the concatenation of the descendants of all prime kernels inside this composite kernel, they are consecutive by Lemma 5.25. In other words, if $\mathrm{Ker}_{k,m}(\mathscr{F}_l) = \alpha_{s_1\dots t_1}\dots\alpha_{s_n\dots t_n}$, where $1 \le m \le \mathrm{nker}_k(\mathscr{F}_l)$ and $\alpha_{s_i\dots t_i}$ is a prime kernel for $1 \le i \le n$, then we say that the descendant of $\mathscr{F}_l$ is $\mathrm{kDc}_k(\alpha_{s_1\dots t_1})\dots\mathrm{kDc}_k(\alpha_{s_n\dots t_n})$, these descendants are consecutive by Lemma 5.25. Lemma 5.24 guarantees that empty prime kernels do not lead to any ambiguity in the notation here since their descendants are also empty. Denote the descendant of a composite kernel $\mathrm{Ker}_{k,m}(\mathscr{F}_l)$ by $\mathrm{kDc}_k(\mathrm{Ker}_{k,m}(\mathscr{F}_l))$.

So, now we have split each stable k-multiblock into a concatenation of alternating (possibly empty) composite kernels and (nonempty) forgetful occurrences of left or right regular parts of k'-blocks ($1 \le k' \le k$) such that Case I holds for their evolutions at the left or at the right, respectively.

We are going to call these left or right regular parts, as well as some concatenations, the inner pseudoregular parts of the k-multiblock. Here is the precise definition: Let $\mathscr{F}_l$ be a stable k-multiblock. Suppose that $\mathrm{Fg}(\mathscr{F}_l) = \alpha_{i\dots j}$. First, if $1 \le m < \mathrm{nker}_k(\mathscr{F}_l)$, $\mathrm{Ker}_{k,m}(\mathscr{F}_l) = \alpha_{s\dots t}$ and $\mathrm{Ker}_{k,m+1}(\mathscr{F}_l) = \alpha_{s'\dots t'}$, then we call the occurrence $\alpha_{t+1\dots s'-1}$ between these two composite kernels the $(m, m+1)$*th inner pseudoregular part* of $\mathscr{F}_l$ and denote it by $\mathrm{IpR}_{k,m,m+1}(\mathscr{F}_l)$. Second, we say that the $(0,1)$*th* (resp. the $(\mathrm{nker}_k(\mathscr{F}_l), \mathrm{nker}_k(\mathscr{F}_l)+1)$*th*) *inner pseudoregular part* of $\mathscr{F}_l$ is the empty occurrence $\alpha_{i\dots i-1}$ (resp. $\alpha_{j+1\dots j}$) at the beginning (resp. at the end) of the forgetful occurrence of $\mathscr{F}_l$. Denote it by $\mathrm{IpR}_{k,0,1}(\mathscr{F}_l)$ (resp. by $\mathrm{IpR}_{k,\mathrm{nker}_k(\mathscr{F}_l),\mathrm{nker}_k(\mathscr{F}_l)+1}$). Finally, choose indices m and m' so that $0 \le m < m' \le \mathrm{nker}_k(\mathscr{F}_l) + 1$. We call the concatenation $\mathrm{IpR}_{k,m,m+1}(\mathscr{F}_l)\,\mathrm{Ker}_{k,m+1}(\mathscr{F}_l)\dots\mathrm{Ker}_{k,m'-1}(\mathscr{F}_l)\,\mathrm{IpR}_{k,m'-1,m'}(\mathscr{F}_l)$ the (m, m')*th inner pseudoregular part* of $\mathscr{F}_l$ and denote it by $\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)$. (By Lemmas 5.22 and 5.23, these words are really consecutive even if $m = 0$ or $m' = \mathrm{nker}_k(\mathscr{F}_l) + 1$.) In particular, $\mathrm{IpR}_{k,0,\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}_l) = \mathrm{Fg}(\mathscr{F}_l)$. If $\mathrm{Ker}_{k,1}(\mathscr{F}_l)$ (resp. $\mathrm{Ker}_{k,\mathrm{nker}_k(\mathscr{F}_l)}(\mathscr{F}_l)$) is an empty occurrence, then it coincides with $\mathrm{IpR}_{k,0,1}(\mathscr{F}_l)$ (resp. with $\mathrm{IpR}_{k,\mathrm{nker}_k(\mathscr{F}_l),\mathrm{nker}_k(\mathscr{F}_l)+1}$) as an occurrence in α, and in this case $\mathrm{IpR}_{k,0,m'}(\mathscr{F}_l) = \mathrm{IpR}_{k,1,m'}(\mathscr{F}_l)$ (resp. $\mathrm{IpR}_{k,m,\mathrm{nker}_k(\mathscr{F}_l)}(\mathscr{F}_l) = \mathrm{IpR}_{k,m,\mathrm{nker}_k(\mathscr{F}_l)+1}(\mathscr{F}_l)$) for $1 < m' \le \mathrm{nker}_k(\mathscr{F}_l) + 1$ (resp. for $0 \le m < \mathrm{nker}_k(\mathscr{F}_l)$) as an occurrence in α.

For example, let us list the composite kernels and the regular parts between them for the 2-multiblock consisting of a single 2-block from the example above. We have already listed its prime kernels, and its composite kernels and regular parts between them are:

1. The first prime kernel from the list above, which is $\mathfrak{ee}$.
2. The left regular part of $\mathfrak{ee}ee..eff..f\mathfrak{ff}$, which is $ee..e$.
3. The second prime kernel from the list above, which is $eeff$.
4. The left regular part of $\mathfrak{ee}ee..eff..f\mathfrak{ff}$, which is $ff..f$.

5. The concatenation of the third, fourth and fifth prime kernels, which is $\mathfrak{ff}d\mathfrak{ff}$.
6. The left regular part of $\mathfrak{ff}ff..fee..ee\mathfrak{ee}$, which is $ff..f$.
7. The sixth element of the list above, ff.
8. The right regular part of $\mathfrak{ff}ff..fee..ee\mathfrak{ee}$, which is $ee..e$.
9. The concatenation of the seventh, eighth and ninth prime kernels, $\mathfrak{ee}c\mathfrak{ee}$.
10. The left regular part of $\mathfrak{ee}ee..ee\mathfrak{ee}$, which is $ee..e$.
11. The tenth prime kernel, ee.
12. The right regular part of $\mathfrak{ee}ee..ee\mathfrak{ee}$, which is $ee..e$.
13. The eleventh prime kernel, $\mathfrak{ee}$.
14. The right regular part of the whole 2-block, an occurrence of the form $c\mathfrak{ee}ee..ee\mathfrak{ee}c\mathfrak{ee}ee..ee\mathfrak{ee}c\dots c\mathfrak{ee}ee..ee\mathfrak{ee}c\dots ee\mathfrak{ee}c\mathfrak{ee}eeeeeeeeee\mathfrak{ee}$.
15. The twelfth prime kernel, $c\mathfrak{ee}eeeeee\mathfrak{ee}c\mathfrak{ee}ee\mathfrak{ee}c\mathfrak{ee}\mathfrak{c}$.

We already know (Remark 5.21) that the operation of taking the descendant of a prime kernel establishes a bijection between the prime kernels of a stable k-multiblock and the prime kernels of its descendant. The same is true for composite kernels:

Remark 5.26. *The operation of taking the descendant of a composite kernel establishes a bijection between the composite kernels of a stable k-multiblock and the composite kernels of its descendant.*

In other words, if $\mathscr{F}$ is an evolution of stable k-multiblocks, then $\mathrm{nker}_k(\mathscr{F}_l)$ does not depend on l and $\mathrm{Ker}_{k,m}(\mathscr{F}_{l+1}) = \mathrm{kDc}_k(\mathrm{Ker}_{k,m}(\mathscr{F}_l))$ for $l \geq 0$ and for $1 \leq m \leq \mathrm{nker}_k(\mathscr{F}_l)$.

So if $\mathscr{F}$ is an evolution of stable k-multiblocks, we can denote the number $\mathrm{nker}_k(\mathscr{F}_l)$ for arbitrary l by $\mathrm{nker}_k(\mathscr{F})$.

Lemma 5.27. *If $\mathscr{F}$ is an evolution of stable k-multiblocks, then $\mathrm{Ker}_{k,m}(\mathscr{F}_l)$ does not depend on l as an abstract word for $1 \leq m \leq \mathrm{nker}_k(\mathscr{F})$.*

Proof. This follows directly from Lemma 5.24 and the definitions of a composite kernel and its descendant. □

So, we can denote the abstract word $\mathrm{Ker}_{k,m}(\mathscr{F}_l)$ for arbitrary $l \geq 0$ by $\mathrm{Ker}_{k,m}(\mathscr{F})$ and call it *the mth composite kernel of $\mathscr{F}$*. The number $\mathrm{nker}_k(\mathscr{F})$ is then called *the number of composite kernels of $\mathscr{F}$*.

Lemma 5.28. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks, $k > 0$. Let $1 \leq m < \mathrm{nker}_k(\mathscr{F})$.*

Then there exist an evolution of k'-blocks $\mathscr{E}$ ($1 \leq k' \leq k$) and a number $l_0 \geq 3k'$ such that one of the following statement holds:

1. *Case I holds for $\mathscr{E}$ at the left, and for all $l \geq 0$ we have $\mathrm{IpR}_{k,m,m+1}(\mathscr{F}_l) = \mathrm{Fg}(\mathrm{LR}_{k'}(\mathscr{E}_{l+l_0}))$.*
2. *Case I holds for $\mathscr{E}$ at the right, and for all $l \geq 0$ we have $\mathrm{IpR}_{k,m,m+1}(\mathscr{F}_l) = \mathrm{Fg}(\mathrm{RR}_{k'}(\mathscr{E}_{l+l_0}))$.*

Proof. This follows directly from the last statement of Lemma 5.25. □

Lemma 5.29. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks, $k > 0$. Let $1 \leq m < m' \leq \mathrm{nker}_k(\mathscr{F})$. For each $l \geq 0$, denote $n_l = |\,\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)|$.*

Then $n_l \geq 2\mathbf{L}$, n_l strictly grows as l grows, and there exists k' ($1 \leq k' \leq k$) such that n_l is $\Theta(l^{k'})$ for $l \to \infty$.

Proof. By the previous lemma, there exist numbers $k_m, \dots, k_{m'-1}$ ($1 \le k_i \le k$), evolutions $\mathscr{E}^{(m)}, \dots \mathscr{E}^{(m'-1)}$ ($\mathscr{E}^{(i)}$ is an evolution of k_i-blocks), and numbers $l_m, \dots, l_{m'-1}$ ($l_i \ge 3k_i$) such that for each i ($m \le i < m'$) one of the following holds:

1. Case I holds for $\mathscr{E}^{(i)}$ at the left, and for each $l \ge 0$ we have $\mathrm{IpR}_{k,i,i+1}(\mathscr{F}_l) = \mathrm{Fg}(\mathrm{LR}_{k_i}(\mathscr{E}^{(i)}_{l+l_i}))$.

2. Case I holds for $\mathscr{E}^{(i)}$ at the right, and for each $l \ge 0$ we have $\mathrm{IpR}_{k,i,i+1}(\mathscr{F}_l) = \mathrm{Fg}(\mathrm{RR}_{k_i}(\mathscr{E}^{(i)}_{l+l_i}))$.

By Lemmas 5.12, 5.13, and 5.15, we see that in each of these cases, $|\,\mathrm{IpR}_{k,i,i+1}(\mathscr{F}_l)| \ge 2\mathbf{L}$, that $|\,\mathrm{IpR}_{k,i,i+1}(\mathscr{F}_l)|$ strictly grows as l grows, and that $|\,\mathrm{IpR}_{k,i,i+1}(\mathscr{F}_l)|$ is $\Theta((l+l_i)^{k_i}) = \Theta(l^{k_i})$ for $l \to \infty$.

Now, n_l is the sum of $m'-m-1 \ge 0$ summands $|\,\mathrm{Ker}_{k,i}(\mathscr{F})|$ that do not depend of l, and of $m'-m \ge 1$ summands $|\,\mathrm{IpR}_{k,i,i+1}(\mathscr{F}_l)|$. Therefore, $n_l \ge 2\mathbf{L}$, and n_l strictly grows as l grows. The asymptotic of n_l for $l \to \infty$ is

$$\sum_{i=m+1}^{m'-1} |\,\mathrm{Ker}_{k,i}(\mathscr{F})| + \sum_{i=m}^{m'-1} \Theta(l^{k_i}) = \Theta(l^{k'}),$$

where $k' = \max(k_m, \dots, k_{m'-1})$. ☐

Corollary 5.30. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks, $k > 0$. Suppose that $\mathrm{nker}_k(\mathscr{F}) > 1$. Let $m, m' \in \mathbb{Z}$ be two indices such that $0 \le m < \mathrm{nker}_k(\mathscr{F})$, $m < m'$ and $1 < m' \le \mathrm{nker}_k(\mathscr{F})+1$. For each $l \ge 0$, denote $n_l = |\,\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)|$. (In particular, if $m = 0$ and $m' = \mathrm{nker}_k(\mathscr{F}) + 1 > 2$, then $n_l = |\,\mathrm{Fg}(\mathscr{F}_l)|$.)*

Then $n_l \ge 2\mathbf{L}$, n_l strictly grows as l grows, and there exists k' ($1 \le k' \le k$) such that n_l is $\Theta(l^{k'})$ for $l \to \infty$.

Proof. The only cases in the statement of this corollary not covered by the previous lemma are the cases when $m = 0$ or $m' = \mathrm{nker}_k(\mathscr{F}) + 1$.

If $m = 0$ and $m' < \mathrm{nker}_k(\mathscr{F}) + 1$, then denote $n'_l = |\,\mathrm{IpR}_{k,1,m'}(\mathscr{F}_l)|$ (recall that we assume that $m' > 1$). By the previous lemma, $n'_l \ge 2\mathbf{L}$, n'_l strictly grows as l grows, and there exists k' ($1 \le k' \le k$) such that n'_l is $\Theta(l^{k'})$ for $l \to \infty$. But $\mathrm{IpR}_{k,0,1}(\mathscr{F}_l)$ is always an empty occurrence, so $\mathrm{IpR}_{k,0,m'}(\mathscr{F}_l) = \mathrm{Ker}_{k,1}(\mathscr{F}_l)\,\mathrm{IpR}_{k,1,m'}(\mathscr{F}_l)$, and $n_l = |\,\mathrm{Ker}_{k,1}(\mathscr{F})| + n'_l$, and the first summand does not depend on l. Hence, $n_l \ge 2\mathbf{L} + |\,\mathrm{Ker}_{k,1}(\mathscr{F})| \ge 2\mathbf{L}$, n_l strictly grows as l grows, and n_l is $\Theta(l^{k'}) + |\,\mathrm{Ker}_{k,1}(\mathscr{F})| = \Theta(l^{k'})$ for $l \to \infty$.

The case when $m > 0$, but $m' = \mathrm{nker}_k(\mathscr{F}) + 1$ is completely analogous.

Finally, if $m = 0$ and $m' = \mathrm{nker}_k(\mathscr{F}) + 1$, then, since $\mathrm{nker}_k(\mathscr{F}) > 1$ by assumption, we can apply the previous lemma to $n'_l = |\,\mathrm{IpR}_{k,1,\mathrm{nker}_k(\mathscr{F})}(\mathscr{F}_l)|$. Again, $n'_l \ge 2\mathbf{L}$, n'_l strictly grows as l grows, and there exists k' ($1 \le k' \le k$) such that n'_l is $\Theta(l^{k'})$ for $l \to \infty$. And again, since $\mathrm{IpR}_{k,0,1}(\mathscr{F}_l)$ and $\mathrm{IpR}_{k,\mathrm{nker}_k(\mathscr{F}),\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}_l)$ are always empty occurrences, we have $n_l = |\,\mathrm{Ker}_{k,1}(\mathscr{F})| + n'_l + |\,\mathrm{Ker}_{k,\mathrm{nker}_k(\mathscr{F})}(\mathscr{F})|$. The first and the last summands do not depend on l, so $n_l \ge 2\mathbf{L} + |\,\mathrm{Ker}_{k,1}(\mathscr{F})| + |\,\mathrm{Ker}_{k,\mathrm{nker}_k(\mathscr{F})}(\mathscr{F})| \ge 2\mathbf{L}$, n_l strictly grows as l grows, and n_l is $\Theta(l^{k'}) + |\,\mathrm{Ker}_{k,1}(\mathscr{F})| + |\,\mathrm{Ker}_{k,\mathrm{nker}_k(\mathscr{F})}(\mathscr{F})| = \Theta(l^{k'})$ for $l \to \infty$. ☐

The following lemma shows how composite kernels and inner pseudoregular parts behave for concatenations of consecutive evolutions.

Lemma 5.31. *Let $\mathscr{F}$ and $\mathscr{F}'$ be two consecutive evolutions of* ***nonempty*** *stable k-multiblocks ($k \ge 0$), and let $\mathscr{F}''$ be the concatenation of $\mathscr{F}$ and $\mathscr{F}'$. Then:*

1. $\mathrm{nker}_k(\mathscr{F}'') = \mathrm{nker}_k(\mathscr{F}) + \mathrm{nker}_k(\mathscr{F}') - 1$

2. *If $1 \le m < \mathrm{nker}_k(\mathscr{F})$, then $\mathrm{Ker}_{k,m}(\mathscr{F}''_l) = \mathrm{Ker}_{k,m}(\mathscr{F}_l)$ for all $l \ge 0$ as an occurrence in α and $\mathrm{Ker}_{k,m}(\mathscr{F}'') = \mathrm{Ker}_{k,m}(\mathscr{F})$ as an abstract word. If $m = \mathrm{nker}_k(\mathscr{F})$, then $\mathrm{Ker}_{k,m}(\mathscr{F}''_l) = \mathrm{Ker}_{k,m}(\mathscr{F}_l)\,\mathrm{Ker}_{k,1}(\mathscr{F}'_l)$ for all $l \ge 0$ as an occurrence in α and $\mathrm{Ker}_{k,m}(\mathscr{F}'') = \mathrm{Ker}_{k,m}(\mathscr{F})\,\mathrm{Ker}_{k,1}(\mathscr{F}')$ as an abstract word. If $\mathrm{nker}_k(\mathscr{F}) < m \le \mathrm{nker}_k(\mathscr{F}'')$, then $\mathrm{Ker}_{k,m}(\mathscr{F}''_l) = \mathrm{Ker}_{k,m-\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}'_l)$ for all $l \ge 0$ as an occurrence in α and $\mathrm{Ker}_{k,m}(\mathscr{F}'') = \mathrm{Ker}_{k,m-\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}')$ as an abstract word.*

3. *If* $0 \le m < m' \le \mathrm{nker}_k(\mathscr{F})$*, then* $\mathrm{IpR}_{k,m,m'}(\mathscr{F}'') = \mathrm{IpR}_{k,m,m'}(\mathscr{F})$ *for all* $l \ge 0$ *as an occurrence in* α*. If* $\mathrm{nker}_k(\mathscr{F}) \le m < m' \le \mathrm{nker}_k(\mathscr{F}'')+1$*, then* $\mathrm{IpR}_{k,m,m'}(\mathscr{F}'') = \mathrm{IpR}_{k,m-\mathrm{nker}_k(\mathscr{F})+1,m'-\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}')$ *for all* $l \ge 0$ *as an occurrence in* α*. If* $0 \le m < \mathrm{nker}_k(\mathscr{F}) < m' \le \mathrm{nker}_k(\mathscr{F}'')+1$*, then* $\mathrm{IpR}_{k,m,m'}(\mathscr{F}'') = \mathrm{IpR}_{k,m,\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F})\,\mathrm{IpR}_{k,0,m'-\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}')$ *for all* $l \ge 0$ *as an occurrence in* α*.*

Proof. The first two claims follow directly from Lemmas 5.22 and 5.23 and the definition of a composite central kernel. The third claim follows from the first two claims and the definition of an inner pseudoregular part. □

A particular case of prime and composite kernels will be especially important for us. If $\alpha_{i\dots j}$ is a stable k-block ($k \ge 1$), we call the prime (resp. composite) kernels of $\mathrm{C}_k(\alpha_{i\dots j})$ the *prime (resp. composite) central kernels* of $\alpha_{i\dots j}$. Denote the number of the composite central kernels of a stable k-block $\alpha_{i\dots j}$ by $\mathrm{ncker}_k(\alpha_{i\dots j})$. Observe that this definition coincides with the definition we gave in the previous section for 1-blocks. Again, we can write the composite central kernels of $\alpha_{i\dots j}$ in a list as they occur in $\mathscr{E}_l$, from the left to the right. We call the elements of this list the first, the second, ..., the $\mathrm{ncker}_k(\alpha_{i\dots j})$th composite central kernel of $\alpha_{i\dots j}$ and denote them by $\mathrm{cKer}_{k,1}(\alpha_{i\dots j}), \mathrm{cKer}_{k,2}(\alpha_{i\dots j}), \dots, \mathrm{cKer}_{k,\mathrm{ncker}_k(\alpha_{i\dots j})}(\alpha_{i\dots j})$. It follows from Lemma 5.22 and from Remark 3.14 that $\mathrm{ncker}_k(\alpha_{i\dots j}) \ge 1$. Note that, for example, the first composite central kernel of $\alpha_{i\dots j}$ can be the first or the second composite kernel of the k-multiblock consisting of the k-block $\alpha_{i\dots j}$ only, depending on whether Case II or Case I holds for the evolution of $\alpha_{i\dots j}$ at the left.

Let $\mathscr{E}$ is an evolution of k-blocks, and let $\mathscr{E}_l$ be a stable k-block. We have defined the descendants of these composite central kernels, and they are composite central kernels of $\mathscr{E}_{l+1}$. By Remark 5.26, $\mathrm{ncker}_k(\mathscr{E}_l) = \mathrm{nker}_{k-1}(\mathrm{C}_k(\mathscr{E}_l)) = \mathrm{nker}_{k-1}(\mathrm{C}_k(\mathscr{E}_{l+1})) = \mathrm{ncker}_k(\mathscr{E}_{l+1})$, and by Lemma 5.27, $\mathrm{cKer}_{k,m}(\mathscr{E}_{l+1})$ is the same abstract word as $\mathrm{cKer}_{k,m}(\mathscr{E}_l)$ for $1 \le m \le \mathrm{ncker}_k(\mathscr{E}_l)$. In other words, the number $\mathrm{ncker}_k(\mathscr{E}_l)$ and the abstract words $\mathrm{cKer}_{k,1}(\mathscr{E}_l), \mathrm{cKer}_{k,2}(\mathscr{E}_l), \dots, \mathrm{cKer}_{k,\mathscr{E}_l}(\mathscr{E}_l)$ do not depend on l if $l \ge 3k$. We call these abstract words *the composite central kernels of* $\mathscr{E}$, denote the number of them by $\mathrm{ncker}_k(\mathscr{E})$, and denote the composite central kernels of $\mathscr{E}$ themselves by $\mathrm{cKer}_{k,1}(\mathscr{E}), \mathrm{cKer}_{k,2}(\mathscr{E}), \dots, \mathrm{cKer}_{k,\mathscr{E}}(\mathscr{E})$.

In the example above, the composite central kernels of the evolution of 2-blocks are: $\mathfrak{ee}$, $eeff$, $\mathfrak{ffdff}$, ff, $\mathfrak{eecee}$, ee, $\mathfrak{ee}$.

The structure of a 2-block in a bit more general case is shown by Fig. 5.

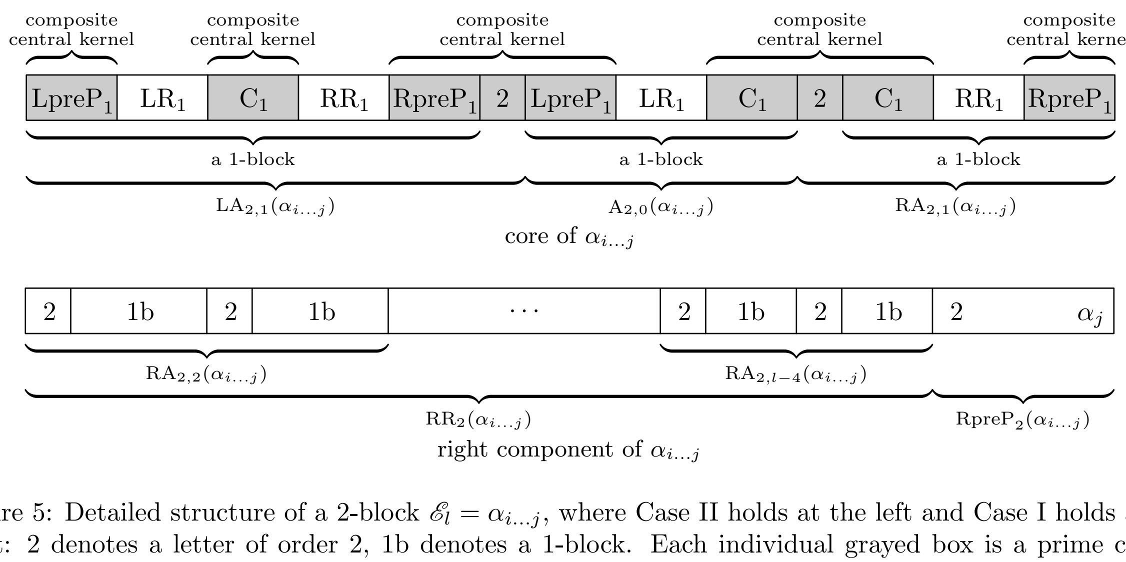

Figure 5: Detailed structure of a 2-block $\mathscr{E}_l = \alpha_{i\dots j}$, where Case II holds at the left and Case I holds at the right: 2 denotes a letter of order 2, 1b denotes a 1-block. Each individual grayed box is a prime central kernel.

Lemma 5.32. *The lengths of all composite central kernels of all evolutions of k-blocks arising in α are bounded by a single constant that depends on Σ, φ, and k only. In particular, only finitely many abstract words can equal central kernels of evolutions of k-blocks arising in α.*

Proof. The proof is similar to the proof of Corollary 5.3. By Corollary 3.4, there are only finitely many sequences of abstract words that can be evolutions in α, so there exists a single constant x that depends on Σ, φ and k only such that if $\mathscr{E}$ is an evolution of k-blocks, then $|\mathscr{E}_{3k}| \le x$. By definition, $\operatorname{cKer}_{k,m}(\mathscr{E}) = \operatorname{cKer}_{k,m}(\mathscr{E}_{3k})$ for $1 \le m \le \operatorname{ncker}_k(\mathscr{E})$, and $\operatorname{cKer}_{k,m}(\mathscr{E}_{3k})$ is a factor of $\mathscr{E}_{3k}$. Therefore, $|\operatorname{cKer}_{k,m}(\mathscr{E})| \le x$. □

Again, as in the proof of Corollary 5.3, we do not claim that if two evolutions of k-blocks equal as sequences of abstract words, then their composite central kernels are equal.

6 Continuously Periodic Evolutions

In this section we will define and study continuously periodic evolutions, which will enable us to formulate a criterion for factor complexity of morphic sequences. We will use the coding ψ a lot as well as the morphism φ, so we will use some obvious properties of codings without mentioning every time that ψ is a coding. For example, if γ is a finite word, then $|\gamma| = |\psi(\gamma)|$, and if $0 \le i \le |\gamma| - 1$, then $\psi(\gamma_i) = \psi(\gamma)_i$. Also, if $i \in \mathbb{Z}_{\ge 0}$, then $\psi(\alpha_i) = \psi(\alpha)_i$. If $\gamma_{i\ldots j}$ is an occurrence in a finite word γ, then $\psi(\gamma_{i\ldots j}) = \psi(\gamma)_{i\ldots j}$. And if $\alpha_{i\ldots j}$ is an occurrence in α, then $\psi(\alpha_{i\ldots j}) = \psi(\alpha)_{i\ldots j}$.

Before we will be able to define continuously periodic evolutions, we need to introduce two more technical notions, namely, we need to define left and right bounding sequences of an evolution of k-blocks and weak left and right evolutional periods of k-multiblocks.

The construction of the left and the right bounding sequences is similar to the construction of pure morphic sequences themselves. Let us construct the right bounding sequence, the construction for the left bounding sequence is symmetric. Let $\mathscr{E}$ be an evolution of k-blocks such that Case II holds at the right. First, consider the following sequence of abstract words: $\operatorname{RB}(\mathscr{E}), \varphi(\operatorname{RB}(\mathscr{E})), \varphi^2(\operatorname{RB}(\mathscr{E})), \ldots$ Since φ is strongly 1-periodic and images of letters of order $\le k$ consist of letters of order $\le k$, the leftmost letter of order $> k$ in each of these words is $\operatorname{RB}(\mathscr{E})$. Temporarily denote by γ_l (resp. δ_l) the prefix (resp. the suffix) of $\varphi^l(\operatorname{RB}(\mathscr{E}))$ to the left (resp. to the right) from the leftmost occurrence of $\operatorname{RB}(\mathscr{E})$ (not including this occurrence of $\operatorname{RB}(\mathscr{E})$). In other words, write $\varphi^l(\operatorname{RB}(\mathscr{E})) = \gamma_l \operatorname{RB}(\mathscr{E})\delta_l$, where γ_l consists of letters of order $\le k$ only. In particular, γ_0 and δ_0 are the empty word.

Remark 6.1. *If $\operatorname{RB}(\mathscr{E}_m) = \alpha_i$, where $m \ge 1$, $\operatorname{RB}(\mathscr{E}_{m_l}) = \alpha_j$, where $l \ge 0$, and $\varphi^l(\alpha_i) = \alpha_{s\ldots t}$ as an occurrence in α, then $\alpha_{j+1\ldots t} = \delta_l$ as an abstract word.*

Lemma 6.2. *For all $l \ge 0$ we have $\delta_{l+1} = \delta_1\varphi(\delta_l)$.*

Proof. We have $\gamma_{l+1} \operatorname{RB}(\mathscr{E})\delta_{l+1} = \varphi^{l+1}(\operatorname{RB}(\mathscr{E})) = \varphi(\varphi^l(\operatorname{RB}(\mathscr{E}))) = \varphi(\gamma_l \operatorname{RB}(\mathscr{E})\delta_l) = \varphi(\gamma_l)\varphi(\operatorname{RB}(\mathscr{E}))\varphi(\delta_l) = \varphi(\gamma_l)\gamma_1 \operatorname{RB}(\mathscr{E})\delta_1\varphi(\delta_l)$. Note that $\varphi(\gamma_l)$ and γ_1 do not contain letters of order $> k$, so the leftmost occurrence of $\operatorname{RB}(\mathscr{E})$ in $\varphi(\gamma_l)\gamma_1 \operatorname{RB}(\mathscr{E})\delta_1\varphi(\delta_l)$ is the occurrence mentioned in this formula explicitly. On the other hand, by the definition of γ_{l+1} and δ_{l+1}, the leftmost occurrence of $\operatorname{RB}(\mathscr{E})$ in $\gamma_{l+1} \operatorname{RB}(\mathscr{E})\delta_{l+1}$ is also the occurrence mentioned in this formula explicitly. Hence, $\delta_{l+1} = \delta_1\varphi(\delta_l)$. □

Lemma 6.3. *For all $l \ge 0$, δ_l is a prefix of δ_{l+1}.*

Proof. Let us prove this by induction on l. For $l = 0$ this is clear since δ_0 is the empty word. Suppose that δ_l is a prefix of δ_{l+1}. Then $\varphi(\delta_l)$ is a prefix of $\varphi(\delta_{l+1})$. By Lemma 6.2, $\delta_{l+1} = \delta_1\varphi(\delta_l)$ and $\delta_{l+2} = \delta_1\varphi(\delta_{l+1})$, so δ_{l+1} is a prefix of δ_{l+2}. □

Lemma 6.4. *For all $l \ge 0$, we have $|\delta_{l+1}| > |\delta_l|$.*

Proof. Recall that we have started with an evolution $\mathscr{E}$ such that Case II holds at the right. By the definitions of Case II and of right atoms, γ_1 cannot contain letters of order k, it consists of letters of smaller orders (or is empty if $k = 1$). But then, if δ_1 also had consisted of letters of order $< k$ only, $\mathrm{RB}(\mathscr{E})$ would have been a letter of order k or less. So, δ_1 contains at least one letter of order $\geq k$, in particular, δ_1 is nonempty. Since φ is nonerasing, $|\varphi(\delta_l)| \geq |\delta_l|$, and $|\delta_{l+1}| = |\delta_1| + |\varphi(\delta_l)| > |\delta_l|$. □

So, we have constructed an infinite sequence of words δ_l, whose lengths strictly increase, and each of them is a prefix of the next one. Let us add $\mathrm{RB}(\mathscr{E})$ at the left of each of these words. We get an infinite sequence of words

$$\mathrm{RB}(\mathscr{E}), \mathrm{RB}(\mathscr{E})\delta_1, \mathrm{RB}(\mathscr{E})\delta_2, \ldots, \mathrm{RB}(\mathscr{E})\delta_l, \ldots$$

whose lengths strictly increase, and each of them is a prefix of the next one. So, there exists a unique infinite (to the right) word such that all these words $\mathrm{RB}(\mathscr{E})\delta_l$ (for all $l \geq 0$) are its prefixes. We call this infinite word *the right bounding sequence of* $\mathscr{E}$ and denote it by $\mathrm{RBS}_k(\mathscr{E})$.

With this definition, Remark 6.1 can be reformulated as follows:

Remark 6.5. *Let $\mathscr{E}$ be an evolution of k-blocks such that Case II holds at the right, $m \geq 1$ and $l \geq 0$. Suppose that $\mathrm{RB}(\mathscr{E}_m) = \alpha_i$, $\mathrm{RB}(\mathscr{E}_{m+l}) = \alpha_j$, and $\varphi^l(\alpha_i) = \alpha_{s\ldots t}$ as an occurrence in α. Then $\alpha_{j+1\ldots t}$ as an abstract word is a prefix of $\mathrm{RBS}_k(\mathscr{E})$.*

$\mathrm{RBS}_k(\mathscr{E})$ is an abstract infinite word, it is not an occurrence in α. However, we can prove the following lemma.

Lemma 6.6. *For all $l \geq 0$, if $\mathscr{E}_{l+2} = \alpha_{i\ldots j}$ as an occurrence in α, then $\alpha_{i\ldots j+|\mathrm{RB}(\mathscr{E})\delta_l|} = \mathscr{E}_{l+2}\,\mathrm{RB}(\mathscr{E})\delta_l$.*

Proof. We prove this by induction on l. If $l = 0$, then $|\delta_l| = 0$ and, by the definition of $\mathrm{RB}(\mathscr{E})$, $\mathrm{RB}(\mathscr{E}_2) = \mathrm{RB}(\mathscr{E})$, and the claim is clear.

Suppose that $\mathscr{E}_{l+2} = \alpha_{i\ldots j}$ and $\alpha_{i\ldots j+|\mathrm{RB}(\mathscr{E})\delta_l|} = \mathscr{E}_{l+2}\,\mathrm{RB}(\mathscr{E})\delta_l$. Let s and t be the indices such that $\alpha_{s\ldots t} = \mathrm{Dc}_k(\alpha_{i\ldots j}) = \mathscr{E}_{l+3}$ as an occurrence in α. Denote also by i' and j' the indices such that $\alpha_{i'\ldots j'} = \varphi(\alpha_{i\ldots j})$. By the definition of the descendant of a k-block, $\alpha_{i'\ldots j'}$ is a suboccurrence of $\alpha_{s\ldots t}$. Consider also the following occurrence in α starting from position i': $\varphi(\alpha_{i\ldots j+|\delta_l|}) = \varphi(\alpha_{i\ldots j}\,\mathrm{RB}(\mathscr{E})\delta_l) = \alpha_{i'\ldots j'}\gamma_1\,\mathrm{RB}(\mathscr{E})\delta_1\varphi(\delta_l) = \alpha_{i'\ldots j'}\gamma_1\,\mathrm{RB}(\mathscr{E})\delta_{l+1}$. Since γ_1 consists of letters of order $\leq k$ only, and $\mathrm{RB}(\mathscr{E})$ is a letter of order $> k$, the occurrence of $\mathrm{RB}(\mathscr{E})$ mentioned explicitly in this formula is the right border of $\mathscr{E}_{l+3}$. In other words, the occurrence of γ_1 mentioned explicitly here is $\alpha_{j'+1\ldots t}$, and the occurrence of $\mathrm{RB}(\mathscr{E})$ mentioned explicitly here is α_{t+1}. Hence, $\alpha_{t+1\ldots t+|\mathrm{RB}(\mathscr{E})\delta_{l+1}|} = \mathrm{RB}(\mathscr{E})\delta_{l+1}$, and $\alpha_{s\ldots t+|\mathrm{RB}(\mathscr{E})\delta_{l+1}|} = \mathscr{E}_{l+3}\,\mathrm{RB}(\mathscr{E})\delta_{l+1}$ □

This lemma still uses the notation δ_l we have introduced temporarily, but the next corollary does not use any temporary notation anymore.

Corollary 6.7. *Let $\mathscr{E}$ be an evolution of k-blocks such that Case II holds at the right, and let δ be an arbitrary finite prefix of $\mathrm{RBS}_k(\mathscr{E})$.*

Then there exists $l_0 \in \mathbb{N}$ (that depends on $\mathscr{E}$ and δ) such that if $l \geq l_0$ and $\mathscr{E}_l = \alpha_{i\ldots j}$ as an occurrence in α, then $\alpha_{i\ldots j+|\delta|} = \mathscr{E}_l\delta$.

Proof. Since the lengths of the words δ_l strictly grow as $l \to \infty$, there exists $l' \in \mathbb{N}$ such that if $l \geq l'$, then δ is a prefix of $\mathrm{RB}(\mathscr{E})\delta_l$.

Set $l_0 = l' + 2$. Then if $\mathscr{E}_l = \alpha_{i\ldots j}$ as an occurrence in α, then by the previous lemma, $\alpha_{i\ldots j+|\mathrm{RB}(\mathscr{E})\delta_{l-2}|} = \mathscr{E}_l\,\mathrm{RB}(\mathscr{E})\delta_{l-2}$. Since δ is a prefix of $\mathrm{RB}(\mathscr{E})\delta_{l-2}$, we have $\alpha_{i\ldots j+|\delta|} = \mathscr{E}_l\delta$. □

Similarly, if Case II holds at the left for an evolution $\mathscr{E}$ of k-blocks, we define its *left bounding sequence* and denote it by $\mathrm{LBS}(\mathscr{E})$. $\mathrm{LBS}_k(\mathscr{E})$ is a word infinite to the left, and the symmetric version of Corollary 6.7 for left bounding sequences can be formulated as follows.

Corollary 6.8. *Let $\mathscr{E}$ be an evolution of k-blocks such that Case II holds at the left, and let δ be an arbitrary finite suffix of* $\mathrm{RBS}_k(\mathscr{E})$.

Then there exists $l_0 \in \mathbb{N}$ (that depends on $\mathscr{E}$ and δ) such that if $l \geq l_0$ and $\mathscr{E}_l = \alpha_{i\ldots j}$ as an occurrence in α, then $i \geq |\delta|$ and $\alpha_{i-|\delta|\ldots j} = \delta\mathscr{E}_l$. □

The symmetric version of Remark 6.5 can be formulated as follows.

Remark 6.9. *Let $\mathscr{E}$ be an evolution of k-blocks such that Case II holds at the left, $m \geq 1$ and $l \geq 0$. Suppose that $\mathrm{LB}(\mathscr{E}_m) = \alpha_i$, $\mathrm{LB}(\mathscr{E}_{m+l}) = \alpha_j$, and $\varphi^l(\alpha_i) = \alpha_{s\ldots t}$ as an occurrence in α. Then $\alpha_{s\ldots j-1}$ as an abstract word is a suffix of $\mathrm{LBS}_k(\mathscr{E})$.*

Now let us define left and right weak evolutional periods for evolutions $\mathscr{F}$ of stable nonempty k-multiblocks ($k \geq 1$). The definition will use two indices, m and m' ($0 \leq m < m' \leq \mathrm{nker}_k(\mathscr{F}) + 1$).

Definition 6.10. A final period λ is called a *left* (resp. *right*) *weak evolutional period* of an evolution $\mathscr{F}$ of stable k-multiblocks ($k \geq 1$) for a pair of indices (m, m') ($0 \leq m < m' \leq \mathrm{nker}_k(\mathscr{F}) + 1$) if

1. For each $l \geq 0$, $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l))$ is a weakly left (resp. right) λ-periodic word.
2. The residue of $|\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)|$ (for $l \geq 0$) modulo $|\lambda|$, i. e. the length of the incomplete occurrence in the previous condition, does not depend on l.

The second condition here enables us to formulate the following remark:

Remark 6.11. *If λ is a left weak evolutional period of an evolution $\mathscr{F}$ of stable nonempty k-multiblocks ($k \geq 1$) for a pair of indices (m, m') ($0 \leq m < m' \leq \mathrm{nker}_k(\mathscr{F}) + 1$) and r is the residue of $|\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)|$ modulo $|\lambda|$ (for any l), then $\lambda' = \mathrm{Cyc}_r(\lambda) = \mathrm{Cyc}_{|\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)|}(\lambda)$ is a right weak evolutional period of $\mathscr{F}$ for the pair (m, m').*

If λ is a right weak evolutional period of an evolution $\mathscr{F}$ of stable nonempty k-multiblocks ($k \geq 1$) for a pair of indices (m, m') ($0 \leq m < m' \leq \mathrm{nker}_k(\mathscr{F}) + 1$) and r is the residue of $|\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)|$ modulo $|\lambda|$ (for any l), then $\lambda' = \mathrm{Cyc}_{-r}(\lambda) = \mathrm{Cyc}_{-|\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)|}(\lambda)$ is a left weak evolutional period of $\mathscr{F}$ for the pair (m, m').

Lemma 6.12. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks ($k \geq 1$). Suppose that $\mathrm{nker}_k(\mathscr{F}) > 1$. Let $m, m', m'' \in \mathbb{Z}$ be three indices such that $0 \leq m < \mathrm{nker}_k(\mathscr{F})$, $m < m'$, $m < m''$, $1 < m' \leq \mathrm{nker}_k(\mathscr{F}) + 1$, and $1 < m'' \leq \mathrm{nker}_k(\mathscr{F}) + 1$.*

If λ is a left weak evolutional period of $\mathscr{F}$ for the pair (m, m'), and λ' is a left weak evolutional period of $\mathscr{F}$ for the pair (m, m''), then $\lambda = \lambda'$, and λ is the minimal left period of $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l))$ for all $l \geq 0$.

Proof. Both λ and λ' are final periods, so $|\lambda| \leq \mathbf{L}$ and $|\lambda'| \leq \mathbf{L}$. Without loss of generality, $m' \leq m''$, so $\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)$ is a prefix of $\mathrm{IpR}_{k,m,m''}(\mathscr{F}_l)$ for all $l \geq 0$. Hence, for all $l \geq 0$, $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l))$ is both a weakly left λ-periodic word and a weakly left λ'-periodic word. By Corollary 5.30, $|\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)| \geq 2\mathbf{L}$. By Corollary 2.12, $\lambda = \lambda'$.

Moreover, λ is the minimal left period of $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l))$ for all $l \geq 0$ by Corollary 2.11. □

Corollary 6.13. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks ($k \geq 1$). Suppose that $\mathrm{nker}_k(\mathscr{F}) > 1$. Let $m, m' \in \mathbb{Z}$ be two indices such that $0 \leq m < \mathrm{nker}_k(\mathscr{F})$, $m < m'$ and $1 < m' \leq \mathrm{nker}_k(\mathscr{F}) + 1$.*

If there exists a left weak evolutional period of $\mathscr{F}$ for the pair (m, m'), then it is unique and is the minimal left period of $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l))$ for all $l \geq 0$. □

Lemma 6.14. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks ($k \geq 1$). Suppose that $\mathrm{nker}_k(\mathscr{F}) > 1$. Let $m'', m', m \in \mathbb{Z}$ be three indices such that $0 \leq m'' < \mathrm{nker}_k(\mathscr{F})$, $0 \leq m' < \mathrm{nker}_k(\mathscr{F})$, $m'' < m$, $m' < m$, and $1 < m \leq \mathrm{nker}_k(\mathscr{F}) + 1$.*

If λ is a right weak evolutional period of $\mathscr{F}$ for the pair (m'', m), and λ' is a right weak evolutional period of $\mathscr{F}$ for the pair (m', m), then $\lambda = \lambda'$, and λ is the minimal right period of $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l))$ for all $l \geq 0$.

Proof. The proof is completely symmetric to the proof of Lemma 6.12. □

Corollary 6.15. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks ($k \ge 1$). Suppose that $\mathrm{nker}_k(\mathscr{F}) > 1$. Let $m', m \in \mathbb{Z}$ be two indices such that $0 \le m' < \mathrm{nker}_k(\mathscr{F})$, $m' < m$ and $1 < m \le \mathrm{nker}_k(\mathscr{F}) + 1$.*

If there exists a right weak evolutional period of $\mathscr{F}$ for the pair (m', m), then it is unique and is the minimal right period of $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l))$ for all $l \ge 0$. □

Now we define left and right pseudoregular parts and continuous evolutional periods for evolutions of k-blocks. The definition is similar to the definition of inner pseudoregular parts and weak evolutional periods for evolutions of k-multiblocks, but is not entirely the same. Let $\mathscr{E}$ be an evolution of k-blocks, and let m be an index ($1 \le m \le \mathrm{ncker}_k(\mathscr{E})$). For each $l \ge 0$ we define the left and the right pseudoregular parts of a stable k-block $\mathscr{E}_l$ for index m as follows. Let i and j be indices such that $\mathscr{E}_l = \alpha_{i\dots j}$. Suppose also that $\mathrm{cKer}_{k,m}(\mathscr{E}_l) = \alpha_{s\dots t}$, $\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{i\dots j})) = \alpha_{i\dots i'}$, and $\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{i\dots j})) = \alpha_{j'\dots j}$. Then *the left (resp. right) pseudoregular part of $\mathscr{E}_l$ for index m* is $\alpha_{i'+1\dots s-1}$ (resp. $\alpha_{t+1\dots j'-1}$). In other words, the left (resp. right) pseudoregular part of $\mathscr{E}_l$ for index m is the suboccurrence of $\mathscr{E}_l$ between the forgetful occurrence of the left preperiod and the mth composite central kernel (resp. between the mth composite central kernel and the forgetful occurrence of the right preperiod). Denote it by $\mathrm{LpR}_{k,m}(\mathscr{E}_l)$ (resp. by $\mathrm{RpR}_{k,m}(\mathscr{E}_l)$).

The following remark shows how the definitions of left and right pseudoregular parts of k-multiblocks and inner pseudoregular parts of k-multiblocks are connected.

Remark 6.16. *Let $\mathscr{E}$ be an evolution of k-blocks, $l_0 \ge 3k$, and let $\mathscr{F}$ be the evolution of k-multiblocks defined by $\mathscr{F}_l = \mathscr{E}_{l+l_0}$. Let m be an index ($1 \le m \le \mathrm{ncker}_k(\mathscr{E})$). Let m' ($1 \le m' \le \mathrm{nker}_k(\mathscr{F})$) be the index such that $\mathrm{cKer}_{k,m}(\mathscr{E}_{l+l_0}) = \mathrm{Ker}_{k,m'}(\mathscr{F}_l)$ as an occurrence in α for all $l \ge 0$. In other words, if Case I holds for $\mathscr{E}$ at the left, then $m' = m + 1$, otherwise $m' = m$.*

Then

1. *If Case I holds for $\mathscr{E}$ at the left, then $\mathrm{LpR}_{k,m}(\mathscr{E}_{l+l_0}) = \mathrm{IpR}_{k,1,m'}(\mathscr{F}_l)$.*
2. *If Case II holds for $\mathscr{E}$ at the left, then $\mathrm{LpR}_{k,m}(\mathscr{E}_{l+l_0}) = \mathrm{IpR}_{k,0,m'}(\mathscr{F}_l)$.*
3. *If Case I holds for $\mathscr{E}$ at the right, then $\mathrm{LpR}_{k,m}(\mathscr{E}_{l+l_0}) = \mathrm{IpR}_{k,m',\mathrm{nker}_k(\mathscr{F})}(\mathscr{F}_l)$.*
4. *If Case II holds for $\mathscr{E}$ at the right, then $\mathrm{LpR}_{k,m}(\mathscr{E}_{l+l_0}) = \mathrm{IpR}_{k,m',\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}_l)$.*

Definition 6.17. A final period λ is called a *left* (resp. *right*) *continuous evolutional period* of an evolution of k-blocks $\mathscr{E}$ for an index m if the following two conditions hold:

1. (a) For each $l \ge 3k$ (i. e. if $\mathscr{E}_l$ is a stable k-block), $\psi(\mathrm{LpR}_{k,m}(\mathscr{E}_l))$ (resp. $\psi(\mathrm{RpR}_{k,m}(\mathscr{E}_l))$) is a weakly left (resp. right) λ-periodic word.
   (b) The residue of $|\,\mathrm{LpR}_{k,m}(\mathscr{E}_l)|$ (resp. of $|\,\mathrm{RpR}_{k,m}(\mathscr{E}_l)|$) (for $l \ge 3k$) modulo $|\lambda|$, i. e. the length of the incomplete occurrence in the previous condition, does not depend on l.
2. If Case II holds for $\mathscr{E}$ at the left (resp. at the right) and $m > 1$ (resp. $m < \mathrm{ncker}_k(\mathscr{E})$) (i. e. $\mathrm{cKer}_{k,m}(\mathscr{E})$ is not the leftmost (resp. rightmost) composite central kernel for $\mathscr{E}$), then $\psi(\mathrm{LBS}_k(\mathscr{E}))$ (resp. $\psi(\mathrm{RBS}_k(\mathscr{E}))$) (this is a sequence infinite to the left (resp. to the right)) is periodic with period λ, i. e. $\psi(\mathrm{LBS}_k(\mathscr{E})) = \dots\lambda\dots\lambda\lambda$ (resp. $\psi(\mathrm{RBS}_k(\mathscr{E})) = \lambda\lambda\dots\lambda\dots$).

The following remark shows connection between the definitions of a continuous evolutional period of an evolution of k-blocks and a weak evolutional period of an evolution of stable k-multiblocks.

Remark 6.18. *Let $\mathscr{E}$ be an evolution of k-blocks, let λ be a final period, and let $\mathscr{F}$ be the evolution of k-multiblocks defined by $\mathscr{F}_l = \mathscr{E}_{l+3k}$. Let m be an index ($1 \le m \le \mathrm{ncker}_k(\mathscr{E})$). Again, let m' ($1 \le m' \le \mathrm{nker}_k(\mathscr{F})$) be the index such that $\mathrm{cKer}_{k,m}(\mathscr{E}_{l+3k}) = \mathrm{Ker}_{k,m'}(\mathscr{F}_l)$ as an occurrence in α for all $l \ge 0$. In other words, if Case I holds for $\mathscr{E}$ at the left, then $m' = m + 1$, otherwise $m' = m$.*

Then

1. (a) *If Case I holds for $\mathscr{E}$ at the left, then λ is a left continuous evolutional period of $\mathscr{E}$ for index m if and only if λ is a left weak evolutional period of $\mathscr{F}$ for pair $(1, m')$.*

   (b) *If Case II holds for $\mathscr{E}$ at the left, then Condition 1 in the definition of a left continuous evolutional period is satisfied for λ if and only if λ is a left weak evolutional period of $\mathscr{F}$ for pair $(0, m')$.*

2. (a) *If Case I holds for $\mathscr{E}$ at the right, then λ is a right continuous evolutional period of $\mathscr{E}$ for index m if and only if λ is a right weak evolutional period of $\mathscr{F}$ for pair $(m', \mathrm{nker}_k(\mathscr{F}))$.*

   (b) *If Case II holds for $\mathscr{E}$ at the right, then Condition 1 in the definition of a right continuous evolutional period is satisfied for λ if and only if λ is a right weak evolutional period of $\mathscr{F}$ for pair $(m', \mathrm{nker}_k(\mathscr{F}) + 1)$.*

Let us make several obvious remarks about this definition. First, if Case II holds at the left for $\mathscr{E}$, then the left component is empty, in particular, the left preperiod is empty. So, $\mathrm{LpR}_{k,m}(\mathscr{E}_l)$ is a prefix of $\mathscr{E}_l$, and by Corollary 6.8, a suffix of the sequence $\psi(\mathrm{LBS}_k(\mathscr{E})\,\mathrm{LpR}_{k,m}(\mathscr{E}_l))$ mentioned in Condition 2 is a factor of $\psi(\alpha)$ if l is large enough. Informally speaking, Condition 2 says that $\psi(\mathrm{LBS}_k(\mathscr{E}))$ is periodic and that the periods of $\psi(\mathrm{LpR}_{k,m}(\mathscr{E}_l))$ and of $\psi(\mathrm{LBS}_k(\mathscr{E}))$ "agree well".

If Case II holds at the left for $\mathscr{E}$ (and the left component is empty), and we are trying to figure out whether λ is a left continuous evolutional period of $\mathscr{E}$ for index 1 (for the first composite central kernel), then it follows from Lemma 5.22 that the word $\psi(\mathrm{LpR}_{k,m}(\mathscr{E}_l))$, whose periodicity is required by Condition 1, is actually empty in this case, and the condition is always satisfied. Condition 2 does not say anything about this case, it only applies if $m > 1$.

Finally, we are ready to give the definition of a continuously periodic evolution.

Definition 6.19. An evolution $\mathscr{E}$ of k-blocks is called *continuously periodic for an index m (*$1 \le m \le \mathrm{nker}_k(\mathscr{E})$*)* if there exist two final periods λ and λ' such that λ is a left continuous evolutional period of $\mathscr{E}$ for the index m and λ' is a right continuous evolutional period of $\mathscr{E}$ for the index m.

An evolution $\mathscr{E}$ of k-blocks is called *continuously periodic* if there exists an index m ($1 \le m \le \mathrm{nker}_k(\mathscr{E})$) such that $\mathscr{E}$ is continuously periodic for m.

Remark 6.20. *By Lemma 4.5, all evolutions of 1-blocks are continuously periodic for index 1 (recall that there is only one composite central kernel of a 1-block).*

One more important case when an evolution of k-blocks is continuously periodic for index 1 is the case when the number of the composite central kernels of an evolution is one, and Case II holds both at the left and at the right. Then all blocks in this evolution in fact consist of letters of order 1, but this does not necessarily mean that $k = 1$, k may be bigger than 1 if the left and the right border of this evolution are letters of order > 2. Condition 1 in the definition of a left and a right continuous evolutional period says in this case that some empty words have to be weakly periodic, and this is always true, and Condition 2 does not apply since there is only one composite central kernel, so such an evolution is always continuously periodic for index 1.

After we gave the definition of a continuously periodic evolution, the statements of Propositions 1.2–1.6 are completely formulated. Before we will be able to prove them, we need to give some more definitions.

A sequence $\mathscr{H} = \mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_2, \dots$ of occurrences in α is called a k-series of obstacles if there exists a number $p \le \mathbf{L}$ such that:

1. Each word $\psi(\mathscr{H}_l)$ is a weakly p-periodic word.
2. The length of $\mathscr{H}_l$ strictly grows as l grows, $|\mathscr{H}_l| \ge 2\mathbf{L}$ for all $l \ge 0$, and there exists k' ($1 \le k' \le k$) such that $|\mathscr{H}_l|$ is $\Theta(l^{k'})$ for $l \to \infty$.
3. If $\mathscr{H}_l = \alpha_{i\dots j}$, then $\psi(\alpha_{i\dots j+1})$ and (if $i > 0$) $\psi(\alpha_{i-1\dots j})$ are not weakly p-periodic words.

We will also need some notion of weak and continuous periodicity for k-multiblocks. Let $\mathscr{F}$ be an evolution of stable nonempty k-multiblocks ($k \geq 1$). The multiblocks $\mathscr{F}_l$ are nonempty in the multiblock sense, in other words, it is not allowed that $\mathscr{F}_l$ contains no k-blocks and no letters of order k, but it is allowed that $\mathscr{F}_l$ consists of a single empty k-block if this k-block is stable.

We call $\mathscr{F}$ *weakly periodic for a sequence of indices* $m_0 = 0, m_1, \ldots, m_{n-1}, m_n = \operatorname{nker}_k(\mathscr{F}) + 1$ ($1 \leq m_1 < m_2 < \ldots < m_{n-1} \leq \operatorname{nker}_k(\mathscr{F})$) if there exist final periods $\lambda^{(0)}, \ldots, \lambda^{(n-1)}$ such that $\lambda^{(i)}$ is a left weak evolutional period of $\mathscr{F}$ for pair (m_i, m_{i+1}) for all i ($0 \leq i < n$). By Remark 6.11, the left weak evolutional periods in this definition can be replaced by right weak evolutional periods, moreover, this can be done independently for each index i. We call $\mathscr{F}$ *weakly periodic* if there exists a sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1}, m_n = \operatorname{nker}_k(\mathscr{F}) + 1$ ($1 \leq m_1 < m_2 < \ldots < m_{n+1} \leq \operatorname{nker}_k(\mathscr{F})$) such that $\mathscr{F}$ is weakly periodic for $m_0, m_1, \ldots, m_{n-1}, m_n$.

Definition 6.21. We call $\mathscr{F}$ *continuously periodic for an index* m ($1 \leq m \leq \operatorname{nker}_k(\mathscr{F})$) if it is weakly periodic for indices $m_0 = 0, m_1 = m, m_2 = \operatorname{nker}_k(\mathscr{F}) + 1$, i. e. if there exist two final periods λ and λ' such that λ is a left weak evolutional period of $\mathscr{F}$ for pair $(0, m)$, and λ' is a right weak evolutional period of $\mathscr{F}$ for pair $(m, \operatorname{nker}_k(\mathscr{F}) + 1)$. We call $\mathscr{F}$ *continuously periodic* if there exists an index m ($1 \leq m \leq \operatorname{nker}_k(\mathscr{F})$) such that $\mathscr{F}$ is continuously periodic for m.

Note that in this definition, unlike in the definition for k-blocks, we don't have any left and right preperiods or a replacement for them, and the words whose periodicity we require (the $(0, m)$th and the $(m, \operatorname{nker}_k(\mathscr{F}) + 1)$th inner pseudoregular parts) are a prefix and a suffix (respectively) of (the forgetful occurrence of) a k-multiblock. Also, we don't introduce any left and right bounding sequence, and speak only about periodicity of factors of (the forgetful occurrence of) a k-multiblock itself.

Finally, a final period λ is called a *total left (resp. right) evolutional period* of $\mathscr{F}$ if it is a weak left (resp. right) evolutional period of $\mathscr{F}$ for the pair $(0, \operatorname{nker}_k(\mathscr{F}) + 1)$. $\mathscr{F}$ is called *totally periodic* if it is weakly periodic for the indices $m_0 = 0, m_1 = \operatorname{nker}_k(\mathscr{F}) + 1$.

Remark 6.22. *Let $\mathscr{F}$ be an evolution of stable nonempty k-blocks ($k \geq 1$) such that* $\operatorname{nker}_k(\mathscr{F}) = 1$*. Then $\mathscr{F}$ is weakly periodic for sequence* $0, 1, 2$ *and is continuously periodic for index 1. The corresponding final periods can be chosen arbitrarily since the inner pseudoregular parts in question in this case are empty occurrences.*

Lemma 6.23. *Let $\mathscr{F}$ be an evolution of stable nonempty k-multiblocks ($k \geq 1$). Suppose that $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1}, m_n = \operatorname{nker}_k(\mathscr{F}) + 1$. Suppose that $m_1 > 1$. Then $\mathscr{F}$ is also weakly periodic for sequence $m_0 = 0, 1, m_1, \ldots, m_{n-1}, m_n = \operatorname{nker}_k(\mathscr{F}) + 1$.*

Proof. Let λ be a right weak evolutional period of $\mathscr{F}$ for pair $(0, m_1)$. By definition this means that $\psi(\operatorname{IpR}_{k,0,m_1}(\mathscr{F}_l))$ is a weakly right λ-periodic word for all $l \geq 0$, and the residue of $|\operatorname{IpR}_{k,0,m_1}(\mathscr{F}_l)|$ modulo $|\lambda|$ does not depend on l. But $\operatorname{IpR}_{k,0,m_1}(\mathscr{F}_l) = \operatorname{Ker}_{k,1}(\mathscr{F}_l) \operatorname{IpR}_{k,1,m_1}(\mathscr{F}_l)$, so $\psi(\operatorname{IpR}_{k,1,m_1}(\mathscr{F}_l))$ is also a weakly right λ-periodic word, and the length of $\operatorname{Ker}_{k,1}(\mathscr{F}_l)$ does not depend on l, so the residue of $|\operatorname{IpR}_{k,1,m_1}(\mathscr{F}_l)| = |\operatorname{IpR}_{k,0,m_1}(\mathscr{F}_l)| - |\operatorname{Ker}_{k,1}(\mathscr{F}_l)|$ modulo $|\lambda|$ does not depend on l either. So, λ is also a right weak evolutional period of $\mathscr{F}$ for pair $(1, m_1)$. Also, any final period is a (right) weak evolutional period of $\mathscr{F}$ for pair $(0, 1)$ since $\operatorname{IpR}_{k,0,1}(\mathscr{F}_l)$ is always an empty occurrence. Hence, we can insert an index 1 in the sequence for which $\mathscr{F}$ is weakly periodic. □

Lemma 6.24. *Let $\mathscr{F}$ be an evolution of stable nonempty k-multiblocks ($k \geq 1$). Suppose that $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1}, m_n = \operatorname{nker}_k(\mathscr{F}) + 1$. Suppose that $m_{n-1} < \operatorname{nker}_k(\mathscr{F})$. Then $\mathscr{F}$ is also weakly periodic for sequence $m_0 = 0, m_1, \ldots, m_{n-1}, \operatorname{nker}_k(\mathscr{F}), m_n = \operatorname{nker}_k(\mathscr{F}) + 1$.*

Proof. The proof is completely similar to the proof of the previous lemma. □

Our next goal is to prove for every $k \in \mathbb{N}$ that if all evolutions of k-blocks present in α are continuously periodic, then either there exists a k-series of obstacles in α, or all evolutions of $(k+1)$-blocks present in α are continuously periodic. In order to prove this, we prove several lemmas.

Let $\mathscr{F}$ and $\mathscr{F}'$ be two consecutive evolutions of stable nonempty k-multiblocks ($k \geq 1$). Suppose that $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1} = \mathrm{nker}_k(\mathscr{F}), m_n = \mathrm{nker}_k(\mathscr{F}) + 1$ and $\mathscr{F}'$ is weakly periodic for a sequence of indices $m'_0 = 0, m'_1 = 1, \ldots, m'_{n'-1}, m'_{n'} = \mathrm{nker}_{k,m}(\mathscr{F}') + 1$. Denote the concatenation of $\mathscr{F}$ and $\mathscr{F}'$ by $\mathscr{F}''$.

Consider the following sequence: $m''_0 = 0, m''_1 = m_1, \ldots, m''_{n-1} = m_{n-1} = \mathrm{nker}_k(\mathscr{F}) = \mathrm{nker}_k(\mathscr{F}) - 1 + m'_1, m''_n = \mathrm{nker}_k(\mathscr{F}) - 1 + m'_2, \ldots, m''_{n-2+n'-1} = \mathrm{nker}_k(\mathscr{F}) - 1 + m'_{n'-1}, m''_{n-2+n'} = \mathrm{nker}_k(\mathscr{F}) - 1 + m'_{n'} = \mathrm{nker}_k(\mathscr{F}) - 1 + \mathrm{nker}_k(\mathscr{F}') + 1 = \mathrm{nker}_k(\mathscr{F}'') + 1$ (the last equality is Lemma 5.31). In other words, we did the following. We removed the last entry $\mathrm{nker}_k(\mathscr{F}) + 1$ from the sequence m and the first entry 0 from the sequence m'. Then we added $\mathrm{nker}_k(\mathscr{F}) - 1$ to each remaining entry of the second sequence. The last remaining entry in the first sequence now coincides with the first remaining entry in the second sequence and equals $\mathrm{nker}_k(\mathscr{F})$. We remove one of these two coinciding entries and take the concatenation of the resulting two sequences.

We are going to prove that $\mathscr{F}''$ is weakly periodic for the sequence $m''_0, \ldots, m''_{n-2+n'}$.

Lemma 6.25. *Let $\mathscr{F}$ and $\mathscr{F}'$ be two consecutive evolutions of stable nonempty k-multiblocks ($k \geq 1$) such that $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1} = \mathrm{nker}_k(\mathscr{F}), m_n = \mathrm{nker}_k(\mathscr{F}) + 1$ and $\mathscr{F}'$ is weakly periodic for a sequence of indices $m'_0 = 0, m'_1 = 1, \ldots, m'_{n'-1}, m'_{n'} = \mathrm{nker}_{k,m}(\mathscr{F}') + 1$.*

Then the concatenation $\mathscr{F}''$ of $\mathscr{F}$ and $\mathscr{F}'$ is weakly periodic for the sequence $m''_0 = 0, m''_1 = m_1, \ldots, m''_{n-1} = m_{n-1}, m''_n = \mathrm{nker}_k(\mathscr{F}) - 1 + m'_2, \ldots, m''_{n-2+n'-1} = \mathrm{nker}_k(\mathscr{F}) - 1 + m'_{n'-1}, m''_{n-2+n'} = \mathrm{nker}_k(\mathscr{F}) - 1 + m'_{n'} = \mathrm{nker}_k(\mathscr{F}) - 1 + \mathrm{nker}_k(\mathscr{F}') + 1 = \mathrm{nker}_k(\mathscr{F}'') + 1$.

Proof. Let $\lambda^{(0)}, \ldots, \lambda^{(n-1)}$ be final periods such that $\lambda^{(i)}$ is a left weak evolutional period of $\mathscr{F}$ for pair (m_i, m_{i+1}), and let $\lambda'^{(0)}, \ldots, \lambda'^{(n'-1)}$ be final periods such that $\lambda'^{(i)}$ is a left weak evolutional period of $\mathscr{F}'$ for pair (m'_i, m'_{i+1}).

Consider the following sequence of final periods: $\lambda''^{(0)} = \lambda^{(0)}, \ldots, \lambda''^{(n-2)} = \lambda^{(n-2)}, \lambda''^{(n-2+1)} = \lambda'^{(1)}, \lambda''^{(n-2+n'-1)} = \lambda'^{(n'-1)}$. In other words, we the sequence of final periods $\lambda^{(0)}, \ldots, \lambda^{(n-1)}$, removed the last entry, took the sequence of final periods $\lambda'^{(0)}, \ldots, \lambda'^{(n'-1)}$, removed the first entry, and took the concatenation of the resulting two sequences.

Set $n'' = n - 2 + n'$. By Lemma 5.31 we see that if $0 \leq i < n - 1$, then $\mathrm{IpR}_{k,m''_i,m''_{i+1}}(\mathscr{F}''_l) = \mathrm{IpR}_{k,m_i,m_{i+1}}(\mathscr{F}_l)$ as an occurrence in α for all $l \geq 0$. And if $n - 1 \leq i < n''$, then $\mathrm{IpR}_{k,m''_i,m''_{i+1}}(\mathscr{F}''_l) = \mathrm{IpR}_{k,m'_{i-(n-2)},m'_{i+1-(n-2)}}(\mathscr{F}'_l)$ as an occurrence in α for all $l \geq 0$. So, if $0 \leq i < n-1$, then $\lambda''^{(i)} = \lambda^{(i)}$ is a left weak evolutional period of $\mathscr{F}$ for pair (m_i, m_{i+1}), hence it is also a a left weak evolutional period of $\mathscr{F}''$ for pair (m''_i, m''_{i+1}). And if $n - 1 \leq i < n''$, then $\lambda''^{(i)} = \lambda'^{(i-(n-2))}$ is a left weak evolutional period of $\mathscr{F}'$ for pair $(m'_{i-(n-2)}, m'_{i+1-(n-2)})$, hence it is also a a left weak evolutional period of $\mathscr{F}''$ for pair (m''_i, m''_{i+1}). Therefore, $\mathscr{F}''$ is weakly periodic for the sequence $m''_0, \ldots, m''_{n''}$. $\square$

Corollary 6.26. *Let $\mathscr{F}$ and $\mathscr{F}'$ be two consecutive evolutions of stable nonempty k-multiblocks ($k \geq 1$). If $\mathscr{F}$ and $\mathscr{F}'$ are weakly periodic, then the concatenation of $\mathscr{F}$ and $\mathscr{F}'$ is also weakly periodic.*

Proof. Let $m_0 = 0, m_1, \ldots, m_{n-1}, m_n = \mathrm{nker}_k(\mathscr{F}) + 1$ be a sequence of indices such that $\mathscr{F}$ is weakly periodic for $m_0, \ldots, m_n$. Let $m'_0 = 0, m'_1, \ldots, m'_{n'-1}, m'_{n'} = \mathrm{nker}_{k,m}(\mathscr{F}') + 1$ be a sequence of indices such that $\mathscr{F}'$ is weakly periodic for $m'_0, \ldots, m'_{n'}$. By Lemma 6.24, without loss of generality (possibly increasing n by 1), we may suppose that $m_{n-1} = \mathrm{nker}_k(\mathscr{F})$. Similarly, by Lemma 6.23, possibly increasing n' by 1, without loss of generality we may suppose that $m'_1 = 1$. The claim now follows from Lemma 6.25. $\square$

Lemma 6.27. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{F}$ be an evolution of nonempty stable k-multiblocks. Then $\mathscr{F}$ is weakly periodic.*

Proof. If each k-multiblock in $\mathscr{F}$ consists of a single periodic letter of order $k+1$, then the claim is clear by Remark 6.22.

If each k-multiblock in $\mathscr{F}$ consists of a single k-block, then there exists an evolution $\mathscr{E}$ of k-blocks and a number $l_0 \geq 3k$ such that $\mathscr{F}_l$ consists of $\mathscr{E}_{l+l_0}$ for all $l \geq 0$. By assumption, $\mathscr{E}$ is continuously periodic, and there exists an index m ($1 \leq m \leq \mathrm{nker}_k(\mathscr{E})$) such that $\mathscr{E}$ is continuously periodic for the index m. Now it follows from Remark 6.18 that there exists an index m' (it can equal m or $m+1$) such that $\mathscr{F}$ is weakly periodic for the sequence $0, m', \mathrm{nker}_k(\mathscr{F})+1$.

Finally, the claim for general $\mathscr{F}$ follows from Lemma 6.25. □

Lemma 6.28. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks. Suppose that there exists a final period λ such that λ is a left weak evolutional period of $\mathscr{F}$ for a pair (m, m') ($0 \leq m < m' \leq \mathrm{nker}_k(\mathscr{F})$). Suppose also that there exists a final period μ such that μ is a left weak evolutional period of $\mathscr{F}$ for a pair (m', m'') ($m' < m'' \leq \mathrm{nker}_k(\mathscr{F})+1$). Then exactly one of the following two statements is true:*

1. *λ is a left weak evolutional period of $\mathscr{F}$ for pair (m, m'').*
2. *There exists a number $s \in \mathbb{N}$ such that the following is true for all $l \geq 0$. Suppose that $\mathrm{Fg}(\mathscr{F}_l) = \alpha_{i\dots j}$ and $\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l) = \alpha_{i'\dots j'}$. Then:*
   - *(a) $\psi(\alpha_{i'\dots j'+s-1})$ is a weakly left λ-periodic word, and $\psi(\alpha_{i'\dots j'+s})$ is not a weakly left λ-periodic word.*
   - *(b) $s \leq |\mathrm{Ker}_{k,m'}(\mathscr{F})| + 2\mathbf{L}$.*
   - *(c) $j' + s \leq j$, i. e. $\alpha_{j'+s}$ is a letter in $\mathrm{Fg}(\mathscr{F}_l)$.*

Proof. Denote the remainder of $|\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)|$ modulo $|\lambda|$ by r (by the definition of a weak left evolutional period, r does not depend on l). Set $\lambda' = \mathrm{Cyc}_r(\lambda)$. By Remark 6.11, this is a right weak evolutional period of $\mathscr{F}$ for the pair (m, m').

We have two possibilities for $\mathrm{Ker}_{k,m'}(\mathscr{F})$: either $\psi(\mathrm{Ker}_{k,m'}(\mathscr{F}))$ is a weakly left λ'-periodic, or $\psi(\mathrm{Ker}_{k,m'}(\mathscr{F}))$ is not weakly left λ'-periodic.

If $\psi(\mathrm{Ker}_{k,m'}(\mathscr{F}))$ is not weakly left λ'-periodic, denote by δ the weakly left λ'-periodic word of length $|\mathrm{Ker}_{k,m'}(\mathscr{F})|$, and denote by t the minimal ("the leftmost") index such that $\delta_t \neq \psi(\mathrm{Ker}_{k,m'}(\mathscr{F}))_t$. Then $\psi(\mathrm{Ker}_{k,m'}(\mathscr{F})_{0\dots t-1})$ is a weakly left λ'-periodic word, and $\psi(\mathrm{Ker}_{k,m'}(\mathscr{F})_{0\dots t})$ is not a weakly left λ'-periodic word. Hence, $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)\,\mathrm{Ker}_{k,m'}(\mathscr{F})_{0\dots t-1})$ is a weakly left λ-periodic word for all $l \geq 0$, and $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)\,\mathrm{Ker}_{k,m'}(\mathscr{F})_{0\dots t})$ is not a weakly left λ-periodic word for all $l \geq 0$. Set $s = t+1$. If $\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l) = \alpha_{i'\dots j'}$, then $\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)\,\mathrm{Ker}_{k,m'}(\mathscr{F})_{0\dots t-1} = \alpha_{i'\dots j'+s-1}$ and $\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)\,\mathrm{Ker}_{k,m'}(\mathscr{F})_{0\dots t} = \alpha_{i'\dots j'+s}$ as an occurrence in α. The largest possible value of t is $|\mathrm{Ker}_{k,m'}(\mathscr{F})|-1$, so $s \leq |\mathrm{Ker}_{k,m'}(\mathscr{F})| \leq |\mathrm{Ker}_{k,m'}(\mathscr{F})| + 2\mathbf{L}$. $\alpha_{j'+s}$ is a letter in $\mathrm{Ker}_{k,m'}(\mathscr{F}_l)$, so $j'+s \leq j$.

Suppose now that $\psi(\mathrm{Ker}_{k,m'}(\mathscr{F}))$ is weakly left λ'-periodic. Then for all $l \geq 0$, $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)\,\mathrm{Ker}_{k,m'}(\mathscr{F}))$ is a weakly left λ-periodic word. If $m' = \mathrm{nker}_k(\mathscr{F})$, then $m'' = \mathrm{nker}_k(\mathscr{F})+1$, and we are done since in this case $\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)\,\mathrm{Ker}_{k,m'}(\mathscr{F}) = \mathrm{IpR}_{k,m,m''}(\mathscr{F}_l)$ for all $l \geq 0$, and $|\mathrm{Ker}_{k,m'}(\mathscr{F})|$ and the remainder of $|\mathrm{Ker}_{k,m'}(\mathscr{F})|$ modulo $|\lambda|$ do not depend on l.

Otherwise, denote $\lambda'' = \mathrm{Cyc}_{|\mathrm{Ker}_{k,m'}(\mathscr{F})|}(\lambda')$. Then $\psi(\mathrm{Ker}_{k,m'}(\mathscr{F}))$ is a weakly right λ''-periodic word and $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)\,\mathrm{Ker}_{k,m'}(\mathscr{F}))$ also is a weakly right λ''-periodic word.

Denote by δ' the weakly left λ''-periodic word of length $2\mathbf{L}$, and denote by γ the weakly left μ-periodic word of length $2\mathbf{L}$ (recall that μ is a left weak evolutional period of $\mathscr{F}$ for the pair (m', m'')). Now we are considering the case when $m' < \mathrm{nker}_k(\mathscr{F})$, and we also have $m'' > m' > m$, so $m'' > 1$, and by Corollary 5.30, $|\mathrm{IpR}_{k,m',m''}(\mathscr{F}_l)| \geq 2\mathbf{L}$ for all $l \geq 0$. Since μ is a left weak evolutional period of $\mathscr{F}$ for the pair (m', m''), γ is a prefix of $\psi(\mathrm{IpR}_{k,m',m''}(\mathscr{F}_l))$ for all $l \geq 0$. Again, we have two possibilities: $\gamma = \delta'$ or $\gamma \neq \delta'$.

If $\gamma \neq \delta'$, denote by t the smallest index such that $\gamma_t \neq \delta'_t$. Then $\gamma_{0\dots t-1}$ is weakly left λ''-periodic, and $\gamma_{0\dots t}$ is not weakly left λ''-periodic. Fix a number $l \geq 0$. Let i and j be the indices such that $\mathrm{Fg}(\mathscr{F}_l) = \alpha_{i\dots j}$, and let i' and j' be the indices such that $\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l) = \alpha_{i'\dots j'}$. Recall that γ is a prefix of $\psi(\mathrm{IpR}_{k,m',m''}(\mathscr{F}_l))$, that $\psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)\,\mathrm{Ker}_{k,m'}(\mathscr{F}))$ is a weakly right λ''-periodic word and is a weakly left λ-periodic word for all $l \geq 0$. Hence, $\psi(\alpha_{i'\dots j'+|\mathrm{Ker}_{k,m'}(\mathscr{F})|+t}) = \psi(\mathrm{IpR}_{k,m,m'}(\mathscr{F}_l)\,\mathrm{Ker}_{k,m'}(\mathscr{F}))\gamma_{0\dots t-1}$ is a weakly left λ-periodic word, and $\psi(\alpha_{i'\dots j'+|\operatorname{Ker}_{k,m'}(\mathscr{F})|+t+1}) = \psi(\operatorname{IpR}_{k,m,m'}(\mathscr{F}_l)\operatorname{Ker}_{k,m'}(\mathscr{F}))\gamma_{0\dots t}$ is not a weakly left λ-periodic word. So, we can set $s = |\operatorname{Ker}_{k,m'}(\mathscr{F})| + t + 1$ and see that the claim is true in this case since t does not depend on l, $s = |\operatorname{Ker}_{k,m'}(\mathscr{F})| + t + 1 \le |\operatorname{Ker}_{k,m'}(\mathscr{F})| + |\gamma| = |\operatorname{Ker}_{k,m'}(\mathscr{F})| + 2\mathbf{L}$, and γ and hence $\gamma_{0\dots t}$ are prefixes of $\psi(\operatorname{IpR}_{k,m',m''}(\mathscr{F}_l))$, so $j' + s \le j$.

Finally, suppose that $\gamma = \delta'$. Since λ'' and μ are final periods, we have $\lambda'' = \mu$ by Corollary 2.12. Hence, $\psi(\operatorname{IpR}_{k,m',m''}(\mathscr{F}_l))$ is weakly left λ''-periodic for all $l \ge 0$ and the remainder of $|\operatorname{IpR}_{k,m',m''}(\mathscr{F}_l)|$ modulo $|\lambda''|$ does not depend on l. Again, recall that $\psi(\operatorname{IpR}_{k,m,m'}(\mathscr{F}_l)\operatorname{Ker}_{k,m'}(\mathscr{F}))$ is a weakly right λ''-periodic word and is a weakly left λ-periodic word for all $l \ge 0$. Therefore, $\psi(\operatorname{IpR}_{k,m,m'}(\mathscr{F}_l)\operatorname{Ker}_{k,m'}(\mathscr{F})\operatorname{IpR}_{k,m',m''}(\mathscr{F}_l)) = \psi(\operatorname{IpR}_{k,m,m''}(\mathscr{F}_l))$ is also a weakly left λ-periodic word. The remainder of $\operatorname{IpR}_{k,m,m''}(\mathscr{F}_l) = |\operatorname{IpR}_{k,m,m'}(\mathscr{F}_l)| + |\operatorname{Ker}_{k,m'}(\mathscr{F})| + |\operatorname{IpR}_{k,m',m''}(\mathscr{F}_l)|$ modulo $|\lambda|$ does not depend on l since the remainder of each summand modulo $|\lambda| = |\lambda''|$ does not depend on l. So, in this case λ is a weak left evolutional period of $\mathscr{F}$ for the pair (m, m'') ☐

Corollary 6.29. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks. Suppose that there exist a final period λ such that λ is a right weak evolutional period of $\mathscr{F}$ for a pair (m, m') $(0 \le m < m' \le \operatorname{nker}_k(\mathscr{F}))$. and λ is a right weak evolutional period of $\mathscr{F}$ for a pair (m_0, m') $(0 \le m_0 < m' \le \operatorname{nker}_k(\mathscr{F}))$.*

Suppose also that there exists a final period μ such that μ is a left weak evolutional period of $\mathscr{F}$ for a pair (m', m'') $(m' < m'' \le \operatorname{nker}_k(\mathscr{F}) + 1)$.

Then there exists a left weak evolutional period of $\mathscr{F}$ for pair (m, m'') if and only if there exists a left weak evolutional period of $\mathscr{F}$ for pair (m_0, m'').

Proof. If $m = m_0$, then everything is clear, so suppose that $m \ne m_0$. Then m' cannot be equal to 1, and $m' > 1$, hence, $m'' > 1$. Also, $m' < m''$, so $m' \le \operatorname{nker}_k(\mathscr{F})$, hence $m < \operatorname{nker}_k(\mathscr{F})$ and $m_0 < \operatorname{nker}_k(\mathscr{F})$.

Denote by r the remainder of $|\operatorname{IpR}_{k,m,m'}(\mathscr{F}_l)|$ modulo $|\lambda|$ (for any $l \ge 0$) and set $\lambda' = \operatorname{Cyc}_{-r}(\lambda)$. By Remark 6.11, λ' is a left weak evolutional period of $\mathscr{F}$ for the pair (m, m'). By Lemma 6.13, if there exists a weak left evolutional period of $\mathscr{F}$ for the pair (m, m''), it equals λ'. By Lemma 6.28, λ' is not a left weak evolutional period of $\mathscr{F}$ for the pair (m, m'') if and only if there exists $s \in \mathbb{N}$ such that the following is true:

For each $l \ge 0$, suppose that $\operatorname{Fg}(\mathscr{F}_l) = \alpha_{i\dots j}$ and $\operatorname{IpR}_{k,m,m'}(\mathscr{F}_l) = \alpha_{i'\dots j'}$. Then:

1. $\psi(\alpha_{i'\dots j'+s-1})$ is a weakly left λ'-periodic word, and $\psi(\alpha_{i'\dots j'+s})$ is not a weakly left λ'-periodic word.
2. $s \le |\operatorname{Ker}_{k,m'}(\mathscr{F})| + 2\mathbf{L}$.
3. $j' + s \le j$, i. e. $\alpha_{j'+s}$ is a letter in $\operatorname{Fg}(\mathscr{F}_l)$.

Since r is the remainder of $|\operatorname{IpR}_{k,m,m'}(\mathscr{F}_l)|$ modulo $|\lambda| = |\lambda'|$, $\lambda = \operatorname{Cyc}_r(\lambda')$, and $\psi(\alpha_{i'\dots j'})$ is a weakly left λ'-periodic word, Condition 1 in the list above is equivalent to the following condition: $\psi(\alpha_{j'+1\dots j'+s-1})$ is a weakly left λ-periodic word, and $\psi(\alpha_{j'+1\dots j'+s})$ is not a weakly left λ-periodic word.

Therefore, a weak left evolutional period of $\mathscr{F}$ for the pair (m, m'') does not exist if and only if there exists $s \in \mathbb{N}$ such that the following is true:

For each $l \ge 0$, suppose that $\operatorname{Fg}(\mathscr{F}_l) = \alpha_{i\dots j}$ and $\operatorname{IpR}_{k,m,m'}(\mathscr{F}_l) = \alpha_{i'\dots j'}$. Then:

1. $\psi(\alpha_{j'+1\dots j'+s-1})$ is a weakly left λ-periodic word, and $\psi(\alpha_{j'+1\dots j'+s})$ is not a weakly left λ-periodic word.
2. $s \le |\operatorname{Ker}_{k,m'}(\mathscr{F})| + 2\mathbf{L}$.
3. $j' + s \le j$, i. e. $\alpha_{j'+s}$ is a letter in $\operatorname{Fg}(\mathscr{F}_l)$.

But these conditions do not use the indices m and i', so we can repeat the arguments above for m_0 instead of m and conclude that the nonexistence of a weak left evolutional period of $\mathscr{F}$ for the pair (m_0, m'') is equivalent to the same list of conditions. ☐

Lemma 6.30. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks. Suppose that there exists a final period λ such that λ is a right weak evolutional period of $\mathscr{F}$ for a pair (m', m) $(1 \le m' < m \le \mathrm{nker}_k(\mathscr{F}) + 1)$. Suppose also that there exists a final period μ such that μ is a right weak evolutional period of $\mathscr{F}$ for a pair (m'', m') $(0 \le m'' < m')$. Then one of the following two statements is true:*

1. *λ is a right weak evolutional period of $\mathscr{F}$ for pair (m'', m).*
2. *There exists a number $s \in \mathbb{N}$ such that the following is true for all $l \ge 0$. Suppose that $\mathrm{Fg}(\mathscr{F}_l) = \alpha_{i\ldots j}$ and $\mathrm{IpR}_{k,m',m}(\mathscr{F}_l) = \alpha_{i'\ldots j'}$. Then:*
   - *(a) $\psi(\alpha_{i'-s+1\ldots j'})$ is a weakly right λ-periodic word, and $\psi(\alpha_{i'-s\ldots j'})$ is not a weakly right λ-periodic word.*
   - *(b) $s \le |\mathrm{Ker}_{k,m}(\mathscr{F})| + 2\mathbf{L}$.*
   - *(c) $i' - s \ge i$, i. e. $\alpha_{i'-s}$ is a letter in $\mathrm{Fg}(\mathscr{F}_l)$.*

Proof. The proof is completely symmetric to the proof of Lemma 6.28. □

Corollary 6.31. *Let $\mathscr{F}$ be an evolution of stable k-multiblocks. Suppose that there exist a final period λ such that λ is a left weak evolutional period of $\mathscr{F}$ for a pair (m', m) $(1 \le m' < m \le \mathrm{nker}_k(\mathscr{F}) + 1)$. and λ is a left weak evolutional period of $\mathscr{F}$ for a pair (m', m_0) $(1 \le m' < m_0 \le \mathrm{nker}_k(\mathscr{F}) + 1)$.*

Suppose also that there exists a final period μ such that μ is a right weak evolutional period of $\mathscr{F}$ for a pair (m'', m') $(0 \le m'' < m')$.

Then there exists a right weak evolutional period of $\mathscr{F}$ for pair (m'', m) if and only if there exists a right weak evolutional period of $\mathscr{F}$ for pair (m'', m_0).

Proof. The proof is completely symmetric to the proof of Corollary 6.29. □

Lemma 6.32. *Let $\mathscr{F}$ be an evolution of stable nonempty k-multiblocks $(k \ge 1)$. Suppose that $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1}, m_n = \mathrm{nker}_k(\mathscr{F}) + 1$, where $n \ge 2$. Let q be an index $(1 \le q \le n-1)$ and let λ be a right weak evolutional period of $\mathscr{F}$ for pair (m_q, m_{q+1}). Suppose also that λ is not a right weak evolutional period of $\mathscr{F}$ for pair (m_{q-1}, m_{q+1}). Then there are two possibilities:*

1. *$\mathscr{F}$ is weakly periodic for the following sequence of indices:*

$$m_0 = 0, m_1, \ldots, m_{q-1}, m_q, \mathrm{nker}_k(\mathscr{F}) + 1.$$

2. *There exists a k-series of obstacles in α.*

Proof. If $q = n - 1$, then everything is clear. Suppose that $q < n - 1$.

Since λ is not a right weak evolutional period of $\mathscr{F}$ for the pair (m_{q-1}, m_{q+1}), it follows from Lemma 6.30 that there exists $s \in \mathbb{N}$ (s does not depend on l) such that for all $l \ge 0$, if $\mathrm{IpR}_{k,m_q,m_{q+1}}(\mathscr{F}_l) = \alpha_{i'\ldots j'}$, then $\psi(\alpha_{i'-s+1\ldots j'})$ is a weakly right λ-periodic word, and $\psi(\alpha_{i'-s\ldots j'})$ is not a weakly right λ-periodic word. Again, denote the residue of $|\mathrm{IpR}_{k,m_q,m_{q+1}}(\mathscr{F}_l)|$ for any $l \ge 0$ by r and denote $\lambda' = \mathrm{Cyc}_{-r}(\lambda)$. Then λ' is a weak left evolutional period of $\mathscr{F}$ for pair (m_q, m_{q+1}), and, if $\mathrm{IpR}_{k,m_q,m_{q+1}}(\mathscr{F}_l) = \alpha_{i'\ldots j'}$, then $\psi(\alpha_{i'-s+1\ldots i'-1})$ is a weakly right λ'-periodic word, and $\psi(\alpha_{i'-s\ldots i'-1})$ is not a weakly right λ'-periodic word.

Now let q' be the maximal index $(q + 1 \le q' \le n)$ such that λ' is a weak left evolutional period of $\mathscr{F}$ for pair $(m_q, m_{q'})$. If $q' = n$, then $\mathscr{F}$ is weakly periodic for the sequence $m_0 = 0, m_1, \ldots, m_{q-1}, m_q, \mathrm{nker}_k(\mathscr{F})+1$, and we are done.

Suppose that $q' < n$. Then, since q' is maximal, it follows from Lemma 6.28 that there exists $s' \in \mathbb{N}$ (s' does not depend on l) such that for all $l \ge 0$, if $\mathrm{IpR}_{k,m_q,m_{q'}}(\mathscr{F}_l) = \alpha_{i''\ldots j''}$, then $\psi(\alpha_{i''\ldots j''+s'-1})$ is a weakly left λ'-periodic word, and $\psi(\alpha_{i''\ldots j''+s'})$ is not a weakly left λ'-periodic word. Note also that if $\mathrm{IpR}_{k,m_q,m_{q+1}}(\mathscr{F}_l) = \alpha_{i'\ldots j'}$ and $\mathrm{IpR}_{k,m_q,m_{q'}}(\mathscr{F}_l) = \alpha_{i''\ldots j''}$, then, by the definition of an inner pseudoregular part, $i' = i''$.

Summarizing, we have the following weakly periodic and non-weakly periodic words. Fix $l \geq 0$ and let i' and j'' be the indices such that $\operatorname{IpR}_{k,m_q,m_{q'}}(\mathscr{F}_l) = \alpha_{i'\dots j''}$. The following two occurrences in $\psi(\alpha)$ are weakly $|\lambda'|$-periodic: $\psi(\alpha_{i'-s+1\dots i'-1})$ with right period λ' and $\psi(\alpha_{i'\dots j''+s'-1})$ with left period λ'. And the following two occurrences are **not** weakly $|\lambda'|$-periodic with right and left period λ', respectively: $\psi(\alpha_{i'-s\dots i'-1})$ with right period λ' and $\psi(\alpha_{i'\dots j''+s'})$ with left period λ'.

Denote $\lambda'' = \operatorname{Cyc}_{-(s-1)}(\lambda')$. Then $\psi(\alpha_{i'-s+1\dots j''+s'-1})$ is a weakly left λ''-periodic word, $\alpha_{i'-s} \neq \lambda''_{|\lambda''|-1}$, and $\psi(\alpha_{i'-s+1\dots j''+s'})$ is not a weakly left λ''-periodic word. In other words, if r' is the residue of $(j''+s') - (i'-s+1)$ modulo $|\lambda''|$, then $\psi(\alpha_{j''+s'}) \neq \lambda''_{r'}$.

Set $p = |\lambda''| = |\lambda|$ and set $\mathscr{H}_l = \alpha_{i'-s+1\dots j''+s'-1}$. Let us prove that $\mathscr{H} = \mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_2, \dots$ is a k-series of obstacles in α. We have $|\mathscr{H}_l| = |\alpha_{i'-s+1\dots i'-1}| + |\alpha_{i'\dots j''}| + |\alpha_{j''+1\dots j''+s'-1}| = (s-1) + |\operatorname{IpR}_{k,m_q,m_{q'}}(\mathscr{F}_l)| + (s'-1)$. By Lemma 5.29 (here we use the fact that $0 < q < q' < n$, so $1 \leq m_q < m_{q'} \leq \operatorname{nker}_k(\mathscr{F})$), $|\operatorname{IpR}_{k,m_q,m_{q'}}(\mathscr{F}_l)|$ strictly grows as l grows, $|\operatorname{IpR}_{k,m_q,m_{q'}}(\mathscr{F}_l)| \geq 2\mathbf{L}$, and there exists $k' \in \mathbb{N}$ $(1 \leq k' \leq k)$ such that $|\operatorname{IpR}_{k,m_q,m_{q'}}(\mathscr{F}_l)|$ is $\Theta(l^{k'})$ for $l \to \infty$. Since $s, s' \in \mathbb{N}$ do not depend on l, $|\mathscr{H}_l|$ also strictly grows as l grows, $|\mathscr{H}_l| \geq 2\mathbf{L}$, and $|\mathscr{H}_l|$ is $\Theta(l^{k'})$ for $l \to \infty$.

Now let us check the required weak p-periodicity for the definition of a k-series of obstacles. We already know that $\psi(\mathscr{H}_l)$ is a weakly p-periodic word with left period λ''. Since λ is a final period, $|\mathscr{H}_l| \geq 2\mathbf{L} \geq |\lambda| = |\lambda''| = p$, and $\psi(\alpha_{i'-s+1+p-1}) = \lambda''_{p-1}$, while $\psi(\alpha_{i'-s}) \neq \lambda''_{p-1}$, so $\psi(\alpha_{i'-s\dots j''+s'-1})$ is not a weakly p-periodic word. Similarly, if r' is the residue of $(j''+s') - (i'-s+1)$ modulo $|\lambda''|$, then $\psi(\alpha_{j''+s'}) \neq \lambda''_{r'}$, but since $|\mathscr{H}_l| \geq p$, we have $\psi(\alpha_{j''+s'-p}) = \lambda''_{r'}$, so $\psi(\alpha_{i'-s+1\dots j''+s'})$ is not a weakly p-periodic word. (We knew before that $\psi(\alpha_{i'-s+1\dots j''+s'})$ is not a weakly p-periodic word with left period λ'', but it is important here that $|\mathscr{H}_l| \geq p$, otherwise we could have $|\psi(\alpha_{i'-s+1\dots j''+s'})| \leq p$, and then $\psi(\alpha_{i'-s+1\dots j''+s'})$ would be a weakly p-periodic word with another left period.) ☐

Lemma 6.33. *Let $\mathscr{F}$ be an evolution of stable nonempty k-multiblocks ($k \geq 1$). Suppose that $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \dots, m_{n-1}, m_n = \operatorname{nker}_k(\mathscr{F}) + 1$, where $n \geq 2$. Let q be an index ($1 \leq q \leq n-1$) and let λ be a left weak evolutional period of $\mathscr{F}$ for pair (m_{q-1}, m_q). Suppose also that λ is not a left weak evolutional period of $\mathscr{F}$ for pair (m_{q-1}, m_{q+1}). Then there are two possibilities:*

1. *$\mathscr{F}$ is weakly periodic for the following sequence of indices:*

$$0, m_q, m_{q+1}, \dots, m_{n-1}, m_n = \operatorname{nker}_k(\mathscr{F}) + 1.$$

2. *There exists a k-series of obstacles in α.*

Proof. The proof is completely symmetric to the proof of the previous lemma. ☐

Lemma 6.34. *Let $\mathscr{F}$ and $\mathscr{F}'$ be two consecutive evolutions of stable nonempty k-multiblocks ($k \geq 1$), and let $\mathscr{F}''$ be their concatenation. Suppose that $\operatorname{nker}_k(\mathscr{F}) > 1$, and $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \dots, m_{n-1}, m_n = \operatorname{nker}_k(\mathscr{F}) + 1$, where $m_{n-1} < \operatorname{nker}_k(\mathscr{F})$ ($n \in \mathbb{N}$, and $n = 1$ is allowed). Suppose that $\mathscr{F}''$ is also weakly periodic for the following sequence: $m'_0 = m_0 = 0, m'_1 = m_1, \dots, m'_{n-1} = m_{n-1}, m'_n = \operatorname{nker}_k(\mathscr{F}'') + 1$.*

Then $\mathscr{F}'$ is totally periodic, moreover, if λ is the (unique by Corollary 6.15) right weak evolutional period of $\mathscr{F}$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F}) + 1)$, then λ is also a total left evolutional period of $\mathscr{F}'$.

Proof. Let λ' be the (unique by Corollary 6.13) left weak evolutional period of $\mathscr{F}$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F}) + 1)$. By Lemma 6.24, $\mathscr{F}$ is weakly periodic for the sequence $m_0, m_1, \dots, m_{n-1}, \operatorname{nker}_k(\mathscr{F}), \operatorname{nker}_k(\mathscr{F}) + 1$, and by Lemma 6.12, the left weak evolutional period of $\mathscr{F}$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F}))$ also equals λ'. By Lemma 5.31, $\operatorname{IpR}_{k,m_{n-1},\operatorname{nker}_k(\mathscr{F})}(\mathscr{F}_l) = \operatorname{IpR}_{k,m_{n-1},\operatorname{nker}_k(\mathscr{F})}(\mathscr{F}''_l)$ as an occurrence in α for all $l \geq 0$, so λ' is also the left weak evolutional period of $\mathscr{F}''$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F}))$. Now, by Lemma 6.12 again, the left weak evolutional period of $\mathscr{F}''$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F}'') + 1)$ (it exists by assumption) also equals λ'.

Denote the residue of $|\operatorname{IpR}_{k,m_{n-1},\operatorname{nker}_k(\mathscr{F})+1}(\mathscr{F}_l)|$ modulo $|\lambda'|$ by r. By Remark 6.11, $\lambda = \operatorname{Cyc}_r(\lambda')$. By Lemma 5.31,

$$\operatorname{IpR}_{k,m_{n-1},\operatorname{nker}_k(\mathscr{F}'')+1}(\mathscr{F}''_l) = \operatorname{IpR}_{k,m_{n-1},\operatorname{nker}_k(\mathscr{F})+1}(\mathscr{F}_l)\operatorname{Fg}(\mathscr{F}'_l).$$

We know that $\psi(\operatorname{IpR}_{k,m_{n-1},\operatorname{nker}_k(\mathscr{F}'')+1}(\mathscr{F}''_l))$ is a weakly left λ'-periodic word, hence $\psi(\operatorname{Fg}(\mathscr{F}'_l))$ is a weakly left λ-periodic word. Since λ' is both the weak left evolutional period of $\mathscr{F}$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F})+1)$ and the weak left evolutional period of $\mathscr{F}''$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F}'')+1)$, the residues of $|\operatorname{IpR}_{k,m_{n-1},\operatorname{nker}_k(\mathscr{F})+1}(\mathscr{F}_l)|$ and of $|\operatorname{IpR}_{k,m_{n-1},\operatorname{nker}_k(\mathscr{F}'')+1}(\mathscr{F}''_l)|$ modulo $|\lambda'| = |\lambda|$ do not depend on l. Hence, the residue of $|\operatorname{Fg}(\mathscr{F}'_l)| = |\operatorname{IpR}_{k,m_{n-1},\operatorname{nker}_k(\mathscr{F}'')+1}(\mathscr{F}''_l)| - |\operatorname{IpR}_{k,m_{n-1},\operatorname{nker}_k(\mathscr{F})+1}(\mathscr{F}_l)|$ modulo $|\lambda|$ does not depend on l, and λ is a left weak evolutional period of $\mathscr{F}'$ for the pair $(0, \operatorname{nker}_k(\mathscr{F}')+1)$. ☐

Lemma 6.35. *Let $\mathscr{F}'$ and $\mathscr{F}$ be two consecutive evolutions of stable nonempty k-multiblocks ($k \geq 1$), and let $\mathscr{F}''$ be their concatenation. Suppose that $\operatorname{nker}_k(\mathscr{F}) > 1$, and $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1}, m_n = \operatorname{nker}_k(\mathscr{F})+1$, where $m_1 > 1$ ($n \in \mathbb{N}$, and $n = 1$ is allowed). Suppose that $\mathscr{F}''$ is also weakly periodic for the following sequence: $m'_0 = m_0 = 0, m'_1 = m_1 + \operatorname{nker}_k(\mathscr{F}) - 1, \ldots, m'_{n-1} = m_{n-1} + \operatorname{nker}_k(\mathscr{F}) - 1, m'_n = \operatorname{nker}_k(\mathscr{F}'') + 1$.*

Then $\mathscr{F}'$ is totally periodic, moreover, if λ is the (unique by Corollary 6.13) left weak evolutional period of $\mathscr{F}$ for the pair $(0, m_1)$, then λ is also a total right evolutional period of $\mathscr{F}'$.

Proof. The proof is completely similar to the proof of the previous lemma. ☐

Lemma 6.36. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{F}$ and $\mathscr{F}'$ be two consecutive evolutions of stable nonempty k-multiblocks, and let $\mathscr{F}''$ be their concatenation. Suppose that $\operatorname{nker}_k(\mathscr{F}) > 1$, and $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1}, m_n = \operatorname{nker}_k(\mathscr{F})+1$, where $n \geq 2$, and $m_{n-1} < \operatorname{nker}_k(\mathscr{F})$. Let λ be the (unique by Corollary 6.15) right weak evolutional period of $\mathscr{F}$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F})+1)$ Suppose also that λ is not a right weak evolutional period of $\mathscr{F}$ for pair (m_{n-2}, m_n). Then there are two possibilities:*

1. *$\mathscr{F}'$ is totally periodic, moreover, λ is a left total evolutional period of $\mathscr{F}'$.*
2. *There exists a k-series of obstacles in α.*

Proof. By Lemma 6.27, $\mathscr{F}'$ is weakly periodic for some sequence of indices $m'_0 = 0, m'_1, \ldots, m'_{n'-1}, m'_{n'} = \operatorname{nker}_k(\mathscr{F}')+1$. By Lemma 6.23, without loss of generality we may suppose that $m'_1 = 1$. By Lemma 6.24, $\mathscr{F}$ is also weakly periodic for the sequence $0, m_1, \ldots, m_{n-1}, \operatorname{nker}_k(\mathscr{F}), \operatorname{nker}_k(\mathscr{F})+1$. By Lemma 6.25, $\mathscr{F}''$ is weakly periodic for the sequence $m''_0 = 0, m''_1 = m_1, \ldots, m''_{n-1} = m_{n-1}, m''_n = \operatorname{nker}_k(\mathscr{F}), m''_{n+1} = \operatorname{nker}_k(\mathscr{F}) - 1 + m'_2, \ldots, m''_{n-1+n'-1} = \operatorname{nker}_k(\mathscr{F}) - 1 + m'_{n'-1}, m''_{n-1+n'} = \operatorname{nker}_k(\mathscr{F}'')+1$. In particular, there exists a right weak evolutional period λ' of $\mathscr{F}''$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F}))$.

We are going to use Lemma 6.32. To use it, we have to prove that λ' is not a weak right evolutional period of $\mathscr{F}''$ for the pair $(m_{n-2}, \operatorname{nker}_k(\mathscr{F}))$. Assume the contrary. By Lemma 5.31, $\operatorname{IpR}_{k,m_{n-2},\operatorname{nker}_k(\mathscr{F})}(\mathscr{F}_l) = \operatorname{IpR}_{k,m_{n-2},\operatorname{nker}_k(\mathscr{F})}(\mathscr{F}''_l)$ as an occurrence in α for all $l \geq 0$. Hence, λ' is then also a weak right evolutional period of $\mathscr{F}$ for the pair $(m_{n-2}, \operatorname{nker}_k(\mathscr{F}))$. We already know that $\mathscr{F}$ is weakly periodic for the sequence $0, m_1, \ldots, m_{n-1}, \operatorname{nker}_k(\mathscr{F}), \operatorname{nker}_k(\mathscr{F})+1$, so there exists a weak right evolutional period of $\mathscr{F}$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F}))$, and by Lemma 6.14, this period must also be λ'. Since $\operatorname{IpR}_{k,\operatorname{nker}_k(\mathscr{F}),\operatorname{nker}_k(\mathscr{F})+1}(\mathscr{F}_l)$ is always an empty occurrence, any final period μ is a left weak evolutional period of $\mathscr{F}$ for the pair $(\operatorname{nker}_k(\mathscr{F}), \operatorname{nker}_k(\mathscr{F})+1)$. Now λ' and μ satisfy the conditions of Corollary 6.29, and it implies that there exists a left weak evolutional period of $\mathscr{F}$ for the pair $(m_{n-2}, \operatorname{nker}_k(\mathscr{F})+1)$ if and only if there exists a left weak evolutional period of $\mathscr{F}$ for pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F})+1)$. By Remark 6.11, we can replace left periods with right ones in this statement, in other words, there exists a right weak evolutional period of $\mathscr{F}$ for the pair $(m_{n-2}, \operatorname{nker}_k(\mathscr{F})+1)$ if and only if there exists a right weak evolutional period of $\mathscr{F}$ for pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F})+1)$. But we know that λ is a weak right evolutional period of $\mathscr{F}$ for the pair $(m_{n-1}, \operatorname{nker}_k(\mathscr{F})+1)$, so a weak right evolutional period of $\mathscr{F}$ for the pair $(m_{n-2}, \operatorname{nker}_k(\mathscr{F})+1)$ also exists, and by Lemma 6.14, it also equals λ, but this contradicts the conditions of the lemma.

Therefore, λ' is not a weak right evolutional period of $\mathscr{F}''$ for the pair $(m_{n-2}, \mathrm{nker}_k(\mathscr{F}))$. By Lemma 6.32, either there exists a k-sequence of obstacles in α, or $\mathscr{F}''$ is weakly periodic for the sequence of indices $m''_0 = 0, m''_1 = m_1, \ldots, m''_{n-1} = m_{n-1}, \mathrm{nker}_k(\mathscr{F}'') + 1$. In the latter case the claim follows directly from Lemma 6.34. ☐

Lemma 6.37. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{F}'$ and $\mathscr{F}$ be two consecutive evolutions of stable nonempty k-multiblocks, and let $\mathscr{F}''$ be their concatenation. Suppose that* $\mathrm{nker}_k(\mathscr{F}) > 1$*, and $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1}, m_n =$* $\mathrm{nker}_k(\mathscr{F}) + 1$*, where $n \geq 2$, and $m_1 > 1$. Let λ be the (unique by Corollary 6.13) left weak evolutional period of $\mathscr{F}$ for the pair $(0, m_1)$ Suppose also that λ is not a left weak evolutional period of $\mathscr{F}$ for pair (m_0, m_2). Then there are two possibilities:*

1. *$\mathscr{F}'$ is totally periodic, moreover, λ is a right total evolutional period of $\mathscr{F}'$.*

2. *There exists a k-series of obstacles in α.*

Proof. The proof is completely symmetric to the proof of the previous lemma. ☐

Lemma 6.38. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{F}$ be an evolution of stable nonempty k-multiblocks. Then at least one of the following is true:*

1. *$\mathscr{F}$ is continuously periodic.*

2. *There exists a k-series of obstacles in α.*

Proof. By Lemma 6.27, $\mathscr{F}$ is weakly periodic for some sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1}, m_n = \mathrm{nker}_k(\mathscr{F}) + 1$. Let q $(1 \leq q \leq n)$ be the maximal index such that $\mathscr{F}$ is weakly periodic for the sequence $m_0, m_q, m_{q+1}, \ldots, m_{n-1}, m_n$.

If $q = n$, in other words, if $\mathscr{F}$ is weakly periodic for the sequence $m_0 = 0, m_n = \mathrm{nker}_k(\mathscr{F}) + 1$, then by Lemma 6.24, $\mathscr{F}$ is also weakly periodic for the sequence $m_0 = 0, \mathrm{nker}_k(\mathscr{F}), m_n = \mathrm{nker}_k(\mathscr{F}) + 1$, and this by definition means that $\mathscr{F}$ is continuously periodic.

Now suppose that $q \leq n-1$, then $n \geq 2$. Let λ' be a final period such that λ' is a weak right evolutional period of $\mathscr{F}$ for the pair (m_q, m_{q+1}). Then λ' cannot be a weak right evolutional period for the pair $(0, m_{q+1})$ as well, otherwise $\mathscr{F}$ would also be weakly periodic for the sequence $m_0, m_{q+1}, \ldots, m_{n-1}, m_n$, and this is a contradiction with the fact that q was chosen as a maximal index such that $\mathscr{F}$ is weakly periodic for the sequence $m_0, m_q, m_{q+1}, \ldots, m_{n-1}, m_n$. So, we can use Lemma 6.32. It implies that either there exists a k-series of obstacles in α, or $\mathscr{F}$ is weakly periodic for the sequence $m_0 = 0, m_q, m_n = \mathrm{nker}_k(\mathscr{F}) + 1$, so $\mathscr{F}$ is continuously periodic for the index m_q directly by the definition. ☐

Corollary 6.39. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks, and let $\mathscr{F}$ be the following evolution of stable nonempty k-multiblocks: $\mathscr{F}_l = \mathrm{C}_k(\mathscr{E}_{l+3k})$. Then at least one of the following is true:*

1. *$\mathscr{F}$ is continuously periodic.*

2. *There exists a k-series of obstacles in α.*

☐

Now we are going to prove some periodicity properties for the (left and right) regular parts of $(k+1)$-blocks or to find a k-series of obstacles provided that we know that all evolutions of k-blocks are continuously periodic.

Lemma 6.40. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{F}$, $\mathscr{F}'$ and $\mathscr{F}''$ be three consecutive evolutions of stable nonempty k-multiblocks. Suppose that* $\mathrm{nker}_k(\mathscr{F}') > 1$*. Then at least one of the following is true:*

1. *At least one of the evolutions $\mathscr{F}$, $\mathscr{F}'$, and $\mathscr{F}''$ is totally periodic.*

2. *There exists a k-series of obstacles in α.*

Proof. First, apply Lemma 6.38 to $\mathscr{F}'$. Either there exists a k-series of obstacles in α (and then we are done), or there exists an index m ($1 \le m \le \mathrm{nker}_k(\mathscr{F}')$) such that $\mathscr{F}'$ is continuously periodic for m.

Suppose that $m > 1$. Denote the (unique by Corollary 6.13) left weak evolutional period of $\mathscr{F}'$ for the pair $(0, m)$ by λ. If λ is also a weak left evolutional period of $\mathscr{F}'$ for the pair $(0, \mathrm{nker}_k(\mathscr{F}')+1)$, $\mathscr{F}'$ is totally periodic, and we are done. Otherwise, by Lemma 6.37, either $\mathscr{F}$ is totally periodic, or there is a k-series of obstacles in α.

Now let us consider the case when $m = 1$. Denote the (unique by Corollary 6.15) right weak evolutional period of $\mathscr{F}'$ for the pair $(1, \mathrm{nker}_k(\mathscr{F}) + 1)$ by λ'. If λ' is also a weak right evolutional period of $\mathscr{F}'$ for the pair $(0, \mathrm{nker}_k(\mathscr{F}') + 1)$, $\mathscr{F}'$ is totally periodic. Otherwise, by Lemma 6.36, either $\mathscr{F}''$ is totally periodic, or there is a k-series of obstacles in α. □

Lemma 6.41. *Let $\mathscr{F}$, $\mathscr{F}'$ and $\mathscr{F}''$ be three consecutive evolutions of stable nonempty k-multiblocks ($k \ge 1$). Suppose that $\mathscr{F}$, $\mathscr{F}'$, and $\mathscr{F}''$ are totally periodic. Suppose also that* $\mathrm{nker}_k(\mathscr{F}) > 1$, $\mathrm{nker}_k(\mathscr{F}') > 1$, *and* $\mathrm{nker}_k(\mathscr{F}'') > 1$. *Let λ (resp. λ') be the (unique by Corollary 6.15) total right evolutional period of $\mathscr{F}$ (resp. of $\mathscr{F}'$), and let μ' (resp μ'') be the (unique by Corollary 6.13) total left evolutional period of $\mathscr{F}'$ (resp. of $\mathscr{F}''$).*

Then at least one of the following is true:

1. $\lambda = \mu'$.

2. $\lambda' = \mu''$.

3. *There exists a k-series of obstacles in α.*

Proof. By Lemma 6.23, $\mathscr{F}$ is also weakly periodic for the sequence

$$0, \mathrm{nker}_k(\mathscr{F}), \mathrm{nker}_k(\mathscr{F}) + 1.$$

By Lemma 6.24, $\mathscr{F}'$ is also weakly periodic for the sequence $0, 1, \mathrm{nker}_k(\mathscr{F}') + 1$. Denote the concatenation of $\mathscr{F}$ and $\mathscr{F}'$ by $\mathscr{F}'''$. By Lemma 6.25, $\mathscr{F}'''$ is weakly periodic for the sequence

$$0, \mathrm{nker}_k(\mathscr{F}), \mathrm{nker}_k(\mathscr{F}''') + 1 = \mathrm{nker}_k(\mathscr{F}) + \mathrm{nker}_k(\mathscr{F}').$$

By Lemma 6.14, the right weak evolutional period of $\mathscr{F}'$ for the pair $(1, \mathrm{nker}_k(\mathscr{F}') + 1)$ equals λ'. By Lemma 5.31, $\mathrm{IpR}_{k,\mathrm{nker}_k(\mathscr{F}),\mathrm{nker}_k(\mathscr{F}''')+1}(\mathscr{F}'''_l) = \mathrm{IpR}_{k,1,\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}''_l)$ as an occurrence in α for all $l \ge 0$, so λ' is also a right weak evolutional period of $\mathscr{F}'''$ for the pair $(\mathrm{nker}_k(\mathscr{F}), \mathrm{nker}_k(\mathscr{F}''') + 1)$.

Suppose first that λ' is also a weak right evolutional period of $\mathscr{F}'''$ for the pair $(0, \mathrm{nker}_k(\mathscr{F}''')+1)$. Then, since $\mathscr{F}'$ is totally periodic, we can use Lemma 6.35. It implies that μ', which is a left weak evolutional period of $\mathscr{F}'$ for the pair $(0, \mathrm{nker}_k(\mathscr{F}') + 1)$, is also a right total evolutional period of $\mathscr{F}$. Then it follows from Corollary 6.15 that $\lambda = \mu'$.

Now let us consider the case when λ' is not a weak right evolutional period of $\mathscr{F}'''$ for the pair $(0, \mathrm{nker}_k(\mathscr{F}''') + 1)$. Then, by Lemma 6.36, either there exists a k-series of obstacles in α, or λ' is a left total evolutional period of $\mathscr{F}''$. In the latter case, $\lambda' = \mu''$ by Corollary 6.13. □

Lemma 6.42. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{F}$ be an evolution of stable nonempty k-multiblocks, and let λ be a final period. Suppose that there exists $l_0 \ge 0$ such that $\psi(\mathrm{Fg}(\mathscr{F}_{l_0}))$ is a weakly left λ-periodic word. Then λ is a left total period of $\mathscr{F}$.*

Proof. By Lemma 6.27, $\mathscr{F}$ is weakly periodic for a sequence of indices $m_0 = 0, m_1, \ldots, m_{n-1}, m_n = \mathrm{nker}_k(\mathscr{F}) + 1$. By Lemma 6.23, without loss of generality we may suppose that $m_1 = 1$. Since $\mathrm{IpR}_{k,0,1}(\mathscr{F}_l)$ is always an empty word, λ is a weak left evolutional period of $\mathscr{F}$ for the pair $(0, 1)$. Let q $(1 \le q \le n)$ be the maximal index such that λ is a weak left evolutional period of $\mathscr{F}$ for the pair $(0, m_q)$. If $q = n$, we are done.

Otherwise, we are going to get a contradiction using Lemma 6.28. Denote $\mathrm{Fg}(\mathscr{F}_{l_0}) = \alpha_{i\ldots j}$ and $\mathrm{IpR}_{k,0,m_q}(\mathscr{F}_{l_0}) = \alpha_{i\ldots j'}$. If λ is not a weak left evolutional period of $\mathscr{F}$ for the pair $(0, m_{q+1})$, then Lemma 6.28 implies that there exists $s \in \mathbb{N}$ such that $\psi(\alpha_{i\ldots j'+s})$ is not a weakly left λ-periodic word, and $j' + s \le j$, i. e. $\alpha_{j'+s}$ is a letter in $\mathrm{Fg}(\mathscr{F}_{l_0})$. But then $\psi(\alpha_{i\ldots j'+s})$ is a prefix of $\psi(\mathrm{Fg}(\mathscr{F}_{l_0}))$, and $\psi(\mathrm{Fg}(\mathscr{F}_{l_0}))$ is weakly left λ-periodic, a contradiction. ☐

The following lemma can be proved symmetrically.

Lemma 6.43. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{F}$ be an evolution of stable nonempty k-multiblocks, and let λ be a final period. Suppose that there exists $l_0 \ge 0$ such that $\psi(\mathrm{Fg}(\mathscr{F}_{l_0}))$ is a weakly right λ-periodic word. Then λ is a right total period of $\mathscr{F}$.* ☐

Lemma 6.44. *Let $k \in \mathbb{N}$. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the left. Let $m \ge 0, m' \ge 0$. Then $\mathscr{F} = (\mathscr{F}_l)_{l \ge 0}$, where $\mathscr{F}_l = \mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m+m'})$, is an evolution of stable nonempty k-multiblocks, and $\mathrm{nker}_k(\mathscr{F}) > 1$.*

Proof. The stability follows from Corollary 5.9 (and from the definition of the left regular part of $\mathscr{E}_{l+3k}$), and the nonemptiness follows from the definition of Case I.

By Lemma 5.10, there exists a k-block $\alpha_{i\ldots j}$ in $\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{3(k+1)+m+m'})$ such that Case I holds at the left or at the right for the evolution of $\alpha_{i\ldots j}$. Denote the evolution $\alpha_{i\ldots j}$ belongs to by $\mathscr{E}'$. Then, by the definitions of the descendant of a k-multiblock and of a left atom, there exists $l_0 \ge 0$ such that $\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m+m'})$ contains $\mathscr{E}'_{3(k+1)+l-l_0}$ for all $l \ge 0$. By Lemma 5.15, $|\mathscr{E}'_{3(k+1)+l-l_0}|$ is $\Theta(l^k)$ for $l \to \infty$. Hence, $|\mathrm{Fg}(\mathscr{F}_l)|$ cannot be bounded for $l \to \infty$. But if $\mathrm{nker}_k(\mathscr{F}) = 1$, then $\mathrm{Fg}(\mathscr{F}_l) = \mathrm{Ker}_{k,1}(\mathscr{F})$ for all $l \ge 0$, and in particular, $|\mathrm{Fg}(\mathscr{F}_l)|$ is bounded for $l \to \infty$. Therefore, $\mathrm{nker}_k(\mathscr{F}) > 1$. ☐

The proof of the following lemma is completely symmetric.

Lemma 6.45. *Let $k \in \mathbb{N}$. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the right. Let $m \ge 0, m' \ge 0$. Then $\mathscr{F} = (\mathscr{F}_l)_{l \ge 0}$, where $\mathscr{F}_l = \mathrm{RA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m+m'})$, is an evolution of stable nonempty k-multiblocks, and $\mathrm{nker}_k(\mathscr{F}) > 1$.* ☐

Lemma 6.46. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the left. Consider the following evolution of stable nonempty (by Lemma 6.44) k-multiblocks: $\mathscr{F}_l = \mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)})$.*

At least one of the following is true:

1. *$\mathscr{F}$ is totally periodic.*

2. *There exists a k-series of obstacles in α.*

Proof. First, consider the following three evolutions of stable nonempty k-multiblocks: $\mathscr{F}'_l = \mathrm{LA}_{k+1,4}(\mathscr{E}_{l+3(k+1)+2})$, $\mathscr{F}''_l = \mathrm{LA}_{k+1,3}(\mathscr{E}_{l+3(k+1)+2})$, and $\mathscr{F}'''_l = \mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)+2})$. By Lemma 6.44 for $m = 2, m' = 0$ (resp. for $m = 1, m' = 1$, for $m = 0, m' = 2$), we have $\mathrm{nker}_k(\mathscr{F}') > 1$ (resp. $\mathrm{nker}_k(\mathscr{F}'') > 1$, $\mathrm{nker}_k(\mathscr{F}''') > 1$). Also, these three evolutions are consecutive, so, by Lemma 6.40, either there exists a k-series of obstacles in α (and then we are done), or at least one of these three evolutions is totally periodic.

Suppose now that at least one of the evolutions $\mathscr{F}'$, $\mathscr{F}''$, and $\mathscr{F}'''$ is totally periodic. In other words, there exists a final period λ and a number m (m can equal 0, 1, or 2) such that for all $l \ge 0$, $\psi(\mathrm{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+2})))$ is a weakly left λ-periodic word, and the residue of $|\operatorname{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+2}))|$ modulo $|\lambda|$ does not depend on l. By Corollary 3.18, $\operatorname{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+2})) = \operatorname{Fg}(\mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)+(2-m)}))$ as an abstract word. Therefore, λ is a left total evolutional period of the following evolution of stable k-multiblocks: $\mathscr{F}_{2-m}, \mathscr{F}_{2-m+1}, \mathscr{F}_{2-m+2}, \ldots$

Therefore, λ is a total left evolutional period of $\mathscr{F}$ by Lemma 6.42. □

Lemma 6.47. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the right. Consider the following evolution of stable nonempty (by Lemma 6.45) k-multiblocks: $\mathscr{F}_l = \mathrm{RA}_{k+1,2}(\mathscr{E}_{l+3(k+1)})$.*

At least one of the following is true:

1. *$\mathscr{F}$ is totally periodic.*

2. *There exists a k-series of obstacles in α.*

Proof. The proof is completely symmetric to the proof of the previous lemma. □

Lemma 6.48. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the left.*

At least one of the following is true:

1. *There exists a unique final period λ and a number r ($0 \le r < |\lambda|$) such that for all $m \ge 0$, λ is a left total period of the evolution $\mathscr{F}$ of stable nonempty (by Lemma 6.45) k-multiblocks defined by $\mathscr{F}_l = \mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m})$, moreover, the residue of $|\operatorname{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m}))|$ always equals r (i. e. it does not depend on m).*

2. *There exists a k-series of obstacles in α.*

Proof. By Lemma 6.46, either there exists a k-series of obstacles in α (and then we are done), or the evolution $\mathscr{F}'$ of stable nonempty k-multiblocks defined by $\mathscr{F}'_l = \mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)})$ is totally periodic.

Suppose that $\mathscr{F}'$ is totally periodic. Since $\operatorname{nker}_k(\mathscr{F}') > 1$ by Lemma 6.44, it follows from Corollary 6.13 that the left total evolutional period of $\mathscr{F}'$ is unique, denote it by λ. Denote by r the remainder of $|\operatorname{Fg}(\mathscr{F}'_l)|$ modulo $|\lambda|$ (it does not depend on l by the definition of a weak left evolutional period). Then $\psi(\operatorname{Fg}(\mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)})))$ is a weakly left λ-periodic word for all $l \ge 0$. By Corollary 3.18, $\operatorname{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m})) = \operatorname{Fg}(\mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)}))$ as an abstract word. Hence, for all $l \ge 0$ and $m \ge 0$, $\psi(\operatorname{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m})))$ is a weakly left λ-periodic word, and the residue of $|\operatorname{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m}))|$ equals r. Therefore, λ is a left total period of the evolution $\mathscr{F}$ of stable nonempty k-multiblocks defined by $\mathscr{F}_l = \mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m})$. □

Lemma 6.49. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the right.*

At least one of the following is true:

1. *There exists a unique final period λ and a number r ($0 \le r < |\lambda|$) such that for all $m \ge 0$, λ is a right total period of the evolution $\mathscr{F}$ of stable nonempty (by Lemma 6.45) k-multiblocks defined by $\mathscr{F}_l = \mathrm{RA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m})$, moreover, the residue of $|\operatorname{Fg}(\mathrm{RA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m}))|$ always equals r (i. e. it does not depend on m).*

2. *There exists a k-series of obstacles in α.*

Proof. The proof is completely symmetric to the proof of the previous lemma. □

Lemma 6.50. *Let λ be a final period. Let γ be a finite word. Suppose that γ is weakly $|\lambda|$-periodic with both left and right period λ, and $|\gamma| \ge 2\mathbf{L}$. Then γ is a completely λ-periodic word.*

Proof. Denote the remainder of $|\gamma|$ modulo $|\lambda|$ by r. Set $\lambda' = \mathrm{Cyc}_r(\lambda) = \mathrm{Cyc}_{|\gamma|}(\lambda)$. Then γ is a weakly right λ'-periodic word. Since λ is a final period, by Corollary 2.12 $\lambda' = \lambda$.

Assume that $r > 0$. Then the equality $\lambda = \lambda'$ means that $\lambda_{0\dots r-1} = \lambda_{|\lambda|-r\dots|\lambda|-1}$ and $\lambda_{0\dots|\lambda|-r-1} = \lambda_{r\dots|\lambda|-1}$. We are going to get a contradiction with Corollary 2.11. Namely, consider the word $\delta = \lambda\lambda$. Clearly, it is (in particular) weakly left λ-periodic, and its length is (in particular) at least $2|\lambda|$. Let us check that δ is also a weakly r-periodic word. To see this, we have to check that $\delta_i = \delta_{i+r}$ for $0 \le i < |\delta| - r$ (in other words, $0 \le i < 2|\lambda| - r$). Consider the following three cases for i:

1. $0 \le i < |\lambda| - r$. Then $\delta_i = \lambda_i = \lambda_{i+r} = \delta_{i+r}$ since $\lambda_{0\dots|\lambda|-r-1} = \lambda_{r\dots|\lambda|-1}$.
2. $|\lambda| - r \le i < \lambda$. Then $\delta_i = \lambda_i = \lambda_{i-(|\lambda|-r)} = \lambda_{i+r-|\lambda|} = \delta_{i+r}$ since $\lambda_{0\dots r-1} = \lambda_{|\lambda|-r\dots|\lambda|-1}$.
3. $|\lambda| \le i < 2|\lambda| - r$. Then $\delta_i = \lambda_{i-|\lambda|} = \lambda_{i-|\lambda|+r} = \delta_{i+r}$ since $\lambda_{0\dots|\lambda|-r-1} = \lambda_{r\dots|\lambda|-1}$.

Therefore, δ is a weakly r-periodic word, and, since $r < |\lambda|$, we have a contradiction with Corollary 2.11. ☐

Lemma 6.51. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the left.*

At least one of the following is true:

1. *There exists a unique final period λ such that for all $l \ge 0$ and $m \ge 0$, $\psi(\mathrm{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m})))$ is a* ***completely*** *λ-periodic word.*
2. *There exists a k-series of obstacles in α.*

Proof. Suppose that k-series of obstacles do not exist in α. Then, by Lemma 6.48, there exists a final period λ and a number r ($0 \le r < |\lambda|$) such that for all $l \ge 0$ and $m \ge 0$, $\psi(\mathrm{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m})))$ is a weakly left λ-periodic word, and the residue of $|\mathrm{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m}))|$ modulo $|\lambda|$ equals r. It is sufficient to prove that $r = 0$.

Again, consider the following three evolutions of stable nonempty k-multiblocks: $\mathscr{F}'_l = \mathrm{LA}_{k+1,4}(\mathscr{E}_{l+3(k+1)+2})$, $\mathscr{F}''_l = \mathrm{LA}_{k+1,3}(\mathscr{E}_{l+3(k+1)+2})$, and $\mathscr{F}'''_l = \mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)+2})$. Then λ is the total left evolutional period of each of them. By Lemma 6.41, λ is also the total right evolutional period of at least one of the evolutions $\mathscr{F}'$ or $\mathscr{F}''$. So, at least one of the words $\psi(\mathrm{Fg}(\mathrm{LA}_{k+1,4}(\mathscr{E}_{3(k+1)+2})))$ and $\psi(\mathrm{Fg}(\mathrm{LA}_{k+1,3}(\mathscr{E}_{3(k+1)+2})))$ is a weakly $|\lambda|$-periodic word with both left and right period λ. Denote this word by γ. By Corollary 5.30, $|\gamma| \ge 2\mathbf{L}$. By Lemma 6.50, γ is a completely λ-periodic word, and $|\gamma|$ is divisible by $|\lambda|$. But we also know that the residue of $|\gamma|$ modulo $|\lambda|$ equals r, so $r = 0$.

Therefore, all words $\psi(\mathrm{Fg}(\mathrm{LA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m})))$ for all $l \ge 0$ and $m \ge 0$ are completely λ-periodic. ☐

Corollary 6.52. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the left.*

At least one of the following is true:

1. *There exists a unique final period λ such that for all $l \ge 0$,*

$$\psi(\mathrm{Fg}(\mathrm{LR}_{k+1}(\mathscr{E}_{l+3(k+1)})))$$

is a completely λ-periodic word. λ is also a total left and a total right period of the evolution $\mathscr{F}$ of stable nonempty k-multiblocks defined by $\mathscr{F}_l = \mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)})$.

2. *There exists a k-series of obstacles in α.* ☐

Lemma 6.53. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the right.*

At least one of the following is true:

1. *There exists a unique final period λ such that for all $l \geq 0$ and $m \geq 0$, $\psi(\mathrm{Fg}(\mathrm{RA}_{k+1,2+m}(\mathscr{E}_{l+3(k+1)+m})))$ is a* ***completely*** *λ-periodic word.*

2. *There exists a k-series of obstacles in α.*

Proof. The proof is completely symmetric to the proof of Lemma 6.51. □

Corollary 6.54. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the right.*

At least one of the following is true:

1. *There exists a unique final period λ such that for all $l \geq 0$,*

$$\psi(\mathrm{Fg}(\mathrm{RR}_{k+1}(\mathscr{E}_{l+3(k+1)})))$$

is a completely λ-periodic word. λ is also a total left and a total right period of the evolution $\mathscr{F}$ of stable nonempty k-multiblocks defined by $\mathscr{F}_l = \mathrm{RA}_{k+1,2}(\mathscr{E}_{l+3(k+1)})$.

2. *There exists a k-series of obstacles in α.* □

Now we are going to prove some facts about the periodicity of left and right bounding sequences of evolutions of $(k+1)$-blocks such that Case II holds at the right or at the left.

Lemma 6.55. *Let $k \in \mathbb{N}$. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case II holds at the right. Let $l_0 \geq 3(k+1)$, and let $\mathscr{F}$ be an evolution of stable nonempty k-multiblocks such that $\mathrm{Fg}(\mathscr{F}_l)$ is a suffix of $\mathscr{E}_{l+l_0}$ for all $l \geq 0$. Suppose that $\mathscr{F}$ is continuously periodic for an index m ($1 \leq m < \mathrm{nker}_k(\mathscr{F})$). Let λ be the (unique by Corollary 6.15) weak right evolutional period of $\mathscr{F}$ for the pair $(m, \mathrm{nker}_k(\mathscr{F})+1)$.*

Then there are three possibilities:

1. *$\mathscr{F}$ is totally periodic.*

2. *$\psi(\mathrm{RBS}_{k+1}(\mathscr{E}))$ is periodic with period λ.*

3. *There exists a k-series of obstacles in α.*

Proof. Suppose that k-series of obstacles do not exist in α and that $\mathscr{F}$ is not totally periodic. Then, since $\mathscr{F}$ is continuously periodic for the index m, Lemma 6.30 implies that there exists a number $s \in \mathbb{N}$ such that for all $l \geq 0$, if $\mathrm{Fg}(\mathscr{F}_l) = \alpha_{i\ldots j}$ and $\mathrm{IpR}_{k,m,\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}_l) = \alpha_{i'\ldots j}$, then $\psi(\alpha_{i'-s+1\ldots j})$ is a weakly right λ-periodic word, and $\psi(\alpha_{i'-s\ldots j})$ is not a weakly right λ-periodic word.

Assume that $\psi(\mathrm{RBS}_{k+1}(\mathscr{E}))$ is not an infinite periodic sequence with period λ. Then there exists a number $s' \geq 0$ such that $\psi(\mathrm{RBS}_{k+1}(\mathscr{E}))_{0\ldots s'-1}$ is a weakly left λ-periodic word, and $\psi(\mathrm{RBS}_{k+1}(\mathscr{E}))_{0\ldots s'}$ is not a weakly left λ-periodic word. We are going to find a k-series of obstacles in α.

By Corollary 6.7, there exists $l_1 \geq 0$ such that if $l \geq 0$ and $\mathscr{E}_{l+l_1} = \alpha_{i''\ldots j}$, then $\alpha_{i''\ldots j+s'+1} = \mathscr{E}_{l+l_1}\,\mathrm{RBS}_{k+1}(\mathscr{E})_{0\ldots s'}$. Without loss of generality, $l_1 \geq l_0$.

Fix a number $l \geq 0$. Suppose that $\mathscr{E}_{l+l_1} = \alpha_{i''\ldots j}$. Then $\mathrm{Fg}(\mathscr{F}_{l+l_1-l_0})$ is a suffix of $\mathscr{E}_{l+l_1}$, so there exists an index $i \geq i''$ such that $\mathrm{Fg}(\mathscr{F}_{l+l_1-l_0}) = \alpha_{i\ldots j}$. Let $i' \geq i$ be the index such that $\mathrm{IpR}_{k,m,\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}_{l+l_1-l_0}) = \alpha_{i'\ldots j}$. Set $\mathscr{H}_l = \alpha_{i'-s+1\ldots j+s'}$.

First, as an abstract word,

$$\psi(\mathscr{H}_l) = \psi(\alpha_{i'-s+1\ldots j}\alpha_{j+1\ldots j+s'}) = \psi(\alpha_{i'-s+1\ldots j})\psi(\underset{k+1}{\mathrm{RBS}}(\mathscr{E})_{0\ldots s'-1}),$$

and $\psi(\alpha_{i'-s+1\ldots j})$ (resp. $\psi(\mathrm{RBS}_{k+1}(\mathscr{E})_{0\ldots s'-1})$) is a weakly right (resp. left) λ-periodic word, hence, $\psi(\mathscr{H}_l)$ is weakly left λ'-periodic, where $\lambda' = \mathrm{Cyc}_{-(j-(i'-s+1)+1)}(\lambda)$. Since $\psi(\alpha_{j+1\ldots j+s'+1}) = \psi(\mathrm{RBS}_{k+1}(\mathscr{E}))_{0\ldots s'}$ is not a weakly left λ-periodic word, $\psi(\alpha_{i'-s+1\ldots j}\alpha_{j+1\ldots j+s'+1}) = \psi(\alpha_{i'-s+1\ldots j+s'+1})$ is not a weakly left λ'-periodic word either. And since $\psi(\alpha_{i'-s\ldots j})$ is not a weakly right λ-periodic word, $\psi(\alpha_{i'-s}) \neq \lambda'_{|\lambda|-1}$.

Now, since $1 \le m < \operatorname{nker}_k(\mathscr{F})$, by Corollary 5.30, $|\operatorname{IpR}_{k,m,\operatorname{nker}_k(\mathscr{F})+1}(\mathscr{F}_{l+l_1-l_0})| \ge 2\mathbf{L}$, hence, $|\mathscr{H}_l| = (s-1) + |\operatorname{IpR}_{k,m,\operatorname{nker}_k(\mathscr{F})+1}(\mathscr{F}_{l+l_1-l_0})| + s' \ge 2\mathbf{L}$. In particular, $|\mathscr{H}_l| \ge |\lambda| = |\lambda'|$. Since $\psi(\alpha_{i'-s+1\dots j+s'})$ is a weakly left λ'-periodic word, $\psi(\alpha_{i'-s+|\lambda|}) = \lambda'_{|\lambda|-1}$. Since $\psi(\alpha_{i'-s}) \ne \lambda'_{|\lambda|-1}$, $\psi(\alpha_{i'-s\dots j+s'})$ is not a weakly $|\lambda|$-periodic word (with any period). Let r be the residue of $|\alpha_{i'-s+1\dots j+s'}|$ modulo $|\lambda|$. Then, since $\psi(\alpha_{i'-s+1\dots j+s'})$ is a weakly left λ'-periodic word, $\psi(\alpha_{j+s'-|\lambda|+1}) = \lambda'_r$. And since $\psi(\alpha_{i'-s+1\dots j+s'+1})$ is not a weakly left λ'-periodic word, $\psi(\alpha_{j+s'+1}) \ne \lambda'_r$. Therefore, $\psi(\alpha_{i'-s+1\dots j+s'+1})$ is not a weakly $|\lambda|$-periodic word (with any period).

Finally, let $l \ge 0$ be arbitrary again. By Corollary 5.30, $|\operatorname{IpR}_{k,m,\operatorname{nker}_k(\mathscr{F})+1}(\mathscr{F}_{l+l_1-l_0})|$ strictly grows as l grows, and there exists $k' \in \mathbb{N}$ ($1 \le k' \le k$) such that $|\operatorname{IpR}_{k,m,\operatorname{nker}_k(\mathscr{F})+1}(\mathscr{F}_{l+l_1-l_0})|$ is $\Theta(l^{k'})$ for $l \to \infty$. Then, since s and s' do not depend on l, $|\mathscr{H}_l| = (s-1) + |\operatorname{IpR}_{k,m,\operatorname{nker}_k(\mathscr{F})+1}(\mathscr{F}_{l+l_1-l_0})| + s'$ also strictly grows as l grows, and $|\mathscr{H}_l|$ is $\Theta(l^{k'})$ for $l \to \infty$. ☐

Lemma 6.56. *Let $k \in \mathbb{N}$. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case II holds at the left. Let $l_0 \ge 3(k+1)$, and let $\mathscr{F}$ be an evolution of stable nonempty k-multiblocks such that $\operatorname{Fg}(\mathscr{F}_l)$ is a prefix of $\mathscr{E}_{l+l_0}$ for all $l \ge 0$. Suppose that $\mathscr{F}$ is continuously periodic for an index m ($1 < m \le \operatorname{nker}_k(\mathscr{F})$). Let λ be the (unique by Corollary 6.13) weak left evolutional period of $\mathscr{F}$ for the pair $(0, m)$.*

Then there are three possibilities:

1. *$\mathscr{F}$ is totally periodic.*
2. *$\psi(\operatorname{LBS}_{k+1}(\mathscr{E}))$ is periodic with period λ.*
3. *There exists a k-series of obstacles in α.*

Proof. The proof is completely symmetric to the proof of the previous lemma. ☐

Lemma 6.57. *Let $k \in \mathbb{N}$. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case II holds both at the left and at the right. Suppose that $\operatorname{ncker}_{k+1}(\mathscr{E}) > 1$. Let $\mathscr{F}$ be the evolution of stable nonempty k-blocks defined by $\mathscr{F}_l = \operatorname{C}_{k+1}(\mathscr{E}_{l+3(k+1)})$. Suppose that $\mathscr{F}$ is totally periodic, and let λ (resp. λ') be the left (resp. the right) total evolutional period of $\mathscr{F}$.*

Then there are three possibilities:

1. *$\psi(\operatorname{LBS}_{k+1}(\mathscr{E}))$ is periodic with period λ.*
2. *$\psi(\operatorname{RBS}_{k+1}(\mathscr{E}))$ is periodic with period λ'.*
3. *There exists a k-series of obstacles in α.*

Proof. Suppose that $\psi(\operatorname{LBS}_{k+1}(\mathscr{E}))$ is not an infinite periodic sequence with period λ, and $\psi(\operatorname{RBS}_{k+1}(\mathscr{E}))$ is not an infinite periodic sequence with period λ'. We are going to find a k-series of obstacles in α.

Let s be the length of the maximal weakly right λ-periodic suffix of $\psi(\operatorname{LBS}_{k+1}(\mathscr{E}))$, in other words, $s \ge 0$ is the number such that $\psi(\operatorname{LBS}_{k+1}(\mathscr{E}))_{-s+1\dots 0}$ is a weakly right λ-periodic word, and $\psi(\operatorname{LBS}_{k+1}(\mathscr{E}))_{-s\dots 0}$ is not a weakly right λ-periodic word. Similarly, let s' be the length of the maximal weakly left λ'-periodic prefix of $\psi(\operatorname{RBS}_{k+1}(\mathscr{E}))$, in other words, $s' \ge 0$ is the number such that $\psi(\operatorname{RBS}_{k+1}(\mathscr{E}))_{0\dots s'-1}$ is a weakly left λ'-periodic word, and $\psi(\operatorname{RBS}_{k+1}(\mathscr{E}))_{0\dots s'}$ is not a weakly left λ'-periodic word. By Corollaries 6.7 and 6.8, there exists $l_0 \ge 0$ such that if $l \ge 0$ and $\mathscr{E}_{l+l_0} = \alpha_{i\dots j}$, then $\alpha_{i-s-1\dots j+s'+1} = \operatorname{LBS}_{k+1}(\mathscr{E})_{-s\dots 0}\mathscr{E}_{l+l_0}\operatorname{RBS}_{k+1}(\mathscr{E})_{0\dots s'}$. Without loss of generality, $l_0 \ge 3(k+1)$.

Fix $l \ge 0$. Suppose that $\mathscr{E}_{l+l_0} = \operatorname{Fg}(\mathscr{F}_{l+l_0-3(k+1)}) = \alpha_{i\dots j}$. Set $\mathscr{H}_l = \alpha_{i-s-1\dots j+s'+1}$. We are going to prove that all $\mathscr{H}_l$ for $l \ge 0$ form a k-series of obstacles. The argument is similar to the proof of Lemma 6.55.

First, it follows from Remark 6.11 and from Corollary 6.15 that $\lambda' = \operatorname{Cyc}_{|\operatorname{Fg}(\mathscr{F}_{l+l_0-3(k+1)})|}(\lambda)$. We also know that $\psi(\alpha_{i\dots j})$ is a weakly left λ-periodic word, $\psi(\alpha_{i-s\dots i-1})$ is a weakly right λ-periodic word, and $\psi(\alpha_{i-s-1\dots i-1})$ is not a weakly right λ-periodic word. Hence, $\psi(\alpha_{i-s\dots j})$ is a weakly right λ'-periodic word, and $\psi(\alpha_{i-s-1\dots j})$ is not a weakly right λ'-periodic word.

Now, $\psi(\alpha_{j+1\dots j+s'})$ is a weakly left λ'-periodic word, and $\psi(\alpha_{j+1\dots j+s'+1})$ is not a weakly left λ'-periodic word. Therefore, if $\lambda'' = \mathrm{Cyc}_{-|\alpha_{i-s\dots j}|}(\lambda')$, then $\psi(\alpha_{i-s\dots j+s'}) = \psi(\mathscr{H}_l)$ is a weakly left λ''-periodic word, $\psi(\alpha_{i-s\dots j+s'+1})$ is not a weakly left λ''-periodic word, and $\psi(\alpha_{i-s-1}) \neq \lambda''_{|\lambda''|-1}$.

Since $\mathrm{ncker}_{k+1}(\mathscr{E}) = \mathrm{nker}_k(\mathscr{F}) > 1$, by Corollary 5.30, $|\mathrm{Fg}(\mathscr{F}_{l+l_0-3(k+1)})| \geq 2\mathbf{L}$, hence, $|\mathscr{H}_l| = s + |\mathrm{Fg}(\mathscr{F}_{l+l_0-3(k+1)})| + s' \geq 2\mathbf{L}$. In particular, $|\mathscr{H}_l| \geq |\lambda''| = |\lambda|$. Since $\psi(\alpha_{i-s\dots j+s'})$ is a weakly left λ''-periodic word, $\psi(\alpha_{i-s+|\lambda|-1}) = \lambda''_{|\lambda''|-1}$. Since $\psi(\alpha_{i-s-1}) \neq \lambda''_{|\lambda''|-1}$, $\psi(\alpha_{i-s-1\dots j+s'})$ is not a weakly $|\lambda''|$-periodic word (with any period). Let r be the residue of $|\alpha_{i-s\dots j+s'}| = (j'+s)-(i-s)+1$ modulo $|\lambda''|$. Then, since $\psi(\alpha_{i-s\dots j+s'})$ is a weakly left λ''-periodic word, $\psi(\alpha_{j+s'-|\lambda''|+1}) = \lambda''_r$. And since $\psi(\alpha_{i-s\dots j+s'+1})$ is not a weakly left λ''-periodic word, $\psi(\alpha_{j+s'+1}) \neq \lambda''_r$. Therefore, $\psi(\alpha_{i-s\dots j+s'+1})$ is not a weakly $|\lambda''|$-periodic word (with any period).

Finally, let $l \geq 0$ be arbitrary again. By Corollary 5.30, $|\mathrm{Fg}(\mathscr{F}_{l+l_0-3(k+1)})|$ strictly grows as l grows, and there exists $k' \in \mathbb{N}$ $(1 \leq k' \leq k)$ such that $|\mathrm{Fg}(\mathscr{F}_{l+l_0-3(k+1)})|$ is $\Theta(l^{k'})$ for $l \to \infty$. Then, since s and s' do not depend on l, $|\mathscr{H}_l| = s + |\mathrm{Fg}(\mathscr{F}_{l+l_0-3(k+1)})| + s'$ also strictly grows as l grows, and $|\mathscr{H}_l|$ is $\Theta(l^{k'})$ for $l \to \infty$. ☐

Finally, we are ready to prove that if all evolutions of k-blocks are continuously periodic, then either there is a k-series of obstacles, or all evolutions of $(k+1)$-blocks are continuously periodic.

Lemma 6.58. *Let $k \in \mathbb{N}$. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the left (resp. at the right). Let λ be a final period, and let m be an index $1 \leq m \leq \mathrm{ncker}_{k+1}(\mathscr{E})$. Denote by $\mathscr{F}$ the following evolution of k-multiblocks: $\mathscr{F}_l = \mathrm{C}_{k+1}(\mathscr{E}_{l+3(k+1)})$.*

Suppose that for all $l \geq 0$, $\psi(\mathrm{Fg}(\mathrm{LR}_{k+1}(\mathscr{E}_{l+3(k+1)})))$ (resp. $\psi(\mathrm{Fg}(\mathrm{RR}_{k+1}(\mathscr{E}_{l+3(k+1)})))$) is a completely λ-periodic word. Suppose also that λ is a left (resp. right) weak evolutional period of $\mathscr{F}$ for the pair $(0, m)$ (resp. $(m, \mathrm{nker}_k(\mathscr{F})+1)$).

Then λ is a left continuous evolutional period of $\mathscr{E}$ for the index m.

Proof. We prove the lemma for the situation when Case I holds at the left. If Case I holds at the right, the proof is completely symmetric.

For all $l \geq 0$, $\psi(\mathrm{IpR}_{k,0,m}(\mathscr{F}_l))$ is a weakly left λ-periodic word, and the residue of $|\psi(\mathrm{IpR}_{k,0,m}(\mathscr{F}_l))|$ modulo $|\lambda|$ does not depend on l. $\psi(\mathrm{Fg}(\mathrm{LR}_{k+1}(\mathscr{E}_{l+3(k+1)})))$ is a completely λ-periodic word, and

$$\mathrm{LpR}_{k+1,m}(\mathscr{E}_{l+3(k+1)}) = \mathrm{Fg}(\mathrm{LR}_{k+1}(\mathscr{E}_{l+3(k+1)}))\,\mathrm{IpR}_{k,0,m}(\mathscr{F}_l),$$

so $\psi(\mathrm{LpR}_{k+1,m}(\mathscr{E}_{l+3(k+1)}))$ is also a weakly left λ-periodic word. $|\mathrm{Fg}(\mathrm{LR}_{k+1}(\mathscr{E}_{l+3(k+1)}))|$ is divisible by $|\lambda|$ for all $l \geq 0$, so the residue of $|\mathrm{LpR}_{k+1,m}(\mathscr{E}_{l+3(k+1)})| = |\mathrm{Fg}(\mathrm{LR}_{k+1}(\mathscr{E}_{l+3(k+1)}))| + |\mathrm{IpR}_{k,0,m}(\mathscr{F}_l)|$ modulo $|\lambda|$ equals the residue of $|\mathrm{IpR}_{k,0,m}(\mathscr{F}_l)|$ modulo $|\lambda|$ and does not depend on l. Therefore, λ is a left continuous evolutional period of $\mathscr{E}$ for the index m. ☐

Lemma 6.59. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the left (resp. at the right). Denote by $\mathscr{F}$ the following evolution of k-multiblocks: $\mathscr{F}_l = \mathrm{C}_{k+1}(\mathscr{E}_{l+3(k+1)})$. Suppose that $\mathscr{F}$ is continuously periodic for an index m $(1 \leq m \leq \mathrm{nker}_k(\mathscr{F}))$, but is **not** totally periodic.*

Then there are two possibilities:

1. *There exists a left (resp. right) continuous evolutional period of $\mathscr{E}$ for the index m.*

2. *There exists a k-series of obstacles in α.*

Proof. We prove the lemma for the situation when Case I holds at the left. If Case I holds at the right, the proof is completely symmetric. Suppose that k-series of obstacles do not exist in α. We have to prove that there exists a left continuous evolutional period of $\mathscr{E}$ for the index m.

By Corollary 6.52 there exists a final period λ such that for all $l \geq 0$, $\psi(\mathrm{Fg}(\mathrm{LR}_{k+1}(\mathscr{E}_{l+3(k+1)})))$ is a completely λ-periodic word, moreover, λ is the unique right total evolutional period of the evolution $\mathscr{F}'$ of stable nonempty k-multiblocks defined by $\mathscr{F}'_l = \mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)})$. Let us check that λ is also a weak left evolutional period of $\mathscr{F}$ for the pair $(0, m)$. If $m = 1$, this is already clear since $\mathrm{IpR}_{k,0,1}(\mathscr{F}_l)$ is always an empty occurrence. If $m > 1$, then since $\mathscr{F}'$ and $\mathscr{F}$ are consecutive, the fact that the left weak evolutional period of $\mathscr{F}$ for the pair $(0, m)$ also equals λ follows from Lemma 6.37. Now, λ is a left continuous evolutional period of $\mathscr{E}$ for the index m by Lemma 6.58. □

Lemma 6.60. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds both at the left and at the right.*

Then there are two possibilities:

1. *$\mathscr{E}$ is continuously periodic.*
2. *There exists a k-series of obstacles in α.*

Proof. Suppose that k-series of obstacles do not exist in α. We have to prove that $\mathscr{E}$ is continuously periodic.

If $\mathrm{ncker}_{k+1}(\mathscr{E}) = 1$, then the claim follows from Corollaries 6.52 and 6.54. Suppose that $\mathrm{ncker}_{k+1}(\mathscr{E}) > 1$.

Denote by $\mathscr{F}$ the evolution of stable nonempty k-multiblocks defined by $\mathscr{F}_l = \mathrm{C}_{k+1}(\mathscr{E}_{l+3(k+1)})$. Our assumption $\mathrm{ncker}_{k+1}(\mathscr{E}) > 1$ means that $\mathrm{nker}_k(\mathscr{F}) > 1$. By Corollary 6.39, $\mathscr{F}$ is continuously periodic. If $\mathscr{F}$ is not totally periodic, then the claim follows from Lemma 6.59.

Let us consider the case when $\mathscr{F}$ is totally periodic. Again, By Corollaries 6.52 and 6.54, there exist final periods λ and μ such that for all $l \geq 0$, $\psi(\mathrm{Fg}(\mathrm{LR}_{k+1}(\mathscr{E}_{l+3(k+1)})))$ is a completely λ-periodic word and $\psi(\mathrm{Fg}(\mathrm{RR}_{k+1}(\mathscr{E}_{l+3(k+1)})))$ is a completely μ-periodic word. Moreover, λ (resp. μ) is the unique right (resp. left) total evolutional period of the evolution $\mathscr{F}'$ (resp. $\mathscr{F}''$) of stable nonempty k-multiblocks defined by $\mathscr{F}'_l = \mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)})$ (resp. by $\mathscr{F}''_l = \mathrm{RA}_{k+1,2}(\mathscr{E}_{l+3(k+1)})$). So, $\mathscr{F}'$, $\mathscr{F}$, and $\mathscr{F}''$ are three consecutive totally periodic evolutions of stable nonempty k-multiblocks, and by Lemmas 6.44 and 6.45, $\mathrm{nker}_k(\mathscr{F}') > 1$ and $\mathrm{nker}_k(\mathscr{F}'') > 1$. We have also assumed that $\mathrm{nker}_k(\mathscr{F}) > 1$. Now we can use Lemma 6.41. It implies that either λ is a total left evolutional period of $\mathscr{F}$, or μ is a total right evolutional period of $\mathscr{F}$. If λ is a total left evolutional period of $\mathscr{F}$, set $m = \mathrm{nker}_k(\mathscr{F})$. Then λ is also a left weak evolutional period of $\mathscr{F}$ for the pair $(0, m)$ by Lemmas 6.24 and 6.12, and μ is a weak right evolutional period of $\mathscr{F}$ for the pair $(m, \mathrm{nker}_k(\mathscr{F})+1)$ since $\mathrm{IpR}_{k,\mathrm{nker}_k(\mathscr{F}),\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}_l)$ is always an empty occurrence. Similarly, if μ is a total right evolutional period of $\mathscr{F}$, then set $m = 1$. Then μ is also a right weak evolutional period of $\mathscr{F}$ for the pair $(m, \mathrm{nker}_k(\mathscr{F}) + 1)$ by Lemmas 6.23 and 6.14, and λ is a left weak evolutional period of $\mathscr{F}$ for the pair $(0, m)$ since $\mathrm{IpR}_{k,0,1}(\mathscr{F}_l)$ is always an empty occurrence.

The claim now follows from Lemma 6.58. □

Lemma 6.61. *Let $k \in \mathbb{N}$. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case II holds at the left (resp. at the right). Denote by $\mathscr{F}$ the following evolution of k-multiblocks: $\mathscr{F}_l = \mathrm{C}_{k+1}(\mathscr{E}_{l+3(k+1)})$.*

*Suppose that $\mathscr{F}$ is continuously periodic for an index m ($1 \leq m \leq \mathrm{nker}_k(\mathscr{F})$), but is **not** totally periodic.*

Then there are two possibilities:

1. *There exists a left (resp. right) continuous evolutional period of $\mathscr{E}$ for the index m.*
2. *There exists a k-series of obstacles in α.*

Proof. We prove the lemma for the situation when Case II holds at the right. If Case II holds at the left, the proof is completely symmetric. Suppose that k-series of obstacles do not exist in α. We have to prove that there exists a right continuous evolutional period of $\mathscr{E}$ for the index m.

If $m = \mathrm{nker}_k(\mathscr{F}) = \mathrm{ncker}_{k+1}(\mathscr{E})$, then, as we have already noted after the definition of a continuous evolutional period of an evolution of k-blocks, any final period is a right continuous evolutional period of $\mathscr{E}$ for the index m.

Suppose that $m < \mathrm{nker}_k(\mathscr{F})$. Denote by λ the (unique by Corollary 6.15) weak right evolutional period of $\mathscr{F}$ for the pair $(m, \mathrm{nker}_k(\mathscr{F})+1)$. Since Case II holds for $\mathscr{E}$ at the right, $\mathrm{Fg}(\mathrm{C}_{k+1}(\mathscr{E}_l))$ is a suffix of $\mathscr{E}_l$ for all $l \geq 1$, and $\mathrm{IpR}_{k,m,\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}_l) = \mathrm{RpR}_{k+1,m}(\mathscr{E}_{l+3(k+1)})$. By Lemma 6.55, $\psi(\mathrm{RBS}_{k+1}(\mathscr{E}))$ is periodic with period λ, so λ is a right continuous evolutional period of $\mathscr{E}$ directly by definition. □

Lemma 6.62. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case I holds at the left and Case II holds at the right.*

Then there are two possibilities:

1. *$\mathscr{E}$ is continuously periodic.*

2. *There exists a k-series of obstacles in α.*

Proof. Suppose that k-series of obstacles do not exist in α. We have to prove that $\mathscr{E}$ is continuously periodic.

If $\mathrm{ncker}_{k+1}(\mathscr{E}) = 1$, then the claim follows from Corollary 6.52. Suppose that $\mathrm{ncker}_{k+1}(\mathscr{E}) > 1$.

Again, denote by $\mathscr{F}$ the evolution of stable nonempty k-multiblocks defined by $\mathscr{F}_l = \mathrm{C}_{k+1}(\mathscr{E}_{l+3(k+1)})$. By Corollary 6.39, $\mathscr{F}$ is continuously periodic. If $\mathscr{F}$ is not totally periodic, then the claim follows from Lemmas 6.59 and 6.61.

Suppose that $\mathscr{F}$ is totally periodic. Denote by λ and μ the (unique by Corollaries 6.13 and 6.15 since $\mathrm{nker}_k(\mathscr{F}) = \mathrm{ncker}_{k+1}(\mathscr{E}) > 1$) left and right (respectively) total evolutional periods of $\mathscr{F}$. It follows from Lemmas 6.24 and 6.12 that λ is also a weak left evolutional period of $\mathscr{F}$ for the pair $(0, \mathrm{nker}_k(\mathscr{F}))$, and it follows from Lemmas 6.23 and 6.14 that μ is also a weak right evolutional period of $\mathscr{F}$ for the pair $(1, \mathrm{nker}_k(\mathscr{F}) + 1)$.

Consider the evolution $\mathscr{F}'$ of stable nonempty k-multiblocks defined by $\mathscr{F}'_l = \mathrm{LA}_{k+1,2}(\mathscr{E}_{l+3(k+1)})$. $\mathscr{F}'$ and $\mathscr{F}$ are consecutive, denote their concatenation by $\mathscr{F}''$. By Corollary 6.52, there exists a final period λ' such that λ' is both left and right total evolutional period of $\mathscr{F}'$. By Lemma 6.24, $\mathscr{F}'$ is also weakly periodic for the sequence $0, \mathrm{nker}_k(\mathscr{F}'), \mathrm{nker}_k(\mathscr{F}') + 1$. Now Lemma 6.25 says that $\mathscr{F}''$ is weakly periodic for the sequence $0, \mathrm{nker}_k(\mathscr{F}'), \mathrm{nker}_k(\mathscr{F}'') + 1$.

First, let us consider the case when $\mathscr{F}''$ is not totally periodic. We know that μ is a weak right evolutional period of $\mathscr{F}$ for the pair $(1, \mathrm{nker}_k(\mathscr{F}) + 1)$. By Lemma 5.31, $\mathrm{IpR}_{k,\mathrm{nker}_k(\mathscr{F}'),\mathrm{nker}_k(\mathscr{F}'')+1}(\mathscr{F}'') = \mathrm{IpR}_{k,1,\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F})$ for all $l \geq 0$ as an occurrence in α. Hence, μ is also a weak right evolutional period of $\mathscr{F}''$ for the pair $(\mathrm{nker}_k(\mathscr{F}'), \mathrm{nker}_k(\mathscr{F}'') + 1)$. By Lemma 6.55, $\psi(\mathrm{RBS}_{k+1}(\mathscr{E}))$ is periodic with period μ. Since Case II holds for $\mathscr{E}$ at the right, $\mathrm{IpR}_{k,1,\mathrm{nker}_k(\mathscr{F})+1}(\mathscr{F}_l) = \mathrm{RpR}_{k+1,1}(\mathscr{E}_{l+3(k+1)})$. Therefore, μ is a right continuous evolutional period of $\mathscr{E}$ for index 1 by definition. It also follows from Corollary 6.52 that λ' is a left continuous evolutional period of $\mathscr{E}$ for index 1, and $\mathscr{E}$ is continuously periodic.

Now suppose that $\mathscr{F}''$ is totally periodic. We know that $\mathrm{nker}_k(\mathscr{F}) = \mathrm{ncker}_{k+1}(\mathscr{E}) > 1$, that $\mathscr{F}'$ and $\mathscr{F}$ are totally periodic, and that λ is a weak left evolutional period of $\mathscr{F}$ for the pair $(0, \mathrm{nker}_k(\mathscr{F}) + 1)$. By Lemma 6.30, λ is also a total right evolutional period of $\mathscr{F}'$. By Lemma 6.44, $\mathrm{nker}_k(\mathscr{F}') > 1$, so by Corollary 6.15, $\lambda = \lambda'$. Now recall that λ is also a weak left evolutional period of $\mathscr{F}$ for the pair $(0, \mathrm{nker}_k(\mathscr{F}))$ and that Corollary 6.52 also says that for all $l \geq 0$, $\psi(\mathrm{Fg}(\mathrm{LR}_{k+1}(\mathscr{E}_{l+3(k+1)})))$ is a completely λ-periodic word. Therefore, by Lemma 6.58, λ is a left continuous evolutional period of $\mathscr{E}$ for the index $\mathrm{nker}_k(\mathscr{F})$. And again, since Case II holds for $\mathscr{E}$ at the right, any final period is a right continuous evolutional period of $\mathscr{E}$ for the index $\mathrm{ncker}_{k+1}(\mathscr{F}) = \mathrm{nker}_k(\mathscr{F})$, and $\mathscr{E}$ is continuously periodic. □

Lemma 6.63. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case II holds at the left and Case I holds at the right.*

Then there are two possibilities:

1. *$\mathscr{E}$ is continuously periodic.*

2. *There exists a k-series of obstacles in α.*

Proof. The proof is completely symmetric to the proof of the previous lemma. □

Lemma 6.64. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Let $\mathscr{E}$ be an evolution of $(k+1)$-blocks such that Case II holds both at the left and at the right.*

Then there are two possibilities:

1. *$\mathscr{E}$ is continuously periodic.*

2. *There exists a k-series of obstacles in α.*

Proof. Suppose that k-series of obstacles do not exist in α. We have to prove that $\mathscr{E}$ is continuously periodic.

Again, consider the evolution $\mathscr{F}$ of stable nonempty k-multiblocks defined by $\mathscr{F}_l = \mathrm{C}_{k+1}(\mathscr{E}_{l+3(k+1)})$. By Corollary 6.39, $\mathscr{F}$ is continuously periodic. If $\mathscr{F}$ is not totally periodic, then the claim follows from Lemma 6.61.

Suppose that $\mathscr{F}$ is totally periodic. If $\mathrm{ncker}_k(\mathscr{E}) = \mathrm{nker}_k(\mathscr{F}) = 1$, then $\mathscr{E}$ is automatically continuously periodic, as we have noted right after the definition of a continuously periodic evolution of k-multiblocks.

If $\mathrm{ncker}_k(\mathscr{E}) = \mathrm{nker}_k(\mathscr{F}) > 1$, denote the (unique by Corollaries 6.13 and 6.15) left and right total evolutional periods of $\mathscr{F}$ by λ and μ, respectively. By Lemma 6.57, either $\psi(\mathrm{LBS}_{k+1}(\mathscr{E}))$ is periodic with period λ, or $\psi(\mathrm{RBS}_{k+1}(\mathscr{E}))$ is periodic with period μ. If $\psi(\mathrm{LBS}_{k+1}(\mathscr{E}))$ is periodic with period λ, then λ is a left continuous evolutional period of $\mathscr{E}$ for the index $\mathrm{ncker}_{k+1}(\mathscr{E})$, and any final period is a right continuous evolutional period of $\mathscr{E}$ for the index $\mathrm{ncker}_{k+1}(\mathscr{E})$. If $\psi(\mathrm{RBS}_{k+1}(\mathscr{E}))$ is periodic with period μ, then μ is a right continuous evolutional period of $\mathscr{E}$ for the index 1, and any final period is a left continuous evolutional period of $\mathscr{E}$ for the index 1. ☐

Proposition 6.65. *Let $k \in \mathbb{N}$. Suppose that all evolutions of k-blocks in α are continuously periodic. Then either all evolutions of $(k+1)$-blocks in α are continuously periodic, or there exists a k-series of obstacles in α.*

Proof. This follows directly from Lemmas 6.60, 6.62, 6.63, and 6.64. ☐

7 Factor complexity

In this section, we will prove Propositions 1.2–1.6 and Theorem 1.1.

Lemma 7.1. *Suppose that there exists a k-series of obstacles $\mathscr{H}$ in α. Then the factor complexity of $\beta = \psi(\alpha)$ is $\Omega(n^{1+1/k})$.*

Proof. Let $k' \in \mathbb{N}$ ($1 \le k' \le k$) be the number such that $|\mathscr{H}_l| = \Theta(l^{k'})$ for $l \to \infty$. This means that there exist $l_0 \in \mathbb{N}$ and $x, y \in \mathbb{R}_{>0}$ such that if $l \ge l_0$, then $xl^{k'} < |\mathscr{H}_l| < yl^{k'}$.

Fix an arbitrary $n \in \mathbb{N}$, $n > 4yl_0^{k'}$. We are going to find a lower estimate for the amount of different factors of β of length n. Set

$$l_1 = \sqrt[k']{\frac{n}{4y}}, \quad l_2 = \sqrt[k']{\frac{n}{2y}}, \quad l_3 = l_2 - l_1 = \left(\sqrt[k']{\frac{1}{2}} - \sqrt[k']{\frac{1}{4}}\right) \sqrt[k']{\frac{n}{y}}.$$

Then $l_0 < l_1 < l_2$, and there exist at least $l_3 - 1$ indices $l \in \mathbb{N}$ such that $l_1 \le l \le l_2$. Consider the occurrences $\mathscr{H}_l$ in α for $l_1 \le l \le l_2$. Since $|\mathscr{H}_l|$ strictly grows as l grows, all these occurrences have different lengths. Moreover, if $l_1 \le l \le l_2$, then

$$|\mathscr{H}_l| > xl^{k'} \ge xl_1^{k'} = \frac{x}{4y}n,$$

and

$$|\mathscr{H}_l| < yl^{k'} \le yl_2^{k'} = \frac{1}{2}n.$$

Denote

$$n_0 = \frac{x}{4y}n.$$

Then if $l_1 \le l \le l_2$, then $n_0 < |\mathscr{H}_l| < n/2$.

For each $l \in \mathbb{N}$, $l_1 \le l \le l_2$, denote by i_l and j_l the indices such that $\mathscr{H}_l = \alpha_{i_l \dots j_l}$. Since $\mathscr{H}_l$ is a series of obstacles, there exists $p \in \mathbb{N}$ ($p \le \mathbf{L}$) such that all words $\psi(\alpha_{i_l \dots j_l})$ are weakly p-periodic, and all words $\psi(\alpha_{i_l \dots j_l+1})$ and (if $i_l > 0$) $\psi(\alpha_{i_l-1 \dots j_l})$ are not. Since all words $\mathscr{H}_l$ have different lengths, i_l cannot coincide with $i_{l'}$ if $l \ne l'$ ($l_1 \le l, l' \le l_2$). Denote by m ($l_1 \le m \le l_2$) the index such that $i_m = \min_{l_1 \le l \le l_2} i_l$. Let us check that if $l \ne m$, $l_1 \le l \le l_2$, then $i_l > n_0 - 2\mathbf{L}$.

Indeed, assume that $i_l \le n_0 - 2\mathbf{L}$. Then $i_m < n_0 - 2\mathbf{L}$ since $i_m < i_l$. But $j_m = j_m - i_m + 1 + i_m - 1 = |\mathscr{H}_m| + i_m - 1 > n_0 - 1$, Similarly, $j_l > n_0 - 1$. So, if $t = \min(j_m, j_l)$, then $t > n_0 - 1$, so $t \ge i_l$ and $|\alpha_{i_l \dots t}| = t - i_l + 1 > n_0 - 1 - n_0 + 2\mathbf{L} + 1 = 2\mathbf{L} \ge 2p$. Denote $t' = \max(j_m, j_l)$. By Corollary 2.5, $\psi(\alpha_{i_m \dots t'})$ is a weakly p-periodic word. In particular, $\psi(\alpha_{i_m \dots j_l})$ is a weakly p-periodic word, but this contradicts the assumption that $\psi(\alpha_{i_l-1 \dots j_l})$ is not a p-periodic word.

Consider the following occurrences in α: $\alpha_{i_l-s \dots i_l-s+n-1}$, where $0 \le s \le n_0 - 2\mathbf{L}$ and $l_1 \le l \le l_2$, $l \ne m$. We already know that if $l_1 \le l \le l_2$, $l \ne m$, then $i_l > n_0 - 2\mathbf{L}$, so if $0 \le s \le n_0 - 2\mathbf{L}$ and $l_1 \le l \le l_2$, $l \ne m$, then $i_l - s > 0$. Clearly, all these occurrences have length n. Let us prove that all words $\psi(\alpha_{i_l-s \dots i_l-s+n-1})$ are different *abstract words*. (If $n_0 - 2\mathbf{L} < 0$, then we have no occurrences, but $n_0 - 2\mathbf{L} \ge 0$ if n is large enough. During the proof that all these abstract words are different, we suppose that $n_0 - 2\mathbf{L} \ge 0$, and we have at least one word.)

Denote $\mathscr{T}_{s,l} = \psi(\alpha_{i_l-s \dots i_l-s+n-1})$. Temporarily fix an index s ($0 \le s \le n_0 - 2\mathbf{L}$) and an index l ($l_1 \le l \le l_2, l \ne m$). Denote $\gamma = \mathscr{T}_{s,l}$. Then $\gamma_v = \psi(\alpha_{i_l-s+v})$ for $0 \le v \le n-1$. We have $j_l - i_l + 1 = |\mathscr{H}_l| < n/2$ and $s \le n_0 - 2\mathbf{L} < n/2$, so $n/2 < n-s$, $j_l - i_l + 1 < n/2 < n-s$, and $s + j_l - i_l + 1 < n$. Hence, $\gamma_{s \dots s+j_l-i_l}$, $\gamma_{s \dots s+j_l-i_l+1}$, and (if $s > 0$) $\gamma_{s-1 \dots s+j_l-i_l}$ are occurrences in γ. We have $\gamma_{s \dots s+j_l-i_l} = \psi(\alpha_{i_l-s+s \dots i_l-s+s+j_l-i_l}) = \psi(\alpha_{i_l \dots j_l})$. Similarly, $\gamma_{s \dots s+j_l-i_l+1} = \psi(\alpha_{i_l \dots j_l+1})$ and (if $s > 0$) $\gamma_{s-1 \dots s+j_l-i_l} = \psi(\alpha_{i_l-1 \dots j_l})$ (the notation $\alpha_{i_l-1 \dots j_l}$ is well-defined since $i_l > n_0 - 2\mathbf{L} \ge 0$). Therefore, $\gamma_{s \dots s+j_l-i_l}$ is a weakly p-periodic word, and $\gamma_{s \dots s+j_l-i_l+1}$ and (if $s > 0$) $\gamma_{s-1 \dots s+j_l-i_l}$ are not.

Now assume that $\mathscr{T}_{s,l} = \mathscr{T}_{s',l'}$ as an abstract word, where $s \ne s'$ or $l \ne l'$. (Here $0 \le s, s' \le n_0 - 2\mathbf{L}$, $l_1 \le l, l' \le l_2$, $l \ne m$, and $l' \ne m$.) Denote $\gamma = \mathscr{T}_{s,l} = \mathscr{T}_{s',l'}$. First, let us consider the case when $s = s'$ and $l \ne l'$. Without loss of generality, $l < l'$, so $j_l - i_l + 1 = |\mathscr{H}_l| < |\mathscr{H}_{l'}| = j_{l'} - i_{l'} + 1$, and $j_l - i_l + 1 \le j_{l'} - i_{l'}$. Then $\gamma_{s \dots s+j_l-i_l+1}$ is a prefix of $\gamma_{s \dots s+j_{l'}-i_{l'}}$, but $\gamma_{s \dots s+j_{l'}-i_{l'}}$ is a p-periodic word, and $\gamma_{s \dots s+j_l-i_l+1}$ is not, so we have a contradiction.

Now suppose that $s \ne s'$. Without loss of generality, $s' < s$, so $s > 0$. Since $|\mathscr{H}_l| = j_l - i_l + 1 \ge 2\mathbf{L}$, $(s + j_l - i_l) - s + 1 \ge 2\mathbf{L}$. Since $|\mathscr{H}_{l'}| > n_0$ and $s' \ge 0$, we also have $s' + j_{l'} - i_{l'} + 1 > n_0$. Since $s \le n_0 - 2\mathbf{L}$, $(s' + j_{l'} - i_{l'}) - s + 1 > n_0 - n_0 + 2\mathbf{L} = 2\mathbf{L}$. Therefore, if $t = \min(s + j_l - i_l, s' + j_{l'} - i_{l'})$, then $t - s + 1 \ge 2\mathbf{L} \ge 2p$. Now we can use Corollary 2.5. Recall that $\gamma_{s \dots s+j_l-i_l}$ and $\gamma_{s' \dots s'+j_{l'}-i_{l'}}$ are weakly p-periodic words. Denote $t' = \max(s + j_l - i_l, s' + j_{l'} - i_{l'})$. By Corollary 2.5, $\gamma_{s' \dots t'}$ is a p-periodic word. Hence, $\gamma_{s-1 \dots t'}$ is also a p-periodic word (recall that $s' < s$ and $s > 0$), and $\gamma_{s-1 \dots s+j_l-i_l}$ is also a p-periodic word. But previously we have seen that $\gamma_{s-1 \dots s+j_l-i_l}$ is not a p-periodic word, so we have a contradiction.

Let us count how many words $\mathscr{T}_{s,l}$ we have. If $n_0 < 2\mathbf{L}$, then we have none of them, and if

$$n_0 \ge 2\mathbf{L} \Leftrightarrow \frac{x}{4y} n \ge 2\mathbf{L} \Leftrightarrow n \ge \frac{8y\mathbf{L}}{x},$$

then there are $n_0 - 2\mathbf{L} + 1$ possibilities for s and at least $l_2 - l_1 - 2 = l_3 - 2$ possibilities for l. Hence, we have at least

$$(n_0 - 2\mathbf{L} + 1)(l_2 - l_1 - 2) = \left(\frac{x}{4y} n - 2\mathbf{L} + 1\right)\left(\left(\sqrt[k']{\frac{1}{2}} - \sqrt[k']{\frac{1}{4}}\right)\sqrt[k']{\frac{n}{y}} - 2\right)$$

different factors of $\beta = \psi(\alpha)$, and the factor complexity of β is $\Omega(n^{1+1/k'})$ for $n \to \infty$. But $k' \le k$, so the factor complexity of β is also $\Omega(n^{1+1/k})$ for $n \to \infty$. □

Note that in the proof of this lemma, we proved in fact that if $|\mathscr{H}_l|$ is $\Theta(l^{k'})$ for $l \to \infty$, then the factor complexity of $\beta = \psi(\alpha)$ is $\Omega(n^{1+1/k'})$ for $n \to \infty$, and $n^{1+1/k'}$ is $\omega(n^{1+1/k})$ if $k' < k$. Later, after we prove

Proposition 1.3, we will see that this is not possible if evolutions of k-blocks really exist in α and all of them are continuously periodic. Therefore, if k-blocks really exist in α, then all k-series of obstacles obtained from Proposition 6.65 actually satisfy $|\mathscr{H}_l| = \Theta(l^k)$ for $l \to \infty$. However, it was not very convenient to prove this directly, so in the definition of a k-series of obstacles we allowed $|\mathscr{H}_l| = \Theta(l^{k'})$ for some $k' \le k$.

Proof of Proposition 1.2. We know that there exists a non-continuously periodic evolution of k-blocks, and we also know (see Remark 6.20) that all evolutions of 1-blocks arising in α are continuously periodic. Let $k' \in \mathbb{N}$ be the largest number such that all evolutions of k'-blocks arising in α are continuously periodic. Then $k' \le k-1$. By Proposition 6.65, there exists a k'-series of obstacles in α. By Lemma 7.1, the factor complexity of $\beta = \psi(\alpha)$ is $\Omega(n^{1+1/k'})$. But $k' \le k-1$, so $n^{1+1/k'} \ge n^{1+1/(k-1)}$, and the factor complexity of β is $\Omega(n^{1+1/(k-1)})$. □

The proof of Proposition 1.3 is based on the following lemma.

Lemma 7.2. *Let $k \in \mathbb{N}$. Suppose that $a \in \Sigma$ is a letter of order at least $k+2$ such that $\varphi(a) = a\gamma$ for some $\gamma \in \Sigma^*$, and all evolutions of k-blocks arising in $\alpha = \varphi^\infty(a)$ are continuously periodic.*

Let $f\colon \mathbb{N} \to \mathbb{R}$ be a function and $n_0 \in \mathbb{N}$, $n_0 > 1$ be a number such that:

1. *If $n \ge n_0$, then $f(n) \ge 3k$.*
2. *If $\mathscr{E}$ is an evolution of k-blocks such that Case I holds at the left (resp. at the right) and $l \ge f(n)$ for some $l, n \in \mathbb{N}$, $n \ge n_0$, then $|\operatorname{Fg}(\operatorname{LR}_k(\mathscr{E}_l))| > n$ (resp. $|\operatorname{Fg}(\operatorname{RR}_k(\mathscr{E}_l))| > n$).*
3. *If b is a letter of order $> k$ and $l \ge f(n)$ for some $l, n \in \mathbb{N}$, $n \ge n_0$, then $|\varphi^l(b)| > n$.*

Then the factor complexity of $\beta = \psi(\alpha)$ is $O(nf(n))$.

Proof. Denote the total number of all abstract words that can equal the forgetful occurrences of all left and right preperiods of stable k-blocks or composite central kernels of k-blocks by M (by Corollary 5.3 and by Lemma 5.32, this number is finite). Denote the maximal length of the forgetful occurrence of a left or a right preperiod or of a composite central kernel of a stable k-block by P. Denote the number of all final periods we have by N.

Fix a number $n \in \mathbb{N}$ ($n \ge n_0$ and $n > P$.) Let $\alpha_{i\ldots j}$ be an occurrence in α of length $n \ge n_0$. Set $l_0 = \lceil f(n) \rceil + 1$. Since $\alpha = \varphi^\infty(a)$, α can be written as $\alpha = \varphi(\alpha)$ and as $\alpha = \varphi^l(\alpha) = \varphi^l(\alpha_0)\varphi^l(\alpha_1)\varphi^l(\alpha_2)\ldots$ for all $l \ge 0$.

Let s' and t' be the indices such that α_i (resp. α_j) is contained in $\varphi^{l_0}(\alpha_{s'})$ (resp. in $\varphi^{l_0}(\alpha_{t'})$) as an occurrence in α. Clearly, $s' \le t'$. If $s' < t'$, then set $s = s'$, $t = t'$, and $q = l_0$.

If $s' = t'$, then for each l ($0 \le l \le l_0$) denote by s''_l and t''_l the indices such that α_i (resp. α_j) is contained in $\varphi^l(\alpha_{s''_l})$ (resp. in $\varphi^l(\alpha_{t''_l})$) as an occurrence in α. Let q be the maximal value of l such that $s''_l < t''_l$. Then $s''_{q+1} = t''_{q+1}$, and both $\alpha_{s''_q}$ and $\alpha_{t''_q}$ are contained in $\varphi(\alpha_{s''_{q+1}})$ as an occurrence in α. So, $|\alpha_{s''_q\ldots t''_q}| = t''_q - s''_q + 1 \le |\varphi|$. Set $s = s''_q$ and $t = t''_q$.

Summarizing, we have found indices s and t and a number $q \in \mathbb{Z}_{\ge 0}$ ($0 \le q \le l_0$) such that:

1. $s < t$.
2. α_i (resp. α_j) is contained in $\varphi^q(\alpha_s)$ (resp. in $\varphi^q(\alpha_t)$) as an occurrence in α.
3. If $q < l_0$, then $t - s < |\varphi|$.

We are going to estimate the amount of different words that can be equal to $\psi(\alpha_{i\ldots j})$ as abstract words (for different i and j such that $|\alpha_{i\ldots j}| = j - i + 1 = n$). We will consider the cases $q = l_0$ and $q < l_0$ separately.

First, suppose that $q = l_0$. Then we don't have any explicit upper estimates for $|\alpha_{s\ldots t}|$ so far, but we can say that if $|\alpha_{s\ldots t}| > 2$, then $\varphi^q(\alpha_{s+1\ldots t-1})$ is a suboccurrence in $\alpha_{i\ldots j}$. So, $|\varphi^q(\alpha_{s+1\ldots t-1})| \le n$, and

$\alpha_{s+1\dots t-1}$ cannot contain letters of order $> k$. Since $\alpha_0 = a$ is a letter of order at least $k+2$, by Lemma 3.5, α can be split into a concatenation of k-blocks and letters of order $> k$, so $\alpha_{s+1\dots t-1}$ is a suboccurrence in a k-block. Denote this k-block by $\alpha_{u'\dots v'}$. Then $\varphi^q(\alpha_{s+1\dots t-1})$ is a nonempty suboccurrence in both $\alpha_{i\dots j}$ and $\mathrm{Dc}_k^q(\alpha_{u'\dots v'})$. Let u and v be the indices such that $\mathrm{Dc}_k^q(\alpha_{u'\dots v'}) = \alpha_{u\dots v}$. If $u > i$, then α_{u-1} is a letter of order $> k$, and this letter must be contained in $\varphi^q(\alpha_s)$, so α_s must be a letter of order $> k$, and $u' = s+1$. Similarly, if $v < j$, then α_{v+1} is a letter of order $> k$, and this letter must be contained in $\varphi^q(\alpha_t)$, so α_t must be a letter of order $> k$, and $v' = t-1$.

Summarizing, we have the following cases:

1. $|\alpha_{s\dots t}| = 2$, and $t = s+1$.
2. $|\alpha_{s\dots t}| > 2$. There exists a (unique) nonempty k-block $\alpha_{u'\dots v'}$ such that $\alpha_{s+1\dots t-1}$ is a suboccurrence in $\alpha_{u'\dots v'}$. Denote $\mathrm{Dc}_k^q(\alpha_{u'\dots v'}) = \alpha_{u\dots v}$. Then there are the following possibilities:
   (a) $u \le i$ and $v \ge j$, so $\alpha_{i\dots j}$ is a suboccurrence in $\alpha_{u\dots v}$.
   (b) $u > i$, but $v \ge j$, then $u' = s+1$ and α_s is a letter of order $> k$.
   (c) $u \le i$, but $v < j$, then $v' = t-1$ and α_t is a letter of order $> k$.
   (d) $u > i$ and $v < j$, then $u' = s+1$, $v' = t-1$, and both α_s and α_t are letters of order $> k$.

Let us consider these cases one by one.

<u>Case 1.</u> There exists $x \in \mathbb{N}$ $(1 \le x \le n-1)$ such that $\alpha_{i\dots j}$ is the concatenation of the suffix of $\varphi^q(\alpha_s)$ of length x and the prefix of $\varphi^q(\alpha_t)$ of length $n-x$. There are at most $|\Sigma|^2(n-1)$ possibilities for $\alpha_{i\dots j}$ as an abstract word. Therefore, there are at most $|\Sigma|^2(n-1)$ possibilities for $\psi(\alpha_{i\dots j})$ as an abstract word.

<u>Case 2.</u> Observe that, since $\alpha_{u\dots v} = \mathrm{Dc}_k^q(\alpha_{u\dots v})$, the evolutional sequence number of $\alpha_{u\dots v}$ is at least $q = l_0 > f(n)$. In particular, $q \ge 3k$, and $\alpha_{u\dots v}$ is a stable k-block. Denote the evolution $\alpha_{u\dots v}$ belongs to by $\mathscr{E}$. If Case I holds for $\mathscr{E}$ at the left (resp. at the right), then $|\operatorname{Fg}(\mathrm{LR}_k(\alpha_{u\dots v}))| > n$ (resp. $|\operatorname{Fg}(\mathrm{RR}_k(\alpha_{u\dots v}))| > n$). If Case I holds for $\mathscr{E}$ at the left or at the right, then $|\operatorname{Fg}(\mathrm{LR}_k(\alpha_{u\dots v}))\operatorname{Fg}(\mathrm{C}_k(\alpha_{u\dots v}))\operatorname{Fg}(\mathrm{RR}_k(\alpha_{u\dots v}))| > n$, and $|\alpha_{u\dots v}| = v-u+1 > n$. We know that all evolutions of k-blocks in α are continuously periodic, so let m $(1 \le m \le \mathrm{ncker}_k(\mathscr{E}))$ be an index such that $\mathscr{E}$ is continuously periodic for the index m. Let λ (resp. μ) be a left (resp. right) continuous evolutional period of $\mathscr{E}$ for the index m.

Let us check that if Case II holds for $\mathscr{E}$ at the left and $u > i$, then $\alpha_{i\dots u-1}$ is a suffix of $\mathrm{LBS}_k(\mathscr{E})$. Recall that in this case, $u'-1 = s$. Denote $\alpha_{u''\dots v''} = \mathrm{Dc}_k(\alpha_{u'\dots v'})$. Then $\alpha_{u''-1}$ is the rightmost letter of order $> k$ in $\varphi(\alpha_s)$. Let α_w be the rightmost letter in $\varphi^{q-1}(\alpha_{u''-1})$, and let $\alpha_{w'}$ be the leftmost letter in $\varphi^q(\alpha_{u'})$. Then α_w is contained in $\varphi^q(\alpha_s)$, so $w < w'$. We have $s < u' < t$, so $\varphi^q(\alpha_{u'})$ is a suboccurrence in $\alpha_{i\dots j}$, hence $\alpha_{w'}$ is contained in $\alpha_{i\dots j}$, $w' \le j$, and $w < j$. On the other hand, α_{w+1} is contained in $\varphi^{q-1}(\alpha_{u''})$, so α_{w+1} is contained in $\mathrm{Dc}_k^{q-1}(\alpha_{u''\dots v''}) = \mathrm{Dc}_k^q(\alpha_{u'\dots v'}) = \alpha_{u\dots v}$, $w+1 \ge u > i$, and $w \ge i$. Therefore, either $\varphi^{q-1}(\alpha_{u''-1})$ is a factor in $\alpha_{i\dots j}$, or $\alpha_{i\dots w}$ is a suffix of $\varphi^{q-1}(\alpha_{u''-1})$. But $\varphi^{q-1}(\alpha_{u''-1})$ cannot be a factor of $\alpha_{i\dots j}$ since $\alpha_{u''-1}$ is a letter of order $> k$ and $q-1 = \lceil f(n) \rceil \ge f(n)$, so $|\varphi^{q-1}(\alpha_{u''-1})| > n$. Therefore, $\alpha_{i\dots w}$ is a suffix of $\varphi^{q-1}(\alpha_{u''-1})$. We have $\alpha_{u''-1} = \mathrm{LB}(\alpha_{u''\dots v''})$, $\alpha_{u-1} = \mathrm{LB}(\alpha_{u\dots v})$, and $\alpha_{u\dots v} = \mathrm{Dc}_k^{q-1}(\alpha_{u''\dots v''})$. We also have $\alpha_{u''\dots v''} = \mathrm{Dc}_k(\alpha_{u'\dots v'})$, so the evolutional sequence number of $\alpha_{u''\dots v''}$ is at least 1. Now it follows from Remark 6.9 that $\alpha_{i\dots u-1}$ is a suffix of $\mathrm{LBS}_k(\mathscr{E})$.

Note that we could not use $\alpha_{u'\dots v'}$ instead of $\alpha_{u''\dots v''}$ in this argument since the evolutional sequence number of $\alpha_{u'\dots v'}$ could equal 0.

Similarly, if Case II holds for $\mathscr{E}$ at the right and $v < j$, then $\alpha_{v+1\dots j}$ is a prefix of $\mathrm{RBS}_k(\mathscr{E})$.

If $u > i$, but Case I holds for $\mathscr{E}$ at the left, then we did not define any left bounding sequence, but we know that independently on whether Case I or II holds for $\mathscr{E}$ at the left, if $u > i$, then α_{u-1} is the rightmost letter of order $> k$ in $\varphi(\alpha_s)$. So, if $u > i$ (and $\alpha_s = \mathrm{LB}(\alpha_{u'\dots v'})$ is a letter of order $> k$), denote by γ the prefix of $\varphi^q(\alpha_s)$ that ends with the rightmost letter of order $> k$ in $\varphi^q(\alpha_s)$. Then $\alpha_{i\dots u-1}$ is a suffix of γ. Clearly, γ as an abstract word depends only on q and on α_s as an abstract letter.

Similarly, if $v < j$, denote by γ' the suffix of $\varphi^q(\alpha_t)$ that begins with the leftmost letter of order $> k$ in $\varphi^q(\alpha_t)$. Then $\alpha_{v+1\dots j}$ is a prefix of γ', and γ' as an abstract word depends only on q and on α_t as an abstract letter.

Now let us consider cases 2a–2d one by one.

Case 2a. We can write $\alpha_{u\dots v}$ as

$$\alpha_{u\dots v} = \mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v})).$$

If Case I holds at the left, then $|\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})| \geq |\,\mathrm{Fg}(\mathrm{LR}_k(\alpha_{u\dots v}))| > n$, and $\alpha_{i\dots j}$ is a suboccurrence either in

$$\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v}),$$

or in

$$\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v})).$$

If Case II holds at the left, then $\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))$ is empty, and $\alpha_{i\dots j}$ is a suboccurrence in

$$\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v})).$$

So, independently on whether Case I or Case II holds at the left, $\alpha_{i\dots j}$ is a suboccurrence either in

$$\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v}),$$

or in

$$\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v})).$$

If $\alpha_{i\dots j}$ is a suboccurrence in

$$\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v})),$$

and Case I holds at the right, then $|\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})| \geq |\,\mathrm{Fg}(\mathrm{RR}_k(\alpha_{u\dots v}))| > n$, and $\alpha_{i\dots j}$ is a suboccurrence either in

$$\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v}),$$

or in

$$\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v})).$$

If $\alpha_{i\dots j}$ is a suboccurrence in

$$\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v})),$$

and Case II holds at the right, then $\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v}))$ is empty, and $\alpha_{i\dots j}$ is a suboccurrence in

$$\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})$$

anyway.

Therefore, independently on whether Case I or II holds at the left or at the right, there are three possibilities for $\alpha_{i\dots j}$: $\alpha_{i\dots j}$ is a suboccurrence either in $\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})$, or in $\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})$, or in $\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v}))$.

If $\alpha_{i\dots j}$ is a suboccurrence of $\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})$, then $\psi(\alpha_{i\dots j})$ is the concatenation of a suffix of $\psi(\mathrm{LpreP}_k(\mathscr{E}))$ of length $x \leq P$ and the weakly $|\lambda|$-periodic word of length $n-x$ with a left period λ', which is a cyclic shift of λ (this cyclic shift can be nontrivial if $x = 0$, and $\alpha_{i\dots j}$ is actually a suboccurrence in $\mathrm{LpR}_{k,m}(\alpha_{u\dots v})$), so λ' is a final period as well. Recall that $n > P$, so $\alpha_{i\dots j}$ cannot be a suboccurrence of $\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))$. We have at most $M(P+1)N$ different words that can equal $\psi(\alpha_{i\dots j})$.

The situation when $\alpha_{i\dots j}$ is a suboccurrence of $\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v}))$ is considered similarly and gives us at most $M(P+1)N$ more possibilities for $\psi(\alpha_{i\dots j})$ as an abstract word.

If $\alpha_{i\dots j}$ is a suboccurrence of

$$\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v}),$$

but is not a suboccurrence of

$$\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})$$

or of

$$\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v})),$$

then $\psi(\alpha_{i\dots j})$ is a factor of the word $\delta\psi(\mathrm{cKer}_{k,m}(\mathscr{E}))\delta'$, where δ (resp. δ') is the weakly $|\lambda|$-periodic (resp. $|\mu|$-periodic) word of length n with right (resp. left) period λ' (resp. μ'), which is a cyclic shift of λ (resp. of μ), and is a final period as well. We have $|\delta\psi(\mathrm{cKer}_{k,m}(\mathscr{E}))\delta'| = 2n + |\psi(\mathrm{cKer}_{k,m}(\mathscr{E}))| \le 2n + P$, and there are at most $N^2M(n+P+1)$ different possibilities for $\psi(\alpha_{i\dots j})$.

Totally, we have $2M(P+1)N + N^2M(n+P+1)$ possibilities for $\psi(\alpha_{i\dots j})$ in Case 2a.

Case 2b. Again write

$$\alpha_{u\dots v} = \mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{Fg}(\mathrm{RpreP}_k(\alpha_{u\dots v})).$$

This time $u > i$, so if Case I holds at the right, then

$$|\,\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})| \ge |\,\mathrm{Fg}(\mathrm{RR}_k(\alpha_{u\dots v}))| > n,$$

and $\alpha_{u\dots j}$ is a suboccurrence in

$$\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v}).$$

And again, if Case II holds at the right, then

$$\alpha_{u\dots v} = \mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v}),$$

and $\alpha_{u\dots j}$ is also a suboccurrence in

$$\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v}).$$

So, independently on whether Case I or Case II holds at the right, $\alpha_{u\dots j}$ is always a suboccurrence in

$$\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v}).$$

If Case I holds at the left, then $|\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})| \ge |\,\mathrm{Fg}(\mathrm{LR}_k(\alpha_{u\dots v}))| > n$, and $\alpha_{u\dots j}$ is a suboccurrence in $\mathrm{Fg}(\mathrm{LpreP}_k(\alpha_{u\dots v}))\,\mathrm{LpR}_{k,m}(\alpha_{u\dots v})$. Therefore, $\psi(\alpha_{i\dots j})$ is the concatenation of the suffix of $\psi(\gamma)$ of length $u-i<n$ and the prefix of length $n-(u-i)$ of the word $\psi(\mathrm{LpreP}_k(\mathscr{E}))\delta$, where δ is the weakly left λ-periodic word of length n. We have at most $|\Sigma|(n-1)MN$ possibilities for $\psi(\alpha_{i\dots j})$.

If Case II holds at the left, then $\alpha_{u\dots j}$ is a suboccurrence in

$$\mathrm{LpR}_{k,m}(\alpha_{u\dots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v}).$$

If $m=1$, then $\alpha_{u\dots j}$ is a suboccurrence in $\mathrm{cKer}_{k,m}(\alpha_{u\dots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\dots v})$. Then $\psi(\alpha_{i\dots j})$ is the concatenation of the suffix of $\psi(\gamma)$ of length $u-i<n$ and the prefix of length $n-(u-i)$ of the word $\psi(\mathrm{cKer}_{k,m}(\mathscr{E}))\delta$,

where δ is the weakly $|\mu|$-periodic word of length n with left period μ', which is a cyclic shift of μ. Again, we have at most $|\Sigma|(n-1)MN$ possibilities for $\psi(\alpha_{i\ldots j})$.

If Case II holds at the left and $m > 1$, then $\psi(\mathrm{LBS}_k(\mathscr{E}))$ is a periodic sequence infinite to the left with period λ. As we have checked previously, $\alpha_{i\ldots u-1}$ is a suffix of $\psi(\mathrm{LBS}_k(\mathscr{E}))$, so $\psi(\alpha_{i\ldots u-1})$ is a weakly right λ-periodic word. $\psi(\mathrm{LpR}_{k,m}(\alpha_{u\ldots v}))$ is a weakly left λ-periodic word, so $\psi(\alpha_{i\ldots j})$ is a factor in a word of the form $\delta\psi(\mathrm{cKer}_{k,m}(\mathscr{E}))\delta'$, where δ (resp. δ') is the weakly $|\lambda|$-periodic (resp. $|\mu|$-periodic) word of length n with a right (resp. a left) period λ' (resp. μ'), which is a cyclic shift of λ (resp. of μ), (so λ' and μ' are final periods). We have at most $N^2M(n+P+1)$ possibilities for $\psi(\alpha_{i\ldots j})$.

In total, Case 2b gives us at most $2|\Sigma|(n-1)MN + N^2M(n+P+1)$ possibilities for $\psi(\alpha_{i\ldots j})$.

Case 2c is symmetric to Case 2b and gives at most $2|\Sigma|(n-1)MN + N^2M(n+P+1)$ more possibilities for $\psi(\alpha_{i\ldots j})$.

Case 2d. This time $\alpha_{u\ldots v}$ is a suboccurrence in $\alpha_{i\ldots j}$, so Case II must hold for $\mathscr{E}$ both at the left and at the right, otherwise, $|\alpha_{u\ldots v}| > n$. So, $\alpha_{u\ldots v} = \mathrm{Fg}(\mathrm{C}_k(\alpha_{u\ldots v})) = \mathrm{LpR}_{k,m}(\alpha_{u\ldots v})\,\mathrm{cKer}_{k,m}(\alpha_{u\ldots v})\,\mathrm{RpR}_{k,m}(\alpha_{u\ldots v})$, and we have several possibilities for the value of m.

First, if $m = 1$ and $\mathrm{ncker}_k(\mathscr{E}) = 1$, then $\alpha_{u\ldots v} = \mathrm{cKer}_{k,1}(\mathscr{E})$ as an abstract word, and there exists a number $x \in \mathbb{N}$ ($1 \le x < n$) such that $\alpha_{i\ldots j}$ is the concatenation of the suffix of γ of length x, the abstract word $\mathrm{cKer}_{k,1}(\mathscr{E})$, and the prefix of γ' of length $n - x - |\,\mathrm{cKer}_{k,1}(\mathscr{E})|$. So, there are at most $|\Sigma|^2(n-1)M$ possibilities for $\alpha_{i\ldots j}$ and at most $|\Sigma|^2(n-1)M$ possibilities for $\psi(\alpha_{i\ldots j})$ in this case.

If $m = 1$, but $\mathrm{ncker}_k(\mathscr{E}) > 1$, then $\alpha_{u\ldots v}$ can be written as $\alpha_{u\ldots v} = \mathrm{cKer}_{k,1}(\alpha_{u\ldots v})\,\mathrm{RpR}_{k,1}(\alpha_{u\ldots v})$, and $\psi(\mathrm{RBS}_k(\mathscr{E}))$ is a periodic sequence infinite to the right with period μ. $\psi(\mathrm{RpR}_{k,1}(\alpha_{u\ldots v}))$ is a weakly right μ-periodic word. As we have checked previously, $\alpha_{v+1\ldots j}$ is a prefix of $\mathrm{RBS}_k(\mathscr{E})$, so $\psi(\alpha_{u\ldots j})$ is a prefix of a word of the form $\psi(\mathrm{cKer}_{k,1}(\mathscr{E}))\delta$, where δ is the weakly $|\mu|$-periodic word of length n with left period μ', which is a cyclic shift of μ (and is a final period as well). And $\psi(\alpha_{i\ldots u-1})$ is the suffix of length $u-i$ ($0 < u-i < n$) of $\psi(\gamma)$. We get at most $|\Sigma|(n-1)MN$ possibilities for $\psi(\alpha_{i\ldots j})$.

The situation when $m = \mathrm{ncker}_k(\mathscr{E})$ and $\mathrm{ncker}_k(\mathscr{E}) > 1$ is symmetric to the situation when $m = 1$ and $\mathrm{ncker}_k(\mathscr{E}) > 1$, so it gives us at most $|\Sigma|(n-1)MN$ more possibilities for $\psi(\alpha_{i\ldots j})$.

Finally, if $1 < m < \mathrm{ncker}_k(\mathscr{E})$, then $\psi(\mathrm{LBS}_k(\mathscr{E}))$ is a periodic sequence infinite to the left with period λ, $\psi(\mathrm{RBS}_k(\mathscr{E}))$ is a periodic sequence infinite to the right with period μ, $\alpha_{i\ldots u-1}$ is a suffix of $\mathrm{LBS}_k(\mathscr{E})$, and $\alpha_{v+1\ldots j}$ is a prefix of $\mathrm{RBS}_k(\mathscr{E})$. Also, $\psi(\mathrm{LpR}_{k,m}(\alpha_{u\ldots v}))$ is a weakly left λ-periodic word, and $\psi(\mathrm{RpR}_{k,m}(\alpha_{u\ldots v}))$ is a weakly right μ-periodic word. Therefore, $\psi(\alpha_{i\ldots j})$ as an abstract word is a factor of a word of the form $\delta\psi(\mathrm{cKer}_{k,m}(\mathscr{E}))\delta'$, where δ is the weakly $|\lambda|$-periodic word of length n with right period λ', which is a cyclic shift of λ, and where δ' is the weakly $|\mu|$-periodic word of length n with left period μ', which is a cyclic shift of μ. λ' and μ' are final periods, $|\delta\psi(\mathrm{cKer}_{k,m}(\mathscr{E}))\delta'| \le 2n + P$, so we have at most $N^2M(n+P+1)$ possibilities for $\psi(\alpha_{i\ldots j})$.

Totally, in Case 2d we have at most $|\Sigma|^2(n-1)M + 2|\Sigma|(n-1)MN + N^2M(n+P+1)$ possibilities for $\psi(\alpha_{i\ldots j})$.

Summarizing, if $q = l_0$, then we have at most $|\Sigma|^2(n-1) + 2M(P+1)N + N^2M(n+P+1) + 2(2|\Sigma|(n-1)MN + N^2M(n+P+1)) + |\Sigma|^2(n-1)M + 2|\Sigma|(n-1)MN + N^2M(n+P+1)$ possibilities for $\psi(\alpha_{i\ldots j})$ as an abstract word. Since $|\Sigma|$, M, N, and P do not depend on n, this number is $O(n)$ for $n \to \infty$.

Now let us consider the case when $q < l_0$. Recall that in this case, $2 \le |\alpha_{s\ldots t}| \le |\varphi|$. Denote by w the index such that α_w is the rightmost letter in $\varphi(\alpha_s)$. Then $\alpha_{i\ldots j}$ is the concatenation of the suffix of $\varphi^q(\alpha_s)$ of length $x = |\alpha_{i\ldots w}| = w - i + 1 < n$ and the prefix of $\varphi^q(\alpha_{s+1\ldots t})$ of length $n - x$. So, $\alpha_{i\ldots j}$ as an abstract word is determined by the following data: a word of length at least two and at most $|\varphi|$, which will be $\alpha_{s\ldots t}$, and two numbers, $q \in \mathbb{Z}_{\ge 0}$ ($q \le \lceil f(n)\rceil + 1 \le f(n) + 2$) and $x \in \mathbb{N}$ ($1 \le x < n$). (This time we need to know q since it is not determined by n uniquely anymore.) There are at most $(|\Sigma|^2 + |\Sigma|^3 + \ldots + |\Sigma|^{|\varphi|})(f(n)+2)n$ possibilities for $\alpha_{i\ldots j}$, and hence at most $(|\Sigma|^2 + \ldots + |\Sigma|^{|\varphi|})(f(n)+2)n$ possibilities for $\psi(\alpha_{i\ldots j})$. $|\Sigma|$ and $|\varphi|$ do not depend on n, so this number is $O(n(f(n)+1))$ for $n \to \infty$.

Overall, we have at most $O(n) + O(n(f(n)+1)) = O(n(f(n)+1))$ possibilities for $\psi(\alpha_{i\ldots j})$ as an abstract word. Since $f(n) \ge 3k$ if $n \ge n_0$, we can say that the function $n \mapsto 1$ is also $O(f(n))$ for $n \to \infty$, and $O(n(f(n)+1)) = O(nf(n))$. ☐

Proof of Proposition 1.3. Consider the following sequences depending on $l \in \mathbb{Z}_{\geq 0}$. Previously we have seen that all of them have asymptotic $\Omega(l^k)$ for $l \to \infty$.

1. The sequences $|\operatorname{Fg}(\operatorname{LR}_k(\mathscr{E}_l))|$ (resp. $|\operatorname{Fg}(\operatorname{RR}_k(\mathscr{E}_l))|$) for all evolutions of k-blocks such that Case I holds for $\mathscr{E}$ at the left (resp. at the right) (see Lemma 5.15).

2. The sequences $|\varphi^l(b)|$, where $b \in \Sigma$ is a letter of order $> k$, including letters of order ∞ (see the definition of the order of a letter).

For the sequences $|\operatorname{Fg}(\operatorname{LR}_k(\mathscr{E}_l))|$ and $|\operatorname{Fg}(\operatorname{RR}_k(\mathscr{E}_l))|$ mentioned here, Lemma 5.15 actually says that the asymptotic of these sequences is $\Theta(l^k)$, and constants in the Θ-notation do not depend on $\mathscr{E}$. So, we may suppose that the constants in the Ω-notation do not depend on $\mathscr{E}$. As for the sequences $|\varphi^l(b)|$, where $b \in \Sigma$ is a letter of order $> k$, there are only finitely many of them since there are only finitely many letters in α, so we may also suppose that the constants in the Ω-notation do not depend on b. Therefore, there exist $l_0 \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}_{>0}$ such that for all $l \geq l_0$ the following is true:

1. If $\mathscr{E}$ is an evolution of k-blocks such that Case I holds for $\mathscr{E}$ at the left (resp. at the right), then $|\operatorname{Fg}(\operatorname{LR}_k(\mathscr{E}_l))| > xl^k$ (resp. $|\operatorname{Fg}(\operatorname{RR}_k(\mathscr{E}_l))| > xl^k$).

2. If $b \in \Sigma$ is a letter of order $> k$, then $|\varphi(b)| > xl^k$.

Without loss of generality, we will suppose that $l_0 \geq 3k$. Set $n_0 = \lceil x l_0^k \rceil$. Consider the following function $f\colon \mathbb{N} \to \mathbb{R}$: $f(n) = \sqrt[k]{n/x}$. If $n \geq n_0$, then $f(n) = \sqrt[k]{n/x} \geq \sqrt[k]{n_0/x} \geq \sqrt[k]{(xl_0^k)/x} = l_0 \geq 3k$. If $n \geq n_0$, $l \in \mathbb{N}$, and $l \geq f(n)$, then $l \geq \sqrt[k]{n/x}$, so $l^k \geq n/x$, $xl^k \geq n$, and, since $l \geq f(n) \geq l_0$, we have the following inequalities:

1. If $\mathscr{E}$ is an evolution of k-blocks such that Case I holds for $\mathscr{E}$ at the left (resp. at the right), then $|\operatorname{Fg}(\operatorname{LR}_k(\mathscr{E}_l))| > xl^k \geq n$ (resp. $|\operatorname{Fg}(\operatorname{RR}_k(\mathscr{E}_l))| > xl^k \geq n$).

2. If $b \in \Sigma$ is a letter of order $> k$, then $|\varphi(b)| > xl^k \geq n$.

Therefore, f satisfies the conditions of Lemma 7.2, and the factor complexity of $\beta = \psi(\alpha)$ is $O(nf(n)) = O(n^{1+1/k})$. □

Now we can check that if evolutions of k-blocks really exist in α and all of them are continuously periodic, then all k-series of obstacles $\mathscr{H}$ in α actually satisfy $|\mathscr{H}_l| = \Theta(l^k)$. (Actually, we do not need this fact to prove any subsequent lemmas, propositions or theorems.) Indeed, if $\alpha = \varphi^\infty(a)$ and k-blocks exist in α, then by Lemma 3.5, a is a letter of order $\geq k+2$. If, in addition, all evolutions of k-blocks are continuously periodic, then by Proposition 1.3, the factor complexity of $\beta = \psi(\alpha)$ is $O(n^{1+1/k})$. But we also have seen in the proof of Lemma 7.1 that if $\mathscr{H}$ is a k-series of obstacles such that $|\mathscr{H}_l|$ is $\Theta(l^{k'})$ for $l \to \infty$ and $k' < k$, then the factor complexity of β is $\Omega(n^{1+1/k'})$, so it is $\omega(n^{1+1/k})$, and we have a contradiction. Therefore, $|\mathscr{H}_l|$ is in fact $\Theta(l^k)$ for $l \to \infty$.

To prove Proposition 1.4, we first prove the following lemma.

Lemma 7.3. *Let $k \in \mathbb{N}$. Suppose that $\alpha = \varphi^\infty(a)$, where a is a letter of order $k+2$. Let $l \in \mathbb{Z}_{\geq 0}$. Let α_s (resp. α_i,α_j) be the rightmost letter of order $\geq k+1$ in $\varphi(a)$ (resp. in $\varphi^l(a)$, in $\varphi^{l+1}(a)$).*

Then $i < j$, $\alpha[<, i+1 \ldots j, <]_k$ and $\alpha[<, 1 \ldots s, <]_k$ are k-multiblocks, each of them begins with a (possibly empty) k-block and ends with a letter of order $k+1$, and $\alpha[<, i+1 \ldots j, <]_k = \operatorname{Dc}_k^l(\alpha[<, 1 \ldots s, <]_k)$.

Proof. First, note that the k-multiblocks in the statement of the lemma really exist and are nonempty. Indeed, Lemma 3.5 says in this case that the only letter of order $k+2$ in α is $\alpha_0 = a$, and all other letters in α are of order $\leq k+1$. If $\varphi(a) = a\gamma$, then for all $n \in \mathbb{N}$ we have $\varphi^n(a) = \varphi^{n-1}(a)\varphi^{n-1}(\gamma) = a\gamma\varphi(\gamma)\ldots\varphi^{n-1}(\gamma)$. Since $\varphi(a)$ contains a, a can only be a periodic letter of order $k+2$, not a preperiodic letter of order $k+2$. Then γ contains at least one letter of order $k+1$, and $\varphi^n(\gamma)$ contains at least one letter of order $k+1$ for all $n \in \mathbb{N}$. Hence, $\varphi^n(a)$ contains at least one letter of order $k+1$. Moreover, for all $n \in \mathbb{N}$, the rightmost letter of order $k+1$ in $\varphi^n(a)$ is the rightmost letter of order $k+1$ in $\varphi^{n-1}(\gamma)$ since $\varphi^{n-1}(\gamma)$ contains at least one letter of order $k+1$. Hence, $i < j$, $1 < s$.

Since α_0 and α_i are letters of order $\ge k+1$, there really exist (possibly empty) k-blocks of the form $\alpha_{i+1\dots j'}$ and $\alpha_{1\dots s'}$, so the notations $<, i+1$ and $<, 1$ really denote delimiters. Since α_j and α_s are also letters of order $\ge k+1$, $j, <$ and $s, <$ also denote delimiters, and the delimiter $j, <$ (resp. $s, <$) is located at the right-hand side of $<, i+1$ (resp. of $<, 1$). Therefore, $\alpha[<, i+1 \dots j, <]_k$ and $\alpha[<, 1 \dots s, <]_k$ are really nonempty k-multiblocks, and each of them begins with a (possibly empty) k-block and ends with a letter of order $k+1$.

If $l = 0$, then $i = 0$ and $j = s$, so the statement of the lemma is trivial. Otherwise, we prove the statement by induction on l. Let α_t be the rightmost letter of order $\ge k+1$ in $\varphi^{l+2}(a)$. We have to prove that $\alpha[<, j+1 \dots t, <]_k = \mathrm{Dc}_k(\alpha[<, i+1 \dots j, <]_k)$. α_t is also the rightmost letter of order $\ge k+1$ in $\varphi^{l+1}(\gamma)$, so, since α_j is the rightmost letter of order $\ge k+1$ in $\varphi^l(\gamma)$, and the image of a letter of order $\le k$ consists of letters of order $\le k$ only, α_t is also the rightmost letter of order $\ge k+1$ in $\varphi(\alpha_j)$. Similarly, α_j is the rightmost letter of order $\ge k+1$ in $\varphi(\alpha_i)$.

Therefore, $\mathrm{Dc}_k(\alpha[>, j \dots j, <]_k)$ is a k-multiblock that ends with the letter α_t of order $\ge k+1$. If $\alpha_{i+1\dots j'}$ is a k-block, then $\alpha_i = \mathrm{LB}(\alpha_{i+1\dots j'})$, and $\mathrm{LB}(\mathrm{Dc}_k(\alpha_{i+1\dots j'}))$ is the rightmost letter of order $> k$ in $\varphi(\alpha_i)$, i. e. $\mathrm{LB}(\mathrm{Dc}_k(\alpha_{i+1\dots j'})) = \alpha_j$. Hence, $\mathrm{Dc}_k(\alpha_{i+1\dots j'})$ is a k-block of the form $\alpha_{j+1\dots t'}$ for some $t' \ge j$, and, by the definition of the descendant of a k-multiblock, $\mathrm{Dc}_k(\alpha[<, i+1 \dots j, <]_k)$ is the k-multiblock that begins with the k-block $\alpha_{j+1\dots t'}$ and ends with the letter α_t of order $\ge k+1$. Therefore,

$$\alpha[<, j+1 \dots t, <]_k = \mathrm{Dc}_k(\alpha[<, i+1 \dots j, <]_k).$$

□

Proof of Proposition 1.4. For each $l \ge 0$, denote by s_l the index such that α_{s_l} is the rightmost letter of order $\ge k+1$ in $\varphi^l(a)$. By Lemma 7.3, $\alpha[<, s_l+1 \dots s_{l+1}, <]_k = \mathrm{Dc}_k^l(\alpha[<, 1 \dots s_1, <]_k)$ for all $l \ge 0$. By Lemma 3.5, the only letter of order $k+2$ in α is $\alpha_0 = a$, so all letters of order $\ge k+1$ in $\alpha[<, 1 \dots s_1, <]_k$ are actually of order $k+1$. All letters of order $\ge k+1$ in $\mathrm{Dc}_k^l(\alpha[<, 1 \dots s_1, <]_k)$ are contained in the images under φ^l of the letters of order $k+1$ from $\alpha[<, 1 \dots s_1, <]_k$, so, if $l \ge 1$, then all letters of order $\ge k+1$ in $\mathrm{Dc}_k^l(\alpha[<, 1 \dots s_1, <]_k) = \alpha[<, s_l+1 \dots s_{l+1}, <]_k$ are actually *periodic* letters of order $k+1$.

In particular, this is true for $l = 1$. Therefore, if $l \ge 1$, then all k-blocks in

$$\alpha[<, s_l+1 \dots s_{l+1}, <]_k = \mathrm{Dc}_k^l(\alpha[<, 1 \dots s_1, <]_k) = \mathrm{Dc}_k^{l-1}(\mathrm{Dc}_k(\alpha[<, 1 \dots s_1, <]_k)) = \mathrm{Dc}_k^{l-1}(\alpha[<, s_1+1 \dots s_2, <]_k)$$

are the $(l-1)$th superdescendants of the k-blocks in $\alpha[<, s_1+1, s_2, <]_k$. Hence, if $l \ge 1$, then the evolutional sequence number of each k-block contained in $\alpha[<, s_l+1 \dots s_{l+1}, <]_k$ is always at least $l-1$. So, if $l \ge 3k+1$, then all k-blocks contained in $\alpha[<, s_l+1 \dots s_{l+1}, <]_k$ are stable, and all letters of order $\ge k+1$ contained in $\alpha[<, s_l+1 \dots s_{l+1}, <]_k$ are periodic letters of order $k+1$. Therefore, if $l \ge 3k+1$, then $\alpha[<, s_l+1 \dots s_{l+1}, <]_k$ is a stable k-multiblock. It is nonempty by Lemma 7.3.

Consider the following evolution $\mathscr{F}$ of stable nonempty k-multiblocks:

$$\mathscr{F}_0 = \alpha[<, s_{3k+1}+1 \dots s_{3k+2}, <]_k,$$
$$\mathscr{F}_l = \mathrm{Dc}_k^l(\alpha[<, s_{3k+1}+1 \dots s_{3k+2}, <]_k) = \mathrm{Dc}_k^l(\mathrm{Dc}_k^{3k+1}(\alpha[<, 1 \dots s_1, <]_k)) = \mathrm{Dc}_k^{l+3k+1}(\alpha[<, 1 \dots s_1, <]_k) = \alpha[<, s_{l+3k+1}+1 \dots s_{l+3k+2}, <]_k \text{ for } l \ge 0.$$

Then we can write

$$\alpha = \alpha_{0\dots s_{3k+1}} \alpha_{s_{3k+1}+1\dots s_{3k+2}} \alpha_{s_{3k+2}+1\dots s_{3k+3}} \dots \alpha_{s_{l+3k+1}+1\dots s_{l+3k+2}} \dots = \alpha_{0\dots s_{3k+1}} \mathrm{Fg}(\mathscr{F}_0)\, \mathrm{Fg}(\mathscr{F}_1) \dots \mathrm{Fg}(\mathscr{F}_l) \dots.$$

Note that $\mathrm{Fg}(\mathscr{F}_l)$ contains a letter (namely, the rightmost letter of order $\geq k+1$ in $\varphi^{l+3k+2}(a)$, which we denote by $\alpha_{s_{l+3k+2}}$), which does not belong to $\varphi^{l+3k+1}(a)$ by Lemma 7.3. Hence, $|\alpha_{0\dots s_{3k+1}}\mathrm{Fg}(\mathscr{F}_0)\mathrm{Fg}(\mathscr{F}_1)\dots\mathrm{Fg}(\mathscr{F}_l)| > |\varphi^{l+3k+1}(a)|$, and the infinite concatenation $\alpha_{0\dots s_{3k+1}}\mathrm{Fg}(\mathscr{F}_0)\mathrm{Fg}(\mathscr{F}_1)\dots\mathrm{Fg}(\mathscr{F}_l)\dots$ is really an infinite word, and it covers the whole α.

First, let us check that $\mathrm{nker}_k(\mathscr{F}) > 1$. Indeed, otherwise $\mathrm{Fg}(\mathscr{F}_l) = \mathrm{Ker}_{k,1}(\mathscr{F}_l)$ for all $l \geq 0$, and by Lemma 5.27, all words $\mathrm{Fg}(\mathscr{F}_l)$ coincide as abstract words. But then $|\alpha_{0\dots s_{3k+1}}\mathrm{Fg}(\mathscr{F}_0)\dots\mathrm{Fg}(\mathscr{F}_l)|$ is $O(l)$ for $l\to\infty$. On the other hand, we know that $|\alpha_{0\dots s_{3k+1}}\mathrm{Fg}(\mathscr{F}_0)\dots\mathrm{Fg}(\mathscr{F}_l)| > |\varphi^{l+3k+1}(a)|$, and a is a letter of order $k+2\geq 3$ since $k\in\mathbb{N}$, so $|\varphi^{l+3k+1}(a)|$ is $\Theta(l^{k-1})$ for $l\to\infty$, and we have a contradiction.

So, $\mathrm{nker}_k(\mathscr{F}) > 1$. Note that the indices α_i and α_j from the statement of the proposition can now be written as $i = s_{3k+1}$ and $j = s_{3k+2}$, so

$$\mathrm{Fg}(\mathscr{F}_0) = \mathrm{Fg}(\alpha[<, s_{3k+1}+1\dots s_{3k+2}, <]_k) = \alpha_{s_{3k+1}+1\dots s_{3k+2}} = \alpha_{i+1\dots j}.$$

First, let us consider the case when there exists a final period λ such that $\psi(\alpha_{i+1\dots j}) = \psi(\mathrm{Fg}(\mathscr{F}_0))$ is a completely λ-periodic word. By Lemma 6.42, in this case λ is a total left evolutional period of $\mathscr{F}$. By the definition of a total left evolutional period, the remainder of $|\mathscr{F}_l|$ modulo $|\lambda|$ does not depend on l, so it equals the remainder of $|\mathscr{F}_0|$ modulo $|\lambda|$, which is zero. Therefore, all words $\psi(\mathrm{Fg}(\mathscr{F}_l))$ are completely λ-periodic words, and $\beta = \psi(\alpha) = \psi(\alpha_{0\dots s_{3k+1}})\psi(\mathrm{Fg}(\mathscr{F}_0))\dots\psi(\mathrm{Fg}(\mathscr{F}_l))\dots$ is an eventually periodic word with period λ, and its factor complexity is $O(1)$.

Now suppose that there exist no final period λ such that $\psi(\alpha_{i+1\dots j})$ is a completely λ-periodic word. Since all evolutions of k-blocks present in α are continuously periodic, Proposition 1.3 guarantees that the factor complexity of β is $O(n^{1+1/k})$. We are going to prove that there exists a k-series of obstacles in α. Assume the contrary.

Consider the following evolutions of k-multiblocks: $\mathscr{F}'$ defined by $\mathscr{F}'_l = \mathscr{F}_{l+1}$, and $\mathscr{F}''$ defined by $\mathscr{F}''_l = \mathscr{F}_{l+2}$. Each k-multiblock $\mathscr{F}'_l = \mathscr{F}_{l+1}$ begins with a k-block, so $\mathscr{F}_l$ and $\mathscr{F}'_l$ are consecutive as k-multiblocks (there cannot be an empty k-block between them). So, the evolutions $\mathscr{F}$ and $\mathscr{F}'$ are consecutive. Similarly, $\mathscr{F}'$ and $\mathscr{F}''$ are consecutive. By Lemma 6.40, at least one of the evolutions $\mathscr{F}$, $\mathscr{F}'$, and $\mathscr{F}''$ is totally periodic. Hence, there exists a final period λ such that at least one of the words $\psi(\mathrm{Fg}(\mathscr{F}_0))$, $\psi(\mathrm{Fg}(\mathscr{F}'_0)) = \psi(\mathrm{Fg}(\mathscr{F}_1))$, or $\psi(\mathrm{Fg}(\mathscr{F}''_0)) = \psi(\mathrm{Fg}(\mathscr{F}_2))$ is a weakly left λ-periodic word.

By Lemma 6.42, this means that λ is a total left evolutional period of $\mathscr{F}$. Now it follows directly form the definition of a total left evolutional period that λ is also a total left evolutional period of $\mathscr{F}'$ and of $\mathscr{F}''$. Then Lemma 6.41 implies that λ is either a right total evolutional period of $\mathscr{F}$, or a right total evolutional period of $\mathscr{F}'$. In particular, at least one of the words $\psi(\mathrm{Fg}(\mathscr{F}_0))$ or $\psi(\mathrm{Fg}(\mathscr{F}'_0)) = \psi(\mathrm{Fg}(\mathscr{F}_1))$ is a weakly right λ-periodic word. By Lemma 6.43, λ is a total right evolutional period of $\mathscr{F}$.

Now we know that λ is both left and right total evolutional period of $\mathscr{F}$. In particular, $\psi(\mathrm{Fg}(\mathscr{F}_0))$ is weakly $|\lambda|$-periodic word with both left and right period λ. Since $\mathrm{nker}_k(\mathscr{F}) > 1$, by Corollary 5.30, $|\mathrm{Fg}(\mathscr{F}_0)| \geq 2\mathbf{L}$. So, by Lemma 6.50, $\psi(\mathrm{Fg}(\mathscr{F}_0)) = \psi(\alpha_{i+1\dots j})$ is a completely λ-periodic word, and we have a contradiction (we are considering the case when there exist no final period λ such that $\psi(\alpha_{i+1\dots j})$ is a completely λ-periodic word).

Therefore, there exists a k-series of obstacles in α, and, by Lemma 7.1, the factor complexity of $\beta = \psi(\alpha)$ is $\Omega(n^{1+1/k})$. We already know that the factor complexity of β is $O(n^{1+1/k})$, so the factor complexity of β is $\Theta(n^{1+1/k})$. □

Proof of Proposition 1.5. Write $\varphi(a) = a\gamma$. Then for all $l\in\mathbb{N}$ we have $\varphi^l(a) = \varphi^{l-1}(a)\varphi^{l-1}(\gamma) = a\gamma\varphi(\gamma)\dots\varphi^{l-1}(\gamma)$ and $\alpha = \varphi^\infty(a) = a\gamma\varphi(\gamma)\dots\varphi^l(\gamma)\dots$. Since a is contained in $\varphi(a)$, a must be a periodic letter of order 2, not a preperiodic letter of order 2. Then all letters in γ have order 1. Since we have assumed that φ is (in particular) weakly 1-periodic morphism, $\varphi(\gamma)$ consists of periodic letters of order 1 only, and $\varphi^2(\gamma) = \varphi(\gamma)$. Therefore, $\alpha = a\gamma\varphi(\gamma)\varphi(\gamma)\dots\varphi(\gamma)\dots$, α is an eventually periodic sequence with period $\varphi(\gamma)$, so $\beta = \psi(\alpha)$ is an eventually periodic sequence with period $\psi(\varphi(\gamma))$, and the factor complexity of β is $O(1)$. □

Proof of Proposition 1.6. By Lemma 3.5, α can be split into a concatenation of k-blocks and letters of order $> k$ (i. e. letters of order ∞ in this case). In particular, k-blocks still exist (although it is possible that all of them are empty occurrences), but Case II must hold for all evolutions of k-blocks since there are no letters of order k.

We are going to use Lemma 7.2. Recall that if b is a letter of order ∞, then there exists $q \in \mathbb{R}$, $q > 1$ such that $|\varphi^l(b)|$ is $\Omega(q^l)$ for $l \to \infty$. Let q_0 be the minimal number q for all letters of order ∞. Since there are finitely many letters in Σ, there exists $l_0 \in \mathbb{N}$ and $x \in \mathbb{R}_{>0}$ such that if $l \geq l_0$ and b is a letter of order ∞, then $|\varphi^l(b)| > xq_0^l$. Without loss of generality, $l_0 \geq 3k$. Set $n_0 = \lceil xq_0^{l_0} \rceil$, and consider the following function $f\colon \mathbb{N} \to \mathbb{R}$: $f(n) = \log_{q_0}(n/x)$. If $n \geq n_0$, then $f(n) = \log_{q_0}(n/x) \geq \log_{q_0}(n_0/x) \geq \log_{q_0}((xq_0^{l_0})/x) = \log_{q_0}(q_0^{l_0}) = l_0 \geq 3k$, so the first condition in Lemma 7.2 is satisfied. We do not have to check the second condition in Lemma 7.2 since Case II holds both at the left and at the right for all evolutions of $k+1$-blocks in α. For the third condition we observe that if $l, n \in \mathbb{N}$, $n \geq n_0$, $l \geq f(n)$, and $b \in \Sigma$ is a letter of order $> k+1$, then b is a letter of order ∞, $l \geq l_0$, so $|\varphi^l(b)| > xq_0^l$. We also have $l \geq \log_{q_0}(n/x)$, so $q_0^l \geq n/x$, $xq_0^l \geq n$, and $|\varphi^l(b)| > n$. Therefore, we can use Lemma 7.2. By Lemma 7.2, the factor complexity of $\beta = \psi(\alpha)$ is $O(n \log_{q_0}(n/x)) = O(n \log_{q_0} n - n \log_{q_0} x) = O(n \log n)$. $\square$

Proof of Theorem 1.1. Let $a \in \Sigma$ be a letter such that $\varphi(a) = a\gamma$ for a nonempty word γ, and let $\alpha = \varphi^\infty(a)$. Consider the following three cases: a can be either a letter of order 2, or a letter of finite order $K > 2$, or a letter of order ∞.

If a is a letter of order 2, then the factor complexity of $\beta = \psi(\alpha)$ is $O(1)$ by Proposition 1.5.

If a is a letter of a finite order $K > 2$, then denote by $k \in \mathbb{N}$ the maximal number among the numbers $1, 2, \ldots, K-2$ such that all evolutions of k-blocks arising in α are continuously periodic. (Recall that all evolutions of 1-blocks are always continuously periodic, see Remark 6.20.) By Proposition 1.3, the factor complexity of $\beta = \psi(\alpha)$ is $O(n^{1+1/k})$. If $k < K-2$, then by Lemma 3.5, α can be split into a concatenation of $(k+1)$-blocks and letters of order $> k+1$, so there exist evolutions of $(k+1)$-blocks in α, and there exists a non-continuously periodic evolution of $(k+1)$-blocks in α. So, by Proposition 1.2, the factor complexity of β is $\Omega(n^{1+1/k})$, therefore it is $\Theta(n^{1+1/k})$. If $k = K-2$, then by Proposition 1.4, the factor complexity of β is either $\Theta(n^{1+1/k})$, or it is $O(1)$.

Finally, suppose that a is a letter of order ∞. Let K be the maximal *finite* order of letters occurring in α (i. e. all letters occurring in α are either of order $\leq K$, or of order ∞). Denote by $k \in \mathbb{N}$ the maximal number among the numbers $1, 2, \ldots, K+1$ such that all evolutions of k-blocks in α are continuously periodic. Again, by Proposition 1.3, the factor complexity of $\beta = \psi(\alpha)$ is $O(n^{1+1/k})$. If $k < K+1$, then there exists an evolution $\mathscr{E}$ of $k+1$-blocks such that $\mathscr{E}$ is not continuously periodic. By Proposition 1.2, the factor complexity of β is $\Omega(n^{1+1/k})$. Therefore, the factor complexity of β is $\Theta(n^{1+1/k})$. If $k = K+1$, then by Proposition 1.6, the factor complexity of β is $O(n \log n)$. $\square$

8 An Example

Here we give an example of a sequence with complexity $\Theta(n^{3/2})$. The easiest way to construct such a sequence is to use Proposition 1.4.

Let $\Sigma = \{1, 2, 3, 4\}$. Consider the following morphism φ: $\varphi(4) = 43, \varphi(3) = 32, \varphi(2) = 21, \varphi(1) = 1$ and the following coding ψ: $\psi(4) = 4, \psi(3) = 3, \psi(2) = \psi(1) = 1$. Here all letters are periodic, 4 is a letter of order 4, 3 is a letter of order 3, 2 is a letter of order 2, and 1 is a letter of order 1. φ is a weakly 1-periodic morphism, but it is not a strongly 1-periodic morphism. However, for φ^4 we have $\varphi^4(4) = 433232213221211$, $\varphi^4(3) = 32212112111$, $\varphi^4(2) = 21111$, and $\varphi^4(1) = 1$, so φ^4 is a strongly 1-periodic morphism. The only final period we have here consists of a single letter 1, so $\mathbf{L} = 1$, and φ^4 is also a strongly 1-periodic morphism with long images.

To construct the pure morphic sequence, we can use φ as well as φ^4, the resulting sequence will be the same.

So, we have $\varphi^\infty(4) = \alpha = 4332322132212113221211211113\ldots$, and $\psi(\varphi^\infty(4)) = \beta = 4331311131111113111111111113\ldots$ Here we have several evolutions of 2-blocks (when we speak about evolutions of 2-blocks, we should use the morphism φ^4, not just φ, to compute the descendants since when we defined descendants, we assumed that φ is a strongly 1-periodic morphism with long images), and it is clear that all left borders and all right borders of all these evolutions equal 3 as abstract letters. Therefore, Case I holds for all evolutions of 2-blocks at the left. The images under ψ of all 2-blocks here consist of letters 1 only, so each evolution $\mathscr{E}$ of 2-blocks here is continuously periodic for the index $\mathrm{nker}_2(\mathscr{E})$. Therefore, we can apply Proposition 1.4 and say that the factor complexity is $\Theta(n^{3/2})$.

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