Boundness of b_2 for hyperkähler manifolds with vanishing odd-Betti numbers

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ABSTRACT. We prove that b_2 is bounded for hyperkähler manifolds with vanishing odd-Betti numbers and condition on $\mathfrak{so}(b_2+2)$-modules. The explicit upper boundary is conjectured.

I also tried to follow the method described by Sawon to prove that b_2 is bounded in dimension eight and ten in the case of vanishing odd-Betti numbers by 24 and 25 respectively. These results are incomplete since some representations are missing.

1 Introduction

A Riemannian manifold (M, g) is called hyperkähler if it admits a triple of a complex structures I, J, K satisfying quaternionic relations and Kähler with a respect to g. A hyperkähler manifold is always holomorphically symplectic. By the Yau Theorem [Y], a hyperkähler structure exists on a compact complex manifold if and only if it is Kähler and holomorphically symplectic.

Definition 1.1: A compact hyperkähler manifold M is called *simple* if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

By Bogomolov decomposition theorem [B] any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds. Huybrechts [H] has proved finiteness of the number of deformation classes of holomorphic symplectic structures on each smooth manifold. In most dimensions we only know two examples of simple hyperkähler manifolds due to Beauville [Bea]. This two infinite series are the Hilbert schemes of $K3$ and the generalized Kummer varieties [Bea]. Except them two sporadic examples of O'Grady in dimension six and ten are known [O1, O2]. It is known [H] that there are only finitely many families with a given second cohomology lattice, hence the bounds on second Betti number

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provide restrictions on the number of deformation classes of hyperkähler manifolds. There is Beauville Conjecture [Bea1]

Conjecture 1.2: The number of deformation types of compact irreducible hyperkähler manifolds is finite in any dimension (at least for given b_2).

In the complex dimension four, Guan [G] proved that the second Betti number of a simple compact hyperkähler manifold of dimension four is bounded above by 23. Guan also have proved boundary conditions for b_3 using Rozansky-Witten invariants [HS]. In dimension six inequality from Rozansky-Witten invariants could also be obtained in dimension six [K].

In dimension six the boundary conditions for b_2 has been obtained by Sawon [S]. In this paper we generalize Sawon's result on Hodge diamond structure for dimensions eight and ten.

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2 Preliminaries

In this Section we recall main properties of cohomology groups of hyperkähler manifolds.

There is the following theorem of Verbitsky [V1]

Theorem 2.1: Let M be an irreducible hyperkähler manifold of complex dimension a $2n$ and let $SH^2(M,\mathbb{C})\subset H^*(M,\mathbb{C})$ be the subalgebra generated by $H^2(M,\mathbb{C})$. Then $SH^2(M,\mathbb{C}) = S^*H^2(M,\mathbb{C})/\left\langle \alpha^{n+1} \middle| \, q(\alpha) = 0 \right\rangle$, where q is a Beaville-Bogomolov-Fujiki–form.

The inclusion $S^nH^2(M) \hookrightarrow H^{2n}$ follows from this Theorem.

It is known that in a hyperkähler case there is an action of $\mathfrak{so}(5)$-algebra [V1], and moreover an action of $\mathfrak{so}(b_2+2,\mathbb{C})$ described by Looijenga and Lunts [LL], and Verbitsky [V2]. By this results one can decompose $\mathrm{H}^{even}(\mathrm{M},\mathbb{C})$ into irreducible representations for this $\mathfrak{so}(b_2+2,\mathbb{C})$-action. The

Hodge diamond is the projection onto a plane of the (higher-dimensional) weight lattice of $\mathfrak{so}(b_2 + 2,\mathbb{C})$ so we can figure out the highest weights which lie in octant of Hodge diamond.

These Lie algebra has type either B_n or D_n, and it is known that all spinor representations belongs to odd dimensional cohomology. Moreover, from Weyl character formula it is easily follows that dimension is polynomial in terms of b and contributions to each even cohomology groups have dimension polynomial in terms of b_2. Each representation is determined by highest weight which is not for necessary sitting in primitive cohomology as it was written in [K1] and Sawon's paper [S]. There are labeled by the Young diagrams. The gap in proof was found by Collen Robles to Sawon, and F. Bogomolov for my work, I am highly thankful for Fedor for his suggestions and all improves of this work.

In early 90-s Salamon [Sa] has used hyperkähler symmetries of Hodge numbers and expression of Euler characteristic to prove the following equation with Betti numbers:

$$nb_{2n} = 2\sum_{i=1}^{2n} (-1)^i \left(3i^2 - n\right) b_{2n-i}.$$

3 Boundary conditions for b_2

In a original paper [K1] we have tried to prove that the second Betti number is bounded if dimension is fixed. Unfortunately, there was a gap in a original idea, i.e. we have assumed that representations are generated by single elements of th Hodge diamond, that is not always the case. We have found a condition which seems to be a good one, i.e finite number of $\mathfrak{so}(b_2 + 2)$-representations for a given n, where $2n$ is a dimension of hyperkähler manifold.

In this Section we prove the following

Theorem 3.1: Let M be a hyperkähler manifold of complex dimension $2n$ with vanishing odd-Betti numbers, and number of irreducible representations which may appear in $H^*(M)$ is finite. Then the second Betti number is bounded.

Proof: Denote a number of possible $\mathfrak{so}(b_2 + 2)$-representations for hyperkähler manifold in dimension n as m. Consider the $2k$-th Betti number. There is inclusion of $\mathrm{Sym}^k(H^2)$ into $H^{2k}(M)$. Thus, its contribution is polynomial of degree k. And also there are contributions of other irreducible $\mathfrak{so}(b_2+2)$-modules into $H^{2k}(M)$. The dimension of each of them is polynomial in terms of b_2. Label the representation which is generated by H^2 as $F(H^2)$.

Then for all even Betti numbers one could write that

$$b_{2k} = \frac{1}{(k-1)!} \prod_{i=0}^{i=k-1} (b_2+i) + P_{k,m}(b_2),$$

where the first term correspond to the dimension of $F(H^2)$, and the second one – to all contributions of irreducible $\mathfrak{so}(b_2+2,\mathbb{C})$-modules into $H^{2k}(M)$.

From Salamon's equation

$$nb_{2n} = 2\sum_{i=1}^{2n} (-1)^i \left(3i^2-n\right) b_{2n-i}.$$

using vanishing of odd-Betti numbers we obtain the following

$$\frac{1}{(n-1)!} \prod_{i=0}^{i=n-1} (b_2+i) + P_{n,m}(b_2) =$$
$$= \sum_{j=1, j \text{ even}}^{j=2n} \left[\left(3j^2-n\right) \frac{1}{(n-j/2)!} \prod_{i=0}^{i=n-j/2-1} (b_2+i) + P_{j,m}(b_2) \right].$$

Rearrangering terms in equation above (we put terms corresponding to contributions of symmetric powers of H^2 (i.e. $F(H^2)$) on the left-hand side and all others on the right-hand side):

$$-\frac{1}{(n-1)!} \prod_{i=0}^{i=n-1} (b_2+i) + \sum_{j=1, j \text{ even}}^{j=2n} \left(3j^2-n\right) \frac{1}{(n-j/2)!} \prod_{i=0}^{i=n-j/2-1} (b_2+i) =$$
$$= P_{2n,m}(b_2) - \sum_{j=1, j \text{ even}}^{j=2n-1} P_{j,m}(b_2).$$

Denote by b_2^l the maximal root (in terms of b_2) of polynomial $P(b_2,n)$ on the left-hand side. Then the polynomial on the left-hand side is negative than $b_2 \geqslant b_2^l$. This polynomial obviously has some positive roots since it is positive for $b_2 = 0$.

The polynomial on the right-hand side consists of m polynomials for each irreducible representation, and the the leading coefficient foe all of them is positive. Thus if we denote by b_2^r the maximum of roots of polynomials for each irreducible representation, then he overall polynomial on the right-hand side is positive then $b_2 \geqslant b_2^r$.

Hence we get $b_2 \leqslant \max\left[b_2^l, b_2^r\right]$. ■

Conjecture 3.2: The second Betti number b_2 is bounded by the maximal root of the following polynomial

$$P\left(b_2, n\right) = -\frac{1}{(n-1)!}\prod_{i=0}^{i=n-1}\left(b_2+i\right) + \sum_{j=1, j \text{ even}}^{j=2n}\left(3j^2-n\right)\frac{1}{(n-j/2)!}\prod_{i=0}^{i=n-j/2-1}\left(b_2+i\right),$$

which is denoted as b_2^l.

To prove this conjecture one need to check that $Q_n\left(b_2, c, d, e, f, \ldots\right)$ is positive than $b_2 \geq b_2^l$. Author does not know the explicit proof of this Conjecture 3.2.

Proposition 3.3: In conditions of Theorem 3.1 we have

$$P\left(b_2, n\right) = -\frac{1}{n!}\left(\prod_{i=3}^{i=n-1}\left(b_2+i\right)\right)\cdot\left(b_2+2n\right)\cdot\left(b_2^2-21b_2+2-96n\right),$$

and

$$b_2^l = \frac{21+\sqrt{433+96n}}{2}.$$

Proof:

We can see that last four terms of $P\left(b_2, n\right)$ are divided by $\left(b_2+3\right)$ and factor is

$$\frac{1}{3}((12n^2-73n+108)b_2^2+3(12n^2-49n+48)b_2+12n(n-1)$$

Then, we can claim by induction (we omit explicit calculations) that sum of last k terms of $P\left(b_2, n\right)$ is

$$2\frac{1}{(k-1)!}\left(\prod_{i=3}^{i=k-1}\left(b_2+i\right)\right)\cdot\left(A_k\cdot b_2^2+3B_k\cdot b_2+12n(n-1)\right),$$

where $A_k = (12n^2 - (73 + 24(k-4))n + 108 + 60k + 24\frac{(k-4)(k-3)}{2})$, and $B_k = (12n^2 - (49 + 16k)n + 48 + 24k + 8\frac{(k-4)(k-3)}{2})$.

Thus, we can write our polynomial $P(b_2, n)$ in the following form

$$P(b_2, n) = \frac{1}{(n-1)!}\prod_{i=0}^{i=n-1}(b_2+i) + 2\frac{1}{(n-1)!}\left(\prod_{i=3}^{i=n-1}(b_2+i)\right)\cdot(A_n \cdot b_2^2 + 3B_n \cdot b_2 + 12n(n-1)).$$

After rearranging of common factors we get the statement. ■

4 Dimension eight and ten

To find explicit boundary condition we shall use the method described by Justin Sawon [S]. In his work he has studied how different $\mathfrak{so}(b_2+2,\mathbb{C})$-modules sit inside Hodge diamond. Unfortunately arguments contain the same gap as the Sawon's proof.

Theorem 4.1: Let M be a eight-dimensional hyperkähler manifold with $H^{2k+1}(M,\mathbb{C}) = 0$. Then $b_2 \leqslant 24$.

Proof:

We will write some explicit splitting of b_4, b_6, and b_8 in terms of b_2 and generators of primitives part of cohomologies.

There is an action of $\mathfrak{so}(b_2+2,\mathbb{C})$ on the complex cohomology of M. Under this action, an element of $H^{3,1}_{\mathrm{pr}}(M)$ will generate an irreducible $\mathfrak{so}(b_2+2,\mathbb{C})$-module of dimension $\frac{(b_2+2)(b_2+1)b_2}{6}$. In Hodge diamond this part sits as

$$\begin{array}{c}
0\\
0\qquad 0\\
0\qquad 0\qquad 0\\
0\qquad 0\qquad 0\qquad 0\\
0\qquad 1\qquad h\qquad 1\qquad 0\\
0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\\
0\qquad 1\qquad h\qquad \frac{h^2-h+4}{2}\qquad h\qquad 1\qquad 0\\
0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\\
0\qquad 0\qquad \mathrm{h}\qquad \frac{h^2-h+4}{2}\qquad \frac{h^3-3h^2-10h-60}{6}\qquad \frac{h^2-h+4}{2}\qquad h\qquad 0\qquad 0\\
0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\\
0\qquad 1\qquad h\qquad \frac{h^2-h+4}{2}\qquad h\qquad 1\qquad 0\\
0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\\
0\qquad 1\qquad h\qquad 1\qquad 0\\
0\qquad 0\qquad 0\qquad 0\\
0\qquad 0\qquad 0\\
0\qquad 0\\
0
\end{array}$$

In the same way we have an irreducible $\mathfrak{so}(b_2+2,\mathbb{C})$-module generated by additional elements of $H^{2,2}_{\mathrm{pr}}$, which sits inside the Hodge diamond as

$$\begin{array}{c}
0\\
0\qquad 0\\
0\qquad 0\qquad 0\\
0\qquad 0\qquad 0\qquad 0\\
0\qquad 0\qquad 1\qquad 0\qquad 0\\
0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\\
0\qquad 0\qquad 1\qquad h\qquad 1\qquad 0\qquad 0\\
0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\\
0\qquad 0\qquad 1\qquad h\qquad \frac{h^2-h-4}{2}\qquad h\qquad 1\qquad 0\qquad 0\\
0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\\
0\qquad 0\qquad 1\qquad h\qquad 1\qquad 0\qquad 0\\
0\qquad 0\qquad 0\qquad 0\qquad 0\qquad 0\\
0\qquad 0\qquad 1\qquad 0\qquad 0\\
0\qquad 0\qquad 0\qquad 0\\
0\qquad 0\qquad 0\\
0\qquad 0\\
0
\end{array}$$

Recall that $b_4 = \frac{(b_2+1)b_2}{2} + cb_2 + d$, where d – part of $H^{2,2}_{\mathrm{pr}}$ that does not come from $H^{3,1}_{\mathrm{pr}}$

So the first module gives

$$c\left(\frac{h^3+3h^2-4h-36}{6}\right)=c\left(\frac{b_2^3-3b_2^2-4b_2-24}{6}\right),$$

the second one – $d\left(\frac{h^2-h-4}{2}+2h+2\right)=d\left(\frac{h^2+3h}{2}\right)=d\left(\frac{b_2^2-b_2-2}{2}\right)$.

Definitely, in H^8 there are elements which come from those part of H^6 which is not generated by $\operatorname{Sym}^3 H^2$ and two $\mathfrak{so}\left(b_2+2,\mathbb{C}\right)$-modules generated by elements of $H^{3,1}_{\mathrm{pr}}$ and $H^{2,2}_{\mathrm{pr}}$.

$$b_6=\frac{(b_2+2)(b_2+1)b_2}{6}+c\left(\frac{b_2^2-b_2+2}{2}\right)+db_2+e$$

Then each element of $H^{3,3}$ part generate $\mathfrak{so}\left(b_2+2,\mathbb{C}\right)$-module of dimension (b_2+2). That gives in b_8 the following term $e\left(b_2\right)$

$$b_8\geqslant\frac{(b_2+3)(b_2+2)(b_2+1)b_2}{24}+c\left(\frac{b_2^3-3b_2^2-4b_2-24}{6}\right)+d\left(\frac{b_2^2-b_2-2}{2}\right)+eb_2$$

From Salamon's relation we have

$$8\cdot\frac{(b_2+2)(b_2+1)b_2}{6}+8c\left(\frac{b_2^2-b_2+2}{2}\right)+8db_2+8e+44\left(\frac{(b_2+1)b_2}{2}+cb_2+d\right)+$$
$$+104b_2+188+b_7-71b_3-23b_5\geqslant$$
$$\geqslant 2\frac{(b_2+3)(b_2+2)(b_2+1)b_2}{24}+2c\left(\frac{b_2^3-3b_2^2-4b_2-24}{6}\right)+2d\left(\frac{b_2^2-b_2-2}{2}\right)+2eb_2$$

Then after rearranging we get

$$-2\frac{(b_2+3)(b_2+2)(b_2+1)b_2}{24}+8\cdot\frac{(b_2+2)(b_2+1)b_2}{6}+44\left(\frac{(b_2+1)b_2}{2}\right)+104b_2+188+b_7\geqslant$$
$$\geqslant c\left(\frac{b_2^3-15b_2^2-124b_2-48}{3}\right)+d\left(b_2^2-9b_2-46\right)+2e\left(b_2-4\right)+71b_3+23b_5$$

The left-part is $\frac{-b_2^4+10b_2^3+301b_2^2+530b_2+2256}{12}+b_7$.

Now recall that all odd Betti numbers are zero. Then the left-hand side is negative for $b_2\geqslant 24$, although the right-hand side is positive. ■

Remark: The second Betti number b_2 is at most 24 for a sufficiently small b_7. Indeed, for the proof of Theorem 4.1 we use the fact that

$$F\left(b_2\right)+b_7 := \frac{-2b_2^4+20b_2^3+602b_2^2+1060b_2+4512}{24}+b_7$$

is negative than $b_2 \geqslant 25$. This is also true for $b_7 \leqslant \left|F\left(b_2\right)|_{b_2=25}\right| = 1281$.

Theorem 4.2: Let M be a ten-dimensional hyperkähler manifold with $H^{2k+1}\left(M,\mathbb{C}\right)=0$. Then $b_2 \leqslant 25$.

Proof: The proof is very similar to the previous one. We could see that they are contributions from $H_{\mathrm{pr}}^{3,1}, H_{\mathrm{pr}}^{2,2}, H_{\mathrm{pr}}^{3,3}, H_{\mathrm{pr}}^{4,4}$. They are isomorphic to $\bigwedge^4 \mathbb{C}^{b_2+2}, \bigwedge^3 \mathbb{C}^{b_2+2}, \bigwedge^2 \mathbb{C}^{b_2+2}$, and $\mathbb{C}^{b_2+2}$ as irreducible $\mathfrak{so}\left(b_2+2,\mathbb{C}\right)-$ modules.

The explicit calculations like in the case of dimension eight give us the following

$$\frac{1}{60}(b_2+3)(b_2+4)(b_2+10)(-b_2^2-21b_2+118) = Q\left(b_2,c,d,e,f,g\right),$$

Polynomial Q(b_2) is positive for $b_2 \geqslant 26$ since c,d,e,f,g are positive constants defined above (f sits in $H^{4,4}$-part, g generates $H^{5,5}$). The left-hand side is negative for $b_2 \geqslant 26$. Hence, $b_2 \leqslant 25$. ■

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