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outline of course

1. Legendre families, period map
2. Hodge structure of curves and abelian varieties
3. Hodge decomposition, Kähler manifolds, mixed Hodge structures
4. Hodge structures for hypersurfaces, K3, Kuga-Satake, period domains
5. Deformation theory intro and variations of the Hodge structure, p-adic Hodge structures

Last time

- Elliptic curves = genus 1 Riemann surfaces, parametrized by $\lambda \in \mathbb{C} \setminus \{0, 1\}$
- $H^1(E_\lambda) = \mathbb{C}[\omega = \frac{dx}{y}] \oplus \mathbb{C}[\overline{\omega}]$ - Hodge structure of weight one
- local period map $P : \mathbb{P}^1 \setminus \{0, 1, \infty\} \to \mathbb{H}$ given by ratio of periods (integrals of ω on the basis of cycles)
- monodromy representation $\rho : \pi_1((\mathbb{P}^1 \setminus \{0, 1, \infty\}), \lambda_0) \to Sl_2(\mathbb{Z})$ (bc we can change the basis)
- global period map $\tilde{P} : \mathbb{P}^1 \setminus \{0, 1, \infty\} \to img\, \rho \backslash \mathbb{H}$

Hodge structure

Definition

A real (rational, integer) Hodge structure of weight k is a real vector space $H_{\mathbb{R}}$ ($H_{\mathbb{Q}}$, free $\mathbb{Z}$-module $H_{\mathbb{Z}}$) together with a decomposition:

$$H_{\mathbb{C}} := H_{\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

for $\mathbb{C}$-subspaces $H^{p,q} \subset H_{\mathbb{C}}$ st $H^{p,q} = \overline{H}^{q,p}$

Definition

A Hodge structure is effective if $H^{p,q} = 0$ unless $p, q \geqslant 0$.

Remark:

We may also consider the Hodge filtration i.e., the descending filtration of $H_{\mathbb{C}}$ given by $F^i H_{\mathbb{C}} = \bigoplus_{p+q=n, p \geqslant i} H^{p,q}$. Hodge structure could be recovered from the filtration as follows: $H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{k-p+1}} H_C, H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}}$.

Remark: There is also a way to describe Hodge structures as algebraic representation. We will need it later to define Kuga-Satake torus.

Examples

trivial

$\mathbb{Z} \subset \mathbb{R}$, then $\mathbb{C} = \mathbb{C}^{0,0}$ is trivial Hodge structure of weight 0. Also it corresponds to the cohomology of a point.

Elliptic curve

Consider the elliptic curve $y^2 = x(x-1)(x-\lambda)$, $\omega = \frac{dx}{y}$.
Recall, we have $H^1(E_\lambda, \mathbb{Z})$ has weight 1 Hodge structure given by form ω.

Tate Hodge structure $\mathbb{Z}(k)$

Consider $H_\mathbb{Z} = (2\pi i)^k \mathbb{Z} \subset \mathbb{C}, H_\mathbb{C} = H^{-k,-k}$.
We need this $(2\pi i)^k$-factor above because we have an algebraically defined cycle class map and a topological one into H^1_{dR}. They differ by this factor. We can shift any Hodge structure by Tate shift:

$$H^{2k}(X, \mathbb{Z})(k) := H^{2k}(X, \mathbb{Z}) \otimes \mathbb{Z}(k)$$

It has weight 0.

Geometry of Tate Hodge structure

$\mathbb{Z}$ might be viewed as the Hodge structure on $H^0(\{x\})$.
Likewise, one can think of $\mathbb{Z}(-1)$ as the Hodge structure on $H^1(\{\mathbb{C} \setminus \{0\}\})$.
Indeed, the motivation is that the cohomology is spanned by dz/z which has integral $2\pi i$ on a loop that makes one counterclockwise turn around the origin.
Note that dz/z is a holomorphic differential on $\mathbb{C} \setminus \{0\}$ and so (counting dz's) has Hodge level 1. Namely, it lies in F^1. Hence, $\mathbb{Z}(-1)$ has type $(1,1)$.

Hodge structure of E_λ as filtration

Recall that the first cohomology $H^1(E, \mathbb{Z})$ has Hodge numbers $h^{1,0} = h^{0,1} = 1$. The Hodge filtration is $H = F_0 \supset F_1 \supset \{0\}$, where $F_1 = H^{1,0}$.

Morphisms

Morphism of Hodge structure of type (r, r)

If V and W are (pure) Hodge structures of weights m and $n = m + 2r$, then a **morphism of pure Hodge structures of type** (r, r) is a morphism of abelian groups $f : V \to W$ such that

$$f_{\mathbb{C}}(V^{p,q}) \subset W^{p+r,q+r}$$

Note: Map $f : V \to W$ of Hodge structures of weights m, n $(n = m + 2r)$ which sends $V^{p,q}$ to $W^{p+r,q+r}$ is not a morphism of Hodge structures. We can make it so with the Tate twist:

$$(2\pi i)^r \cdot f : V \to W(r)$$

Categorification

The category of $\mathbb{R}$-Hodge structures is an abelian tensor category.

Hodge decomposition

For a compact Kähler manifold X the torsion free part of the singular cohomology $H^n(X, \mathbb{Z})$ comes with a natural Hodge structure of weight n given by the standard Hodge decomposition:

$$H^n(X, \mathbb{Z}) \otimes \mathbb{C} = H^n(X, \mathbb{C}) = \bigoplus H^{p,q}(X)$$

We are going to study it a bit later!

The even part $H^{2n}(X, \mathbb{Q})$ contains all algebraic classes (classes obtained as fundamental classes of subvarieties in X). These classes are integral and they are contained in $H^{k,k}(X)$. The Hodge conjecture asserts that the space spanned by those is determined entirely by the Hodge structure itself.

Hodge conjecture

For a smooth projective variety X over $\mathbb{C}$ the subspace of $H^{2k}(X, \mathbb{Q})$ spanned by all algebraic classes $[Z]$ coincides with the space of Hodge classes, ie $H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X) = \langle [Z] | Z \subset X \rangle_{\mathbb{Q}}$.

Polarization

Definition

Let Q be a quadratic form on $H_{\mathbb{R}}$ which is symmetric (anti-symmetric) if k even (odd). We say that Q polarizes a Hodge structure if
(a) $Q(H^{p,q}, H^{p',q}) = 0$ unless $p = q', q = p'$
(b) $i^{p-q} Q(x, \overline{x}) > 0$ for any $0 \neq x \in H^{p,q}$

Question: Is $H^1(E_\lambda, \mathbb{Z})$ polarized?

Yes, it is. There is an intersection form Q.

categorification

The category $\mathbb{R}\mathrm{HS}^{pol}$ is *semisimple* abelian tensor category

Smooth genus g curve X

X topologically is a sphere with g handles.

One can define a Riemann surface of this kind by the equation

$$y^2 = (x - t_1) \cdot \ldots \cdot (x - t_n), n = 2g + 2$$

Remark: For $g > 2$ there are Riemann surfaces which are not given by this equation. Ones which have are called **hyperelliptic**.

Claim

$H^1(X, \mathbb{Z})$ carries a weight 1 Hodge structure polarized by the intersection form Q.

Weight one Hodge structure on X

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Idea: Show that $\Omega^1(X) \oplus \overline{\Omega}^1(X) \subset H^1(X)$ where $\Omega^1(X)$ are holomorphic 1-forms and then use Riemann-Roch for the dimension counting.
First define $H^{1,0}(X)$ as closed holomorphic 1-forms.
Proof:
1. *Global holomorphic 1-forms are closed and cohomologous holomorphic 1-forms are equal.*
Indeed, holomorphic 1-form α locally is $f(z)dz$. Then $d\alpha = \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial f}{d\overline{z}} dz \wedge d\overline{z} = 0$. And if $\alpha = df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{d\overline{z}} d\overline{z}$ then f is global holomorphic function on compact X, so it is constant.

Weight one Hodge structure on X

2. It gives $H^{1,0}(X) \simeq H^0(X, \Omega^1_X)$. If we define $H^{0,1}(X)$ as $\overline{H^{1,0}(X)}$ then we need to prove that $H^{1,0} \cap H^{0,1} = 0$.
Namely, let α be form from the intersection. Then $[\alpha] = [\beta], [\overline{\alpha}] = [\gamma]$ for holomorphic 1-forms β, γ.
If $\alpha \neq 0$ then $0 = i \int \beta \wedge \gamma = i \int \beta \wedge \overline{\beta} > 0$. Contradiction.

3. *(Dimension counting)* By Riemann-Roch $h^0(\mathcal{L}) - h^0(\mathcal{L}^* \otimes K) = deg(\mathcal{L}) + 1 - g$. When $\mathcal{L} \simeq \mathcal{O}$ we have $1 - h^0(K) = 1 - g$. So $H^0(K)$=g.
Hence, we have Hodge structure of weight 1 on $H^1_{dR}(X)$.

4. *Polarization* given by an intersection form.

Explicit description of holomorphic forms

$\dim\Omega^1(X) = g$

For hyperelliptic Riemann surfaces we can construct 1-forms explicitly as

$$\omega_i = \frac{x^i dx}{y}, \quad i = 0, ..., g-1$$

These forms are independent

Indeed, for any polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_{g-1} x^{g-1}$ of degree $\leqslant (g-1)$. The 1-form $p(x)dx/y$ is zero iff $p(x)$ is zero polynomial. So they give a basis.

Period map

What is the period map here?
Consider hyperelliptic curve X with the symplectic basis $(\gamma_1, ..., \gamma_g, \delta_1, ..., \delta_g)$ of $(H_1(X), U)$
Where do periods go?
This basis (marking) gives an isomorphism $\mathfrak{m} : H^1(X, \mathbb{Z}) \to \mathbb{Z}^{2g}$ which extends to $H^1(X, \mathbb{C}) \xrightarrow{\simeq} \mathbb{C}^{2g}$. The subspace $\mathfrak{m}(H^{1,0})$ defines a point in the $Gr(g, 2g)$.
Remark: It depends on marking and the choice of $\vec{t} = (t_1, ..., t_g)$ from $U = \mathbb{C} \setminus \Delta$ (parameter space for X) where Δ is the *discriminant locus* – union of hypersurfaces $t_i = t_j$.

Period map

Consider the universal cover $\tilde{U}$ of U, and the pullback of local system formed by the cohomology groups $H^1(X_u)$ will be isomorphic to the trivial local system $\mathbb{Z}^{2g} \times \tilde{U}$.
We could define **period map** as $\tilde{P} : \tilde{U} \to Gr(g, 2g)$ which as equivariant $\tilde{P}(\gamma x) = \rho(\gamma)\tilde{P}(x)$

Period map

Indeed,

$$\mathfrak{m}(\tilde{u}) : H^1(X_{\tilde{u}}) \to \mathbb{Z}^{2g} \times \tilde{U} \to \mathbb{Z}^{2g}$$

denote the isomorphism of the fibers over $\tilde{u}$, composed with the projection to the first factor.
So we have an isomorphism $\mathfrak{m}(\tilde{u}) : H^1(X_{\tilde{u}}) \simeq \mathbb{Z}^{2g}$ such that $\mathfrak{m}(\gamma \cdot \tilde{u}) = \rho(\gamma)\mathfrak{m}(\tilde{u})$,where ρ is the monodromy representation and $\gamma \cdot \tilde{u}$ is the action of $\pi(U, u_0)$ on $\tilde{U}$ by covering transformations.
Remark: Monodromy representation $\rho : \pi_1(U) \to Sp(g, \mathbb{Z}) = \{M \in GL(2g, \mathbb{Z}) : M^T JM = J\}$ (to integer symplectic group).

Period map

Claim

Map $\tilde{P} : \tilde{U} \to Gr(g, 2g)$ is holomorphic.

Proof.

Note that $\tilde{P}$ is the span of rows of $[A_{ij}|B_{ij}]$ which is $g \times 2g$ matrix with $A_{ij} = \int_{\delta_i} \omega_j$ and $B_{ij} = \int_{\gamma_i} \omega_j$. They are holomorphic functions of $\vec{t}$. □

Claim

(1) (A_{ij}) is invertible

(2) If $A_{ij} = Id$ then (B_{ij}) is symmetric and has a positive definite imaginary part.

Proof.

If $\int_\delta \omega = 0$ for all ω and some $\delta = \sum a_i \delta_i \neq 0$, then $\omega \in Span(\gamma_1, ..., \gamma_g$. And so is $\overline{\omega}$. Hence, $[\omega] \wedge [\overline{\omega}] = 0$. But it contradicts with $i \int_X \omega \wedge \overline{\omega} \geqslant 0$. □

Siegel upper half-space

Now choose the basis ω_i such that A-periods are now δ_{ij}. Then B-periods transform to $Z := \left(\int_{\gamma_i} \omega_j\right) \cdot \left(\int_{\delta_i} \omega_j)^{-1}\right)$.

Now let us prove (2) from above. Namely that $Z = Z^T$ and ImZ is positive definite.

Proof.

1. $[\omega_j] = \sum_i Z_{ij}\gamma^\vee + \delta_j^\vee$
2. Then $\int_X [\omega_j] \wedge [\omega_k] = -Z_{kj} + Z_{jk} = 0$. Hence, symmetric.
3. Consider $\sum_j a_j \omega_j$. Then $0 < i \int_X \left(\sum_j \omega_j\right) \wedge \left(\sum_{j'} a_{j'} \overline{\omega_{j'}}\right) =$

$i\left(\sum_{jj'} a_j a_{j'} \left(-Z_{j'j} + \overline{Z_{jj'}}\right)\right) = \sum_{jj'} a_j a_{j'} \cdot 2ImZ_{jj'}$ □

Definition

Siegel upper half-space $\mathfrak{H}_g$ is $\{Z \in Mat_{g\times g}(\mathbb{C}) | Z = Z^T, ImZ > 0\}$.

Theorem (Period map)

The period map takes values in $\mathfrak{H}_g$ viewed as a subset of the Grassmann manifold via the map $Z \to$ row space of (Id_g, Z). So

$\mathfrak{H}_g$

Remark: 1. $\mathfrak{H}_1 = \mathbb{H}$
2. $\dim \mathfrak{H}_g = \frac{g(g+1)}{2}$ (recall $\dim Gr(g, 2g) = g^2$).

As we mentioned integer symplectic group $Sp(g, \mathbb{Z})$ acts on $\mathfrak{H}_g$. Let as describe that action:

Description of $Sp(g, \mathbb{Z})$-action

Let T be in $Sp(g, \mathbb{Z})$. Call its four $g \times g$-blocks K, L, M, N as follows: $T = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$. Then it acts on $(A|B)$ as $(A|B) \cdot T$. In terms of normalized basis it means
$(Id_g|Z) \mapsto (K + ZM)^{-1}(L + ZN)$.

Properties of $Sp(g, \mathbb{Z})$-action

1. For $g = 1$ reduces to the standard $SL(2, \mathbb{R})$-action on upper half-plane.
2. This action is transitive.
3. The quotient $Sp(g, \mathbb{Z}) \backslash \mathfrak{H}_g$ is an analytic space.

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Indeed, action is transitive: consider the image of $Z = iId_g$ by action of $T = \begin{pmatrix} Id_g & A \\ 0 & Id_g \end{pmatrix} \begin{pmatrix} B^T & 0 \\ 0 & B^{-1} \end{pmatrix}$. This image is $A + iB^TB$.
Recall that any hermitian matrix could be written as B^TB. q.e.d
With the analytic space: note that the action $T \mapsto T(iId_g)$ is proper (in terms of sequences). Then it is proper discontinuous (check). By Cartan's argument we are done.
So there is a quotient map

$$P : U \to Sp(g,\mathbb{Z})\backslash\mathfrak{H}_g$$

which is holomorphic map of analytic spaces.
Remark: $Sp(g,\mathbb{Z})\backslash\mathfrak{H}_g$ is not complex manifold (bc of fixed points)

Injectivity

Given a family $(C_s)_{s\in S}$ of curves of genus g parametrized by a complex variety S, this construction defines a single-valued period map

$$S \to Sp(g,\mathbb{Z})\backslash\mathfrak{H}_g, s \mapsto T(C_s)$$

which is still holomorphic, but never surjective

Remark: for $g \geqslant 4$ it is not even dominant (ie its image is not dense)

Schottky problem

This leads to other questions, in particular, such as the Schottky problem of characterizing the possible images.

Torelli problem

The question of injectivity (called the Torelli problem) needs to be asked more carefully (injectivity will trivially be false if $(C_s)_{s\in S}$ is a constant family), but the answer is positive in the sense that a smooth projective curve of genus g is completely determined (up to isomorphism) by its period point in $Sp(g,\mathbb{Z})\backslash\mathfrak{H}_g$.

Is P surjective?

The dimension of moduli space M_g of curves of genus g is $3g-3$.
It is less than $g(g+1)/2$. So..
A lot of points in $\mathfrak{H}_g$ do not come from the curves

Question: Where they do come from (if they do)?
Answer: They come from **abelian varieties**!

Abelian variety

An abelian variety $X_{/\mathbb{C}}$ is a projective complex torus $X = V/\Lambda$, where $V \simeq \mathbb{C}^g$ and $\Lambda \simeq \mathbb{Z}^{2g}$ is a lattice.

We have $\Lambda = \pi_1(X,0) = H_1(X,\mathbb{Z})$ canonically and $Hom_{\mathbb{Z}}(\Lambda,\mathbb{Z}) = H^1(X,\mathbb{Z})$.
Moreover, $X \simeq (S^1)^{2g}$, so in fact $\Lambda^\bullet Hom_{\mathbb{Z}}(\Lambda,\mathbb{Z}) = H^*(X,\mathbb{Z})$ and preserves the ring structure.

Hodge structures of weight 1

Correspondence

complex tori $\longleftrightarrow$ integral Hodge structures of weight one
abelian varieties $\longleftrightarrow$ polarizable integral Hodge structures of weight one

Hodge structures of weight one are all of geometric origin.

Proof.

For an integral Hodge structure H of weight one, $H \subset H_{\mathbb{C}}$ can be projected injectively into $H^{1,0}$. This yields a lattice $H \subset H^{1,0}$. Consider the quotient $H^{1,0}/H$, it is a complex torus.
Conversely, if C^n/Γ is a complex torus, then $\mathbb{C}^n$ can be regarded as $\Gamma_{\mathbb{R}}$ endowed with an almost complex structure. This yields a decomposition $(\Gamma_{\mathbb{R}})_{\mathbb{C}} = (\Gamma_{\mathbb{R}})^{1,0} \oplus (\Gamma_{\mathbb{R}})^{0,1}$ with $(\Gamma_{\mathbb{R}})^{1,0}$ and $(\Gamma_{\mathbb{R}})^{0,1}$ being the eigenspaces on which $i \in \mathbb{C}$ acts by multiplication by i and $-i$, respectively. It defines an integral Hodge structure of weight one. $\square$

Hodge decomposition: forms

One of the main applications of Hodge structures is to the study of the cohomology of Kähler manifolds, via the **Hodge decomposition**. This decomposition will be described now.

Basics: Let X be C^∞-manifold. If X is a complex manifold, then the space of smooth $\mathbb{R}$-valued n-forms on X $\mathcal{C}^n_{\mathbb{C}} = \bigoplus_{p+q=n} \mathcal{C}^{p,q}$ ((p,q)-forms which locally have form $\sum f_{I,J} dz_I \wedge d\overline{z}_J$ for holomorphic coordinates z_i).
We have exterior derivative d (determined on functions and 1-forms): $\mathcal{C}^{p,q} \xrightarrow{\partial,\overline{\partial}} \mathcal{C}^{p+1,q} \oplus \mathcal{C}^{p,q+1}$.
$d^2 = \partial^2 = \overline{\partial}^2 = 0$

Chain complexes

$(\mathcal{C}^\bullet_{\mathbb{R}}, d)$ $((\mathcal{C}^{p,\bullet}_{\mathbb{R}}, \overline{\partial}))$ and $(\Gamma(X, \mathcal{C}^\bullet_{\mathbb{R}}), d)$ $((\Gamma(X, \mathcal{C}^{p,\bullet}_{\mathbb{R}}), \overline{\partial}))$ are chain complexes. The cohomology complexes of the latter two:
$H^\bullet_{dR}(X, \mathbb{R}) := H^\bullet(\Gamma(X, \mathcal{C}^\bullet_{\mathbb{R}}), d)$ **de Rham cohomology** and
$H^{p,\bullet}_{\overline{\partial}}(X) := H^\bullet(\Gamma(X, \mathcal{C}^{p,\bullet}_{\mathbb{R}}), \overline{\partial})$ **Dolbeault cohomology**.

de Rham *vs* Dolbeault

Remark: There is no obvious map between $H^n_{dR}(X, \mathbb{C})$ and $H^{p,q}_{\overline{\partial}}$ in either direction:

- if α is d-closed form its (p, q)-part may not be $\overline{\partial}$-closed
- a $\overline{\partial}$-closed (p, q)-form may not be d-closed

However, both are true in some cases. Later we will study Hodge decomposition in Kähler case, now let us consider complex tori as a an example when we have Hodge structure

Hodge decomposition for tori

Let $T = V/\Lambda$ be a complex torus, where V is n-dimensional vector space and $\Lambda \subset V$ is a lattice isomorphic to $\mathbb{Z}^{2n}$. Then

$$H^n(T, \mathbb{Z}) \text{ carries a Hodge structure of weight } n$$

Hodge decomposition for tori

Translation-invariant forms

$$\begin{array}{ccc} Hom_{\mathbb{R}}(V,\mathbb{R}) & \longrightarrow & \Gamma(T,\mathcal{C}^1_{\mathbb{R}}) \\ \alpha : V \to \mathbb{R} & \mapsto & d\alpha(\Lambda - invariant) \\ Hom_{\mathbb{C}}(V,\mathbb{C}) & \longrightarrow & \Gamma(T,\mathcal{C}^{1,0}) \\ \alpha & \mapsto & \partial\alpha(\Lambda - invariant) \end{array}$$

and

$$\begin{array}{ccccc} \Lambda^\bullet Hom_{\mathbb{R}}(V,\mathbb{R}) & \longrightarrow & \Gamma(T,\mathcal{C}^\bullet_{\mathbb{R}}) & \xrightarrow{\varphi} & H^\bullet_{dR}(T,\mathbb{R}) \\ \alpha & \mapsto & \omega_\alpha & & \end{array}$$

$$\begin{array}{ccccc} \Lambda^\bullet Hom_{\mathbb{C}}(V,\mathbb{C}) \otimes \Lambda^\bullet Hom_{\mathbb{C}}(\overline{V},\mathbb{C}) & \longrightarrow & \Gamma(T,\mathcal{C}^{\bullet,\bullet}) & \xrightarrow{\psi} & H^{\bullet,\bullet}_{\overline{\partial}}(T) \\ \alpha & \mapsto & \omega_\alpha & & \end{array}$$

These forms are translation-invariant and first one is d-closed, the second one is $\overline{\partial}$-closed. It induces maps φ, ψ respectively. Every class in $H^\bullet_{dR}(T)$ and $H^{\bullet,\bullet}_{\overline{\partial}}(T)$ is uniquely determined by a translation-invariant forms.

Decomposition

$$H^n_{dR}(T,\mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}_{\overline{\partial}}(T)$$

Indeed, the first is $\Lambda^n\left(Hom_{\mathbb{C}}(V,\mathbb{C}) \oplus Hom_{\mathbb{C}}(\overline{V},\mathbb{C})\right)$ and the second is $\bigoplus_{p+q=n} \Lambda^p Hom_{\mathbb{C}}(V,\mathbb{C}) \otimes \Lambda^q Hom_{\mathbb{C}}(\overline{V},\mathbb{C})$, and they are equal.

It gives a weight n Hodge structure on $H^n(T,\mathbb{Z})$.

Next lecture

- Hodge decomposition for tori $\rightsquigarrow$ Hodge decomposition in Kähler case
- Polarization and decomposition on primitive cohomology
- Filtration $\rightsquigarrow$ Mixed Hodge structures
- Hodge structures of hypersurfaces, $K3$
- Hodge structure as an algebraic representation
- Kuga-Satake construction

Thanks!