Attractor Conjecture for CYVHS.

$X/\mathbb{C}$ a Calabi-Yau threefold, i.e. $\wedge^3 \Omega \cong \mathcal{O}$ sm. proper

(sometimes more restrictions but I'll take this as my defn).

<u>Defn</u>. X is an attractor point if

$\exists\ \gamma \in H^3(X, \mathbb{Z})$ s.t.

$$\gamma \in \underbrace{H^{3,0} \oplus H^{0,3}}_{\text{Hodge decomp.}} \subset H^3(X,\mathbb{Z}) \otimes \mathbb{C}.$$

~~In 1998~~ Moore made the following remarkable conjecture

<u>Conj</u>. (Moore '98) If X is attractor, then it is defined $/\overline{\mathbb{Q}}$.

(i.e. has model $/\overline{\mathbb{Q}}$)

Known cases: $X = K3 \times E$, $E_1 \times E_2 \times E_3$, etc

Also for "Dolgachev CY's :
cyclic covers of $\mathbb{P}^3$ branched along
6 hyperplanes" (L - Tripathy)

Observe all of these have a "Torelli theorem",
i.e. period map is ~~an~~ open embedding

More fancy : moduli space is Shimura variety
(~~e.g. for~~ ($\Leftrightarrow$ locally symmetric space)

Probably not true for general CY3's, (L - Tripathy)
(have counterexamples for n odd, > 3 .

unless moduli is Shimura .

Unfortunately not many examples of such CY3's known .

But next best thing :
~~every Shim~~
every Shimura variety has a (unique) CY
variation of Hodge structure .

Thm (2) for wt 3 VHS of CY type, the attractor conjecture holds.

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pure Q-Hodge structure of wt n. V $\mathbb{Q}$-v.sp,

$$V \otimes \mathbb{C} = H^{n,0} \oplus H^{n-1,1} \oplus \cdots \oplus H^{0,n},$$

satisfying some condition. (polarization)

~~Def~~

VHS. these in families, so over base S cx mfd,

* $\mathcal{V}$ a $\mathbb{Q}$-local system,

each fiber $\mathcal{V}_s$ a VHS + "Griffiths transversality".

$$\boxed{\nabla : F^p \longrightarrow F^{p-1} \otimes \Omega^1}.$$

Example. for a family of alg varieties $X \to S$,

local system is $H^n(X, \mathbb{Q})$ + its hodge str,

GT is a "geometric condition".

Def. A QHS is of CY type if $h^{n,0} = 1$. (wt n).

Primer for Shimura varieties

$D \hookrightarrow \check{D}$ compact dual, projective variety

a space of the form $\underbrace{G(\mathbb{Z}) \backslash \overbrace{G(\mathbb{R})/K}^{D}}$

$\mathbb{Z}$-pts of G.

real pts of reductive gp

maximal compact

not all G.

in nice cases, $G(\mathbb{R})/K$ has hermitian str

(when K has $U(1)$-factor)

e.g. $G = SL_2$, $G(\mathbb{R})/K = SL_2(\mathbb{R})/U(1) \simeq \mathfrak{h}$.

(1) has str of quasi-proj variety

(2) has canonical descent to # field E.

using CM points.

((3) integral str ...)

Example. $D = \mathfrak{h} \hookrightarrow \mathbb{P}^1(\mathbb{C}) = \check{D}$

$G = SL_2(\mathbb{Q})$, $G(\mathbb{Z}) = SL_2(\mathbb{Z})$.

$$\begin{array}{ccc} \mathfrak{h} & \hookrightarrow & \mathbb{P}^1 . \\ \downarrow & & \\ SL_2(\mathbb{Z}) \backslash \mathfrak{h} & = & X, \end{array}$$ modular curve $/\mathbb{Q}$

<u>Warning</u>. the $\mathbb{Q}$-str on top and bottom are completely different.

<u>Thm</u>. (Gross, Zhang-Zuo) every Shimura variety S has <u>canonical</u> lowest weight cycles.

<u>canonical means</u>. composition

$$\nabla : T_X \longrightarrow \mathrm{Hom}(H^{n,0}, H^{n,0} \oplus H^{n-1,1})$$
$$\longrightarrow \mathrm{Hom}(H^{n,0}, H^{n-1,1})$$

$$H^{n,0} \oplus H^{n-1,1} \oplus \cdots \oplus H^{0,n}$$

T_X : shifts up by one unit + project.

$\longleftrightarrow$ Maximal family.

weight 3 examples.

real. 1) $G = SO(2,n) / SO(2) \times SO(n)$

compact duals are homog. Legendrian varieties (studied by Manivel)

+ 4 exceptional ones, biggest of which is

$E_{7(-25)} / U(1) \times E_{6(-78)}$

+ $\mathbb{Q}$-forms of these groups.

complex.

⋮

~~bigg~~ exceptional one E_6-type Shimura variety.

Compact duals

$IG(3,6) \subset \mathbb{P}^{13}$, $G(3,6) \subset \mathbb{P}^{19}$, $S_{12} \subset \mathbb{P}^{31}$, $E_7/P_1 \subset \mathbb{P}^{55}$

(Legendrian homog. varieties (Zak, Gross-Wallach, Landsberg-Manivel, ...))

We do not know ~~the~~ E_7, E_6 SL varieties come from geometry. Focus on "real case".

Physics. no known string embedding of a supergravity th., for the E_7-case although there are some suggestion exotic string theory?

V / S, ... of ...

~~Then~~ So have a local system

and attractor mechanism:

$\gamma \in V(\mathbb{Z}) \rightsquigarrow$ ~~a function~~

$$\rightsquigarrow \quad V_\gamma : \tilde{S} \longrightarrow \mathbb{R}$$
$$p \longmapsto \frac{|\langle \Omega, \gamma \rangle|^2}{\Omega \wedge \overline{\Omega}}$$

$\rightsquigarrow$ minima of this fn.

Example. for $\mathfrak{h}$ have homog. coordinate x^0, x^1 $(x^1/x^0 = \tau)$

$V = \mathrm{Sym}^3 \mathrm{Std}_{\mathbb{Q}}$, 4-dimensional

$$V \otimes \mathbb{C} = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

$$\mathfrak{h}' = \left\{ (p, \Omega) \;\middle|\; p \in \mathfrak{h},\ \Omega \in H^{3,0} \setminus \{0\} \right\}.$$

$(F = (x^1)^3 / \cancel{x^0}\, 3x^0)$

write $\quad \Omega = x^0 \gamma_0 + x^1 \gamma_1 + F_0 \gamma^0 + F_1 \gamma^1$

Fact. $\quad F_0 = \dfrac{-(x^1)^3}{3(x^0)^2}, \quad F_1 = (x^1)^2 / x^0$

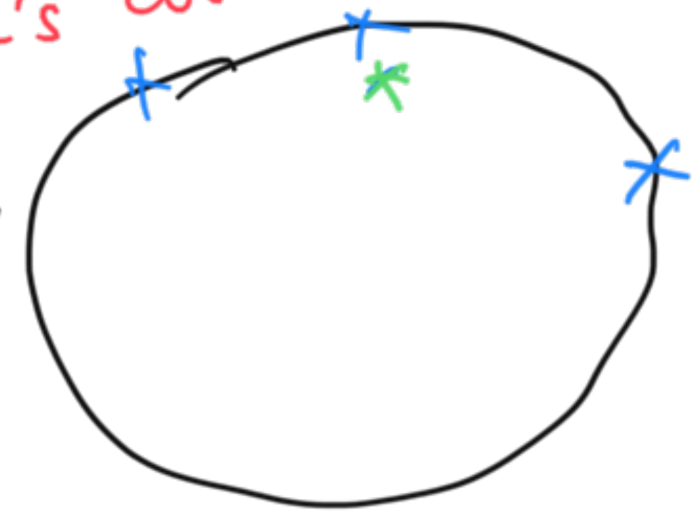
Lemma (L) for $\gamma \in \mathrm{Sym}^3 \mathrm{Std} \rightsquigarrow p(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ (with real roots)

... of ...

"center of mass" of
attractor point is given by
the three roots of $p(X,1)$

Julia's covariant map

disk model for h.



proof. compute $V_\gamma = \frac{|p(\tau)|^2}{y^{3/2}}$,

which matches the sum of distance fn of Cremona–Stoll. $\square$

Prop. $\iff$ attractor condition stated earlier.

Actually more subtle: have action (as in the Example above)

$$G(\mathbb{Z}) \circlearrowright V(\mathbb{Z}) \quad \text{prehomog., s.t.}$$

$V/\!/G \cong \mathbb{A}^1$, and invariant is I_4 quartic form.

Say $\gamma \in V(\mathbb{Z})$ BPS if $I_4(\gamma) > 0$.

<u>Example</u>. $G = \cancel{SL_2(\mathbb{Q})}$ $SL_2/\mathbb{Q}$,

$V \rightsquigarrow \mathrm{Sym}^3\, \mathrm{Std}_{\mathbb{Q}} \simeq \{\text{binary cubic form}\}$

$SL_2(\mathbb{R}) \curvearrowright V(\mathbb{R})$, I_4 = discriminant of cubic form.

one way to view h, or rather

$$h' = \{(p, \Omega) \mid p \in h,\ \Omega \in H^{3,0} \setminus \{0\}\} \text{ — 4-real dim}$$

$$h' \longrightarrow V(\mathbb{R})$$
$$(p, \Omega) \longmapsto \mathrm{Re}\,\Omega \in V(\mathbb{R}).$$

$$\text{image} = \{v \in V(\mathbb{R}) \mid I_4(v) > 0\}.$$

$$\text{attractors} = V(\mathbb{Z}) \subset V(\mathbb{R}).$$

<u>Thm</u> (precise form)(L.) for $\gamma \in BPS$, there is unique attractor point which is a CM point.

"hodge str has unexpected symmetries"

proof of Thm.

Will show attractor points are CM.

have $\gamma \in H^{3,0} \oplus H^{0,3}$, point is to find γ' w/ same property, linearly indep from γ

$$\Rightarrow \underbrace{\langle \gamma, \gamma' \rangle}_{\text{sub } \mathbb{Q}\text{-Hodge str.}} \otimes \mathbb{C} = H^{3,0} \oplus H^{0,3}$$

Use coordinates on $D' = \{ (p, \Omega) \mid p \in D,\ \Omega \in H^{3,0} \}$

(X^I, F_I) : the X^I are coord fns, F_I are auxiliary.

"special geometry". E.g. $\left(X^0, X^1, \frac{-X^1}{3(X^0)^2}, (X^1)^2/X^0\right)$

Given by: pick symplectic basis γ^I, γ_I for V, and for $(p, \Omega) \in D'$ write

$$\Omega = X^I \gamma_I + F_I \gamma^I . \quad \text{(cx coords)}$$

Geometrically $D' \longrightarrow V(\mathbb{C})$ Lagrangian

(can check dimension.)

Real coordinates.

$$\boxed{\begin{aligned} X^I &= p^I + i\phi^I, \\ F_I &= q_I + i\chi_I \end{aligned}}$$

now p, q, ϕ, χ are real

(hence take as many)

use (p, q) as real coords.

Lemma (1) (Cecotti - Ferrara - Girardello, Hitchin)

$\exists$ real fn $S(p, q)$, and that

$$\phi^I = \frac{\partial S(p,q)}{\partial q_I}, \qquad \text{(for any special geom.)}$$

$$\chi^I = -\frac{\partial S(p,q)}{\partial p^I}.$$

(2) in our case $S = \sqrt{I_4}$.

~~Simple calculation~~ $\Rightarrow$

τ coord on h

In Sl_2 case, have $X^1/X^0 = \cancel{\text{a}}\ \cancel{\theta}$

τ ~~satisfies~~ $\in \mathbb{Q}(\sqrt{D})$

and attractor $\Rightarrow$ ...

in general simple calc shows that
periods of $\Omega \in \mathbb{Q}(\sqrt{D})$ for some D
then can find γ.

But for us $\ast$ attractor $\Rightarrow p^I, q_I \in \mathbb{Z}$,
and Lemma (2) $\Rightarrow \Phi^I, \Psi_I \in \mathbb{Q}(\sqrt{-I_6})$. $\square$

Cor. have explicit param. of (certain) CM points of E_7
Shimura var. $\square$.

<u>An amusing question</u>.

more on special geometry

$$H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

given local coordinates x^i, have a
cubic form. This is the B model

$Sp(?, ?) / U(1) \times SU(3) \times SU(3)$

For the case $SU(3,3)/\ldots$

<u>physicists</u>. A-model on $\underbrace{T^6/\mathbb{Z}/3}_{\text{has resolution called } Z\text{-mfd, rigid.}}$

$E \times E \times E$, $H^1 = H^1_\zeta \oplus H^1_{\bar\zeta}$. $\zeta^3 = 1$

$9 = 1 + 1 + 1 + 3 \times 2$

Z has hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 36 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 36 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$\Leftrightarrow$ cubic 7-fold H^7 $0 \; 0 \; 1 \; 36 \; 36 \; 1 \; 0 \; 0$
(or rather a quotient)

So there is a 9-dim subspace which is somehow param by a Sh. var.

Hodge conj $\Rightarrow$ this 9-dim'l subspace has extra alg cycles coming from Weil-type AV's.

$V(\mathbb{Z}) \circlearrowleft G(\mathbb{Z})$ and recall

<u>Recap</u>. have

$$V(\mathbb{Z})/G(\mathbb{Z}) \longrightarrow Sh.$$

More on what these reps are: in each case

$\exists$ cubic norm structure, i.e. W $\mathbb{Z}$-module +

cubic form $N : W \to \mathbb{Z}$.

<u>Exceptional case</u>. $W = \mathrm{herm}_3(A)$, where A is $\mathbb{Z}$-form of ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$)

N = determinant.

<u>Thm</u> (<u>Bhargava, Pollack</u>)

$V(\mathbb{Z})/G(\mathbb{Z})$ param.

$$(S, I)$$

good ring — fract. ideal $\subset S \otimes A$.

<u>Prop</u>. ~~In case~~ the good ring matches the ones good field $\mathbb{Q}(\sqrt{D})$ in the proof

of Thm.

~~View~~ (?) Thm as generalization of CM points on $\mathfrak{h}$
param. fractional ideals.

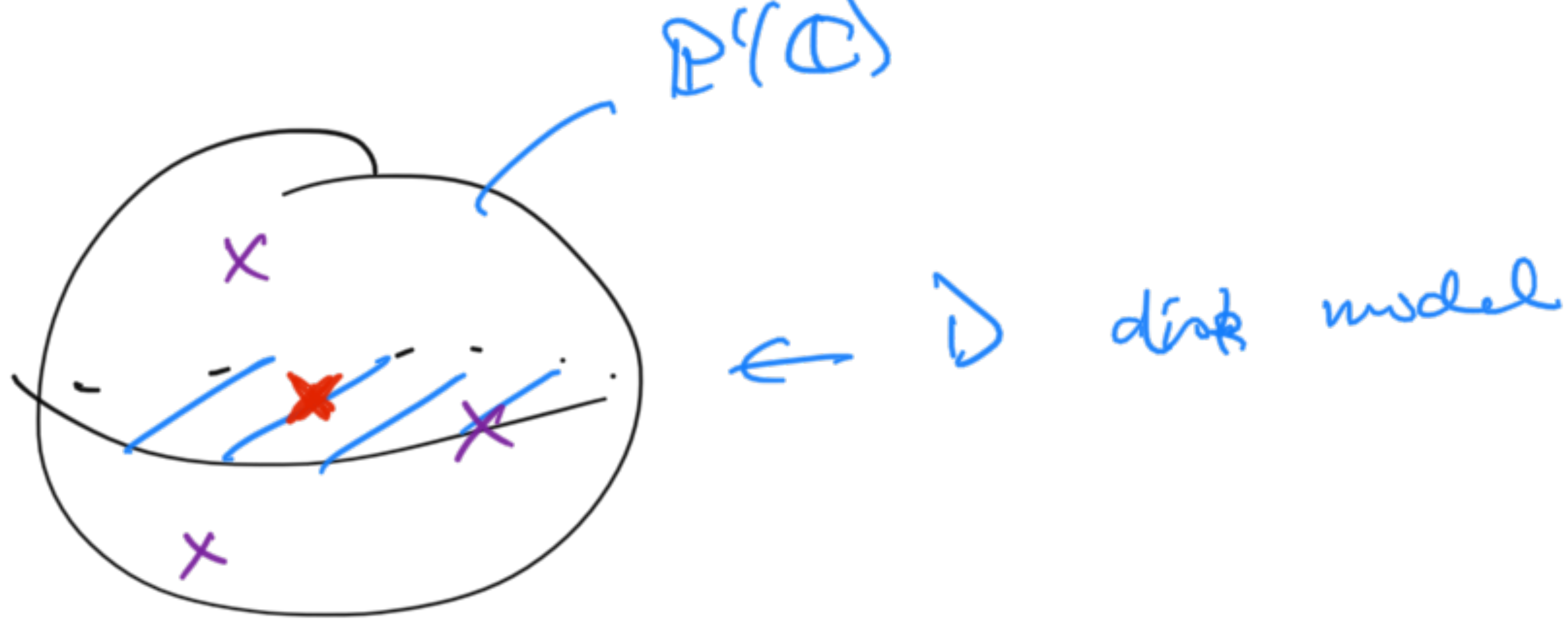
How about non-BPS (?) / other types of attractor?

Thm (L) (1) non-BPS attractor ~~mod~~ mod space are
totally geodesic $\subset \mathcal{D}$

(2) in 5d, BPS attractor are analogue of
KM cycles.

(3) in the case sl_2: $I_{\mathbb{R}} \subset O(E)$ catches real rt + pair of cx rts

take 3 pts on



attract pt = center of mass.

$\rightsquigarrow$ first conceptual description of these.