On orbits of antichains of positive roots

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Main definitions

$$(\mathcal{P}, \preceq)$$ is an arbitrary finite poset. For any $S \subset \mathcal{P}$, let $S_{\text{min}}$ and $S_{\text{max}}$ denote the set of minimal and maximal elements of $S$, respectively.

**Definitions**

- An **antichain** in $\mathcal{P}$ is a subset of mutually incomparable elements.
- An **upper ideal** (or **filter**) is a subset $I \subset \mathcal{P}$ such that if $\gamma \in I$ and $\gamma \preceq \beta$, then $\beta \in I$.

The set of all antichains in $\mathcal{P}$ is denoted by $\mathcal{A}_n(\mathcal{P})$.

- $\Gamma \in \mathcal{A}_n(\mathcal{P})$ if and only if $\Gamma = \Gamma_{\text{min}}$ (or $\Gamma = \Gamma_{\text{max}}$).
- If $\Gamma \in \mathcal{A}_n(\mathcal{P})$, then $I(\Gamma)$ denotes the upper ideal of $\mathcal{P}$ generated by $\Gamma$. That is, $I(\Gamma) = \{ \varepsilon \in \mathcal{P} \mid \exists \gamma \in \Gamma \text{ such that } \gamma \preceq \varepsilon \}$.
- If $I$ is an upper ideal of $\mathcal{P}$, then $I_{\text{min}} \in \mathcal{A}_n(\mathcal{P})$.

This yields a natural bijection between the upper ideals and antichains of $\mathcal{P}$.
Letting \( \Gamma' \preceq \Gamma \) if \( \mathcal{I}(\Gamma') \subset \mathcal{I}(\Gamma) \), we make \( \mathcal{A}_n(\mathcal{P}) \) a poset.

Example

\( \Gamma = \emptyset \) is an antichain and \( \mathcal{I}(\emptyset) \) is the empty upper ideal.

For \( \Gamma \in \mathcal{A}_n(\mathcal{P}) \), we set \( X(\Gamma) := (\mathcal{P} \setminus \mathcal{I}(\Gamma))_{\text{max}} \). This defines the map \( X = X_\mathcal{P} : \mathcal{A}_n(\mathcal{P}) \to \mathcal{A}_n(\mathcal{P}) \). Clearly, \( X \) is one-to-one, i.e., it is a permutation of the finite set \( \mathcal{A}_n(\mathcal{P}) \).

We say that \( X \) is the reverse operator for \( \mathcal{P} \).

If \( \#\mathcal{A}_n(\mathcal{P}) = m \), then \( X \) is an element of the symmetric group \( \Sigma_m \). The order of \( X \), \( \text{ord}(X) \), is the order of the group generated by \( X \).

Main problem: Study connections between combinatorial properties of \( \mathcal{P} \) and algebraic properties of \( X \).
**Definition**

\( \mathcal{P} \) is graded (of level \( r \)) if there is a function \( d : \mathcal{P} \rightarrow \{1, 2, \ldots, r\} \) such that both \( d^{-1}(1) \) and \( d^{-1}(r) \) are non-empty, and \( d(y) = d(x) + 1 \) whenever \( y \) covers \( x \). Then \( d^{-1}(1) \subset \mathcal{P}_{\text{min}} \) and \( d^{-1}(r) \subset \mathcal{P}_{\text{max}} \).

**Lemma**

Suppose \( \mathcal{P} \) is graded of level \( r \), \( d^{-1}(1) = \mathcal{P}_{\text{min}} \) and \( d^{-1}(r) = \mathcal{P}_{\text{max}} \). Then \( \mathcal{X} \) has an orbit of cardinality \( r + 1 \).

- \( \mathcal{P}(i) := d^{-1}(i) \) is an antichain for any \( i \).
- By our hypotheses, \( \mathcal{X}(\mathcal{P}(i)) = \mathcal{P}(i-1) \) for \( i = 2, \ldots, r \), \( \mathcal{X}(\mathcal{P}(1)) = \emptyset \), and \( \mathcal{X}(\emptyset) = \mathcal{P}(r) \).
- Thus, \( \{\emptyset, \mathcal{P}(r), \ldots, \mathcal{P}(1)\} \) is an \( \mathcal{X} \)-orbit.

Such an orbit of \( \mathcal{X} \) is said to be **standard**.
Notation for root systems

- $\Delta$ is a reduced irreducible root system in $V$ (dim $V = n$).
- $\Delta^+$ is a set of positive roots, with the corresponding simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$.
- $W \subset GL(V)$ is the Weyl group of $\Delta$; $w_0 \in W$ is the longest element.

Definition

The root order in $\Delta^+$ is given by letting $x \preceq y$ if $y - x$ is a non-negative integral combination of positive roots. In particular, $y$ covers $x$ if $y - x$ is a simple root.

- $\theta \in \Delta^+$ is the highest root. It is the maximal element of $(\Delta^+, \preceq)$.
- If $\gamma = \sum_{i=1}^{n} a_i \alpha_i \in \Delta^+$, then $ht(\gamma) := \sum a_i$ is the height of $\gamma$.
- $h = h(\Delta)$ is the Coxeter number of $\Delta$. 
Conjectures for $\mathcal{P} = \Delta^+$

$\theta = \alpha_1 + \cdots + \alpha_6$

**Figure:** The poset $\Delta^+(\mathfrak{sl}_7)$
Some properties of $\Delta^+$ and $\mathcal{A}_n(\Delta^+)$

- The function $\alpha \mapsto \text{ht}(\alpha)$ makes $\Delta^+$ the graded poset of level $h-1$.
- If $e_1, \ldots, e_n$ are the exponents of $\Delta$, then
  $$\#(\mathcal{A}_n(\Delta^+)) = \prod_{i=1}^{n} \frac{h + e_i + 1}{e_i + 1} \quad \text{(Cellini-Papi, 2002)}.$$  
- $\#\Gamma$ equals the number of elements of $\mathcal{A}_n(\Delta^+)$ covered by $\Gamma$.
  (For, $\Gamma$ covers $\Gamma'$ with respect to the order $'<'$ described above if and only if $\Gamma' = (J(\Gamma) \setminus \{\gamma_i\})_{\text{min}}$ for some $\gamma_i \in \Gamma$.)
  Hence
  $$\sum_{\Gamma \in \mathcal{A}_n(\Delta^+)} \#\Gamma$$
  equals the total number of edges in the Hasse diagram of $(\mathcal{A}_n(\Delta^+), <)$.
- $$\sum_{\Gamma \in \mathcal{A}_n(\Delta^+)} \frac{\#\Gamma}{\#\mathcal{A}_n(\Delta^+)} = \frac{\#\Delta^+}{h} \quad \text{(Panyushev, 2006)}$$
Figure: Antichains $\Gamma$ and $\mathcal{X}(\Gamma)$ for $\Delta^+(\mathfrak{sl}_{n+1})$
Some orbits of $\mathcal{X} = \mathcal{X}_{\Delta^+}$

Set $\Delta(i) = \{\alpha \in \Delta^+ \mid \text{ht}(\alpha) = i\}$.

Then $\Delta(1) = \Pi = \Delta_{\text{min}}^+$ and $\Delta(h-1) = \{\theta\} = \Delta_{\text{max}}^+$.

**Example**

There are two specific orbits of $\mathcal{X} = \mathcal{X}_{\Delta^+}$:

- By Lemma, there is an orbit of cardinality $h$. Namely, 
  $\{\emptyset, \Delta(h-1), \ldots, \Delta(2), \Delta(1)\}$ is the **standard** $\mathcal{X}$-orbit in $\mathcal{A}_n(\Delta^+)$.  

- There is an orbit of cardinality 2. Let $\mathcal{A} \subset \Pi$ a set of mutually orthogonal roots such that $\Pi \setminus \mathcal{A}$ also has that property. The partition $\{\mathcal{A}, \Pi \setminus \mathcal{A}\}$ is uniquely determined, since the Dynkin diagram of $\Delta$ is a tree. Then $\mathcal{X}(\mathcal{A}) = \Pi \setminus \mathcal{A}$ and $\mathcal{X}(\Pi \setminus \mathcal{A}) = \mathcal{A}$.

**Remark**

If $\Delta$ is of rank 2, then these two orbits exhaust $\mathcal{A}_n(\Delta^+)$.  

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Conjecture 1 (for $\mathcal{X} = \mathcal{X}_{\Delta^+}$)

(i) If $w_0 = -1$, then $\text{ord}(\mathcal{X}) = h$;

(ii) If $w_0 \neq -1$, then $\mathcal{X}^h$ is the involution of $\mathcal{A}_n(\Delta^+)$ induced by $-w_0$ and $\text{ord}(\mathcal{X}) = 2h$;

(iii) Let $O$ be an arbitrary $\mathcal{X}$-orbit in $\mathcal{A}_n(\Delta^+)$. Then

$$\frac{1}{\#O} \sum_{\Gamma \in O} \# \Gamma = \frac{\#\Delta^+}{h} = \frac{n}{2}.$$ 

- $w_0 \neq -1$ if and only if $\Delta$ is of type $\mathbf{A}_n \ (n \geq 2)$, $\mathbf{D}_{2n+1}$, $\mathbf{E}_6$.
- Conjecture 1 has been verified for $\mathbf{A}_n \ (n \leq 5)$, $\mathbf{C}_n \ (n \leq 4)$, $\mathbf{D}_4$, $\mathbf{F}_4$.
- $\Delta^+(\mathbf{B}_n) \simeq \Delta^+(\mathbf{C}_n)$.
- Part (iii) is a refinement of the formula for the number of edges in the Hasse diagram.
Reverse operators for posets associated with root systems

Example (for $\Delta^+(A_n)$)

Usual notation: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2, \ldots, n$, and $\theta = \varepsilon_1 - \varepsilon_{n+1}$.

Suppose $\Gamma = \{\alpha_1\}$ and $n \geq 3$. Then $\mathcal{X}(\{\alpha_1\}) = \alpha_2 + \ldots + \alpha_n$ and

$$\mathcal{X}^k(\{\alpha_1\}) = \{\gamma \in \Delta(\alpha_1, \ldots, \alpha_{n-1}) \mid \text{ht}(\gamma) = n + 1 - k\} \sqcup \{\alpha_{k+1} + \ldots + \alpha_n\}$$

for $1 \leq k \leq n$. In particular, $\mathcal{X}^n(\{\alpha_1\}) = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and hence $\mathcal{X}^{n+1}(\{\alpha_1\}) = \{\alpha_n\}$. Therefore the $\mathcal{X}$-orbit of $\{\alpha_1\}$ is of cardinality $2n+2$.

- For this orbit, we have $\frac{1}{\# \mathcal{O}} \sum_{\Gamma \in \mathcal{O}} \# \Gamma = n/2$, as required.

Challenging problem: construct “invariants” of $\mathcal{X}$, i.e., functions on $A_n(\Delta^+)$ that are constant on the $\mathcal{X}$-orbits. Ideally, one could ask for a family of invariants that separates the orbits. Below, we describe one invariant in the case of type $A_n$. 
- $\Delta^+ \setminus \Pi = \Delta(\geq 2)$ is a subposet of $\Delta^+$.
- $\Delta^+ \setminus \Pi$ is the graded poset of level $h-2$. (Use $\alpha \mapsto \text{ht}(\alpha)-1$.)
- The theory of antichains in $\Delta^+ \setminus \Pi$ resembles that for $\Delta^+$. In particular, $\#(\mathcal{A}_{\Delta^+ \setminus \Pi}) = \prod_{i=1}^{n} \frac{h + e_i - 1}{e_i + 1}$ (Sommers, 2005).
- $\mathcal{X}_{\Delta^+ \setminus \Pi}$ has the standard orbit of cardinality $h-1$.

**Conjecture 2 (for $\mathcal{X}_0 = \mathcal{X}_{\Delta^+ \setminus \Pi}$)**

(i) If $w_0 = -1$, then $\text{ord}(\mathcal{X}_0) = h - 1$;

(ii) If $w_0 \neq -1$, then $\mathcal{X}_0^{h-1}$ is the involution of $\mathcal{A}_{\Delta^+ \setminus \Pi}$ induced by $-w_0$ and $\text{ord}(\mathcal{X}_0) = 2h-2$;

(iii) For any $\mathcal{X}_0$-orbit $\mathcal{O} \subset \mathcal{A}_{\Delta^+ \setminus \Pi}$, we have

$$\frac{1}{\#\mathcal{O}} \sum_{\Gamma \in \mathcal{O}} \#\Gamma = \frac{\#(\Delta^+ \setminus \Pi)}{h-1} = \frac{n \cdot h-2}{2 \cdot h-1}.$$
Again, part (iii) is a refinement of a formula for the number of edges in the Hasse diagram of $\mathfrak{A}_n(\Delta^+ \setminus \Pi)$.

**Empirical evidences supporting Conjecture 2:**

$\Delta^+ \setminus \Pi(\mathbf{A}_{n+1}) \simeq \Delta^+(\mathbf{A}_n)$. Therefore Conjecture 2 holds for $\mathbf{A}_n$ ($n \leq 6$). It has also been verified for $\mathbf{C}_n$ ($n \leq 5$), $\mathbf{D}_n$ ($n \leq 5$), and $\mathbf{F}_4$.

**Warning**

One might have thought that posets $\Delta(\geq j)$ enjoy similar good properties for any $j$. However, this is not the case!

**Example**

For $\mathbf{F}_4$ and $\Delta(\geq 3)$, the reverse operator has orbits of cardinality 10 and 8. Hence its order equals 40, while $h - 2 = 10$. Furthermore, the mean value of the size of antichains along the orbits is not constant.
Reverse operators for posets associated with root systems

Conjectures for $\mathcal{P} = \mathcal{D}^+_s$

- If $\Delta$ has two root lengths, then $\mathcal{D}^+_s = \{ \alpha \in \Delta^+ \mid \alpha \text{ is short} \}$.
- $\mathcal{D}^+_s$ is regarded as subposet of $\Delta^+$.
- $\theta_s$ is the only maximal element of $\mathcal{D}^+_s$ and $(\mathcal{D}^+_s)_{\text{min}} = \Pi \cap \mathcal{D}^+_s = \Pi_s$.

Let $h^*(\Delta)$ be the dual Coxeter number of $\Delta$. If $\Delta^\vee = \{ \frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in \Delta \}$ is the dual root system, then $h^*(\Delta^\vee) - 1 = ht(\theta_s)$.

- $\mathcal{D}^+_s$ is a graded poset of level $h^*(\Delta^\vee) - 1$.
- $\mathcal{X}_s$ has the standard orbit of cardinality $h^*(\Delta^\vee)$.

**Conjecture 3 (for $\mathcal{X}_s = \mathcal{X}_{\mathcal{D}^+_s}$)**

(i) $\text{ord}(\mathcal{X}_s) = h^*(\Delta^\vee)$;

(ii) Let $\mathcal{O}$ be an $\mathcal{X}_s$-orbit in $\mathcal{X}_{\mathcal{D}^+_s}$. Then $\frac{1}{\#\mathcal{O}} \sum_{\Gamma \in \mathcal{O}} \#\Gamma = \frac{\#(\mathcal{D}^+_s)}{h^*(\Delta^\vee)}$. 
Conjecture 3 is true for $\mathbf{B}_n$, $\mathbf{F}_4$, and $\mathbf{G}_2$, where the number of $\mathcal{X}_s$-orbits equals 1, 3, and 1, respectively.

It is also verified for $\mathbf{C}_n$, $n \leq 5$.

$\Delta^+ \setminus \Pi(\mathbf{C}_n) \simeq \Delta^+_s(\mathbf{C}_n)$ (hence $\mathcal{A}\Pi(\Delta^+ \setminus \Pi)$ and $\mathcal{A}\Pi(\Delta^+_s)$ are also isomorphic). There is a more precise conjecture in this case:

**Conjecture 4**

For $\Delta^+_s(\mathbf{C}_n)$, every $\mathcal{X}_s$-orbit is of cardinality $2n - 1 = h^*(\mathbf{B}_n)$. Each $\mathcal{X}_s$-orbit contains a unique antichain lying in $\Delta^+(\alpha_1, \ldots, \alpha_{n-2}) \simeq \Delta^+(\mathbf{A}_{n-2})$.

Since $\#(\mathcal{A}\Pi(\Delta^+_s)) = \binom{2n-1}{n}$ for $\mathbf{C}_n$ (Panyushev, 2004), Conjecture 4 would imply that the number of $\mathcal{X}_s$-orbits equals $\frac{1}{2n-1} \binom{2n-1}{n}$, the $(n-1)$-th Catalan number. This conjecture also provides a canonical representative in each $\mathcal{X}_s$-orbit in $\mathcal{A}\Pi(\Delta^+_s(\mathbf{C}_n))$. 
Fact: $\Delta^+ \setminus \Pi \cong \Delta_s^+$
More possibilities

- Similar conjecture can be formulated for $\Delta_s^+ \setminus \Pi_s$:
  - Everything is easy for $B_n, F_4, G_2$.
  - We also have $\Delta_s^+ \setminus \Pi_s(C_n) \simeq \Delta^+(C_{n-1})$;

- There is a unique non-reduced irreducible root system $BC_n$, where $\Delta^+(BC_n) \simeq \Delta^+ \setminus \Pi(C_{n+1})$. 
The OY-number

Here $\Delta = \Delta(A_n) = \Delta(\mathfrak{sl}_{n+1})$. We describe an $\mathfrak{X}$-invariant function $Y : \mathfrak{A}n(\Delta^+) \to \mathbb{N}$, which is found by Oksana Yakimova.

Let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be an arbitrary antichain in $\Delta^+$ and $I = I(\Gamma)$ the corresponding upper ideal, so that $\Gamma = I_{\text{min}}$. To each $\gamma_s$, we attach certain integer as follows. Clearly, $I \setminus \{\gamma_s\}$ is again an upper ideal. Set

$$r_\Gamma(\gamma_s) := \#(I \setminus \{\gamma_s\})_{\text{min}} - \#I_{\text{min}} + 1.$$

For $\mathfrak{sl}_{n+1}$, the difference between the numbers of minimal elements of $I$ and $I \setminus \{\gamma_s\}$ always belongs to $\{-1, 0, 1\}$. Therefore $r_\Gamma(\gamma_s) \in \{0, 1, 2\}$. The OY-number of $\Gamma$ is defined by

$$Y(\Gamma) := \sum_{s=1}^{k} r_\Gamma(\gamma_s).$$

We specially set $Y(\emptyset) = 0$. 
Example

- For $\Gamma = \Pi = \{\alpha_1, \ldots, \alpha_n\}$, we have $y(\Pi) = 0$. More generally, the same is true for $\Gamma = \Delta(i)$.
- For $\Gamma = \{\alpha_1, \alpha_3, \ldots\}$ (all simple roots with odd numbers) or $\Gamma = \{\alpha_2, \alpha_4, \ldots\}$ (all simple roots with even numbers), we have $y(\Gamma) = n - 1$.

Theorem (O. Yakimova)

The OY-number is $\mathcal{X}$-invariant, i.e., $y(\Gamma) = y(\mathcal{X}(\Gamma))$ for all $\Gamma \in A_n(\Delta^+)$. 

- The minimal (resp. maximal) value of $y$ is 0 (resp. $n - 1$).
- Each of them is attained on a unique $\mathcal{X}$-orbit.

The above definition of $y(\Gamma)$ can be repeated verbatim for any other root system. However, such a function will not be $\mathcal{X}$-invariant.
A duality for $\mathcal{A}n(\Delta^+)$

- For $\Delta(\mathcal{A}_n)$, there is an involutory map ("duality")
  $\ast : \mathcal{A}n(\Delta^+) \rightarrow \mathcal{A}n(\Delta^+)$ (Panyushev, 2004).

- $\Gamma^*$ is called the dual antichain for $\Gamma$.

For $i \leq j$, the root $\alpha_i + \ldots + \alpha_j$ is denoted by $(i, j)$. If
$\Gamma = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ with $i_1 < \cdots < i_k$, then it is represented as an array:

$$\Gamma = \begin{pmatrix} i_1 & \cdots & i_k \\
                     j_1 & \cdots & j_k \end{pmatrix}.$$  

Set $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$. That is, $\Gamma = (I, J)$ is
determined by two strictly increasing sequences of equal cardinalities
lying in $[n] := \{1, \ldots, n\}$ such that $I \leq J$ (componentwise). Then
$\Gamma^* = (I^*, J^*)$ is defined by

$$I^* := [n] \setminus J \text{ and } J^* := [n] \setminus I.$$
The duality has the following properties:

1. \( \# \Gamma + \#(\Gamma^*) = n; \)
2. If \( \Gamma \subset \Pi \), then \( \Gamma^* = \Pi \setminus \Gamma; \)
3. \( \Delta(i)^* = \Delta(n+2-i). \)

**Theorem**

- For any \( \Gamma \in A_n(\Delta^+) \), we have \( \mathcal{X}(\Gamma)^* = \mathcal{X}^{-1}(\Gamma^*). \)
- \( y(\Gamma) = y(\Gamma^*). \)
Appendix: computations for $F_4$

$F_4$, $\mathcal{An}(\Delta^+)$

We use the numbering of simple roots from [Vinberg–Onishchik]. The positive root $\beta = \sum_{i=1}^{4} n_i \alpha_i$ is denoted by $(n_1 n_2 n_3 n_4)$. For instance, $\theta = (2432)$ and $\theta_s = (2321)$.

$\#\mathcal{An}(\Delta^+) = 105$ and $h = 12$. There are eleven $\mathcal{X}$-orbits: eight orbits of cardinality 12 and orbits of cardinality 2, 3, and 4.

We indicate representatives and cardinalities for all $\mathcal{X}$-orbits:

- $\{1000\} - 12$
- $\{0100\} - 12$
- $\{0010\} - 12$
- $\{0001\} - 12$
- $\{0011\} - 12$
- $\{1100\} - 12$
- $\{1111\} - 12$
- $\{2432\} - 12$ (the standard orbit)
- $\{1000, 0010\} - 2$
- $\{0110\} - 3$
- $\{0001, 1110\} - 4$. 

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Appendix: computations for $F_4$

All antichains

- long root
- short root

$\alpha_4 \Rightarrow \alpha_3 \Rightarrow \alpha_2 \Rightarrow \alpha_1 = [0000]$
F₄, An(Δ⁺ \ \ Π)

#An(Δ⁺ \ \ Π) = 66 and h − 1 = 11. The notation Γ → Γ’ means Γ’ = x₀(Γ). The x₀-orbits are:

1) The standard one:
Δ(11) = {2432} → {2431} → ⋯ → Δ(2) → ∅ → Δ(11);

2) \{1321\} → \{2221\} → \{1321, 2211\} → \{1221, 2210\} → \{0221, 1211\} → \{0211, 1111, 2210\} → \{0111, 1210\} → \{0011, 0210, 1110\} → \{0110, 1100\} → \{0011\} → \{2210\} → \{1321\};

3) \{1221\} → \{0221, 2211\} → \{1211, 2210\} → \{0221, 1111, 1210\} → \{0211, 1110\} → \{0111, 0210, 1100\} → \{0011, 0110\} → \{1100\} → \{0221\} → \{2211\} → \{1321, 2210\} → \{1221\};

4) \{1211\} → \{0221, 1111, 2210\} → \{0211, 1210\} → \{1111, 0210\} → \{0111, 1110\} → \{0011, 0210, 1100\} → \{0110\} → \{0011, 1100\} → \{0210\} → \{1111\} → \{0221, 2210\} → \{1211\};
Appendix: computations for $F_4$

Strictly positive antichains

5) \{1210\} \leadsto \{0221, 1111\} \leadsto \{0211, 2210\} \leadsto \{1111, 1210\} \leadsto
\{0221, 1110\} \leadsto \{0211, 1100\} \leadsto \{0111, 0210\} \leadsto \{0011, 1110\} \leadsto
\{0210, 1100\} \leadsto \{0111\} \leadsto \{0011, 2210\} \leadsto \{1210\};

6) \{1110\} \leadsto \{0221, 1100\} \leadsto \{0211\} \leadsto \{1111, 2210\} \leadsto
\{0221, 1210\} \leadsto \{0211, 1111\} \leadsto \{0111, 2210\} \leadsto \{0011, 1210\} \leadsto
\{0210, 1110\} \leadsto \{0111, 1100\} \leadsto \{0011, 0210\} \leadsto \{1110\}.

Each orbit consists of 11 antichains.
F_4, \mathcal{An}(\Delta_s^+) \\

\#\mathcal{An}(\Delta_s^+) = 21 \text{ and } h^* = 9. \text{ The } \mathcal{X}_s\text{-orbits are:} \\
1) \text{ standard: } \Delta_s(8) = \{2321\} \leadsto \{1321\} \leadsto \cdots \leadsto \Delta_s(1) = \{1000, 0100\} \leadsto \emptyset \leadsto \Delta_s(8); \\
2) \{0100\} \leadsto \{1000\} \leadsto \{0111\} \leadsto \{1210\} \leadsto \{1111\} \leadsto \{0111, 1210\} \leadsto \{1110\} \leadsto \{0111, 1100\} \leadsto \{0110, 1000\} \leadsto \{0100\}; \\
3) \{1100\} \leadsto \{0111, 1000\} \leadsto \{0110\} \leadsto \{1100\}. 
Bonus: $H_3$

Question

Is there a "poset of positive roots" for $H_3$ and $H_4$?

The exponents of $H_3$ are 1, 5, 9. Therefore one should have

$$
\#\mathcal{A}_n(\Delta^+) = \frac{12 \cdot 16 \cdot 20}{2 \cdot 6 \cdot 10} = 32 \quad \text{and} \quad \#\mathcal{A}_n(\Delta^+ \setminus \Pi) = \frac{10 \cdot 14 \cdot 18}{2 \cdot 6 \cdot 10} = 21
$$

- The (generalised) Narayana polynomial for $\mathcal{A}_n(\Delta^+)$ should be $1 + 15t + 15t^2 + t^3$.
- Analogues of Conjectures 1–3 should hold.

The answer for $H_3$ is "yes"!
Positive roots for $H_3$

Coxeter graph of $H_3$

Figure: The Hasse diagram of $\Delta^+ (H_3)$