Linear Response and Periodic Orbits

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Mathematics is the part of physics, where experiments are cheap

V. Arnold

Motivation

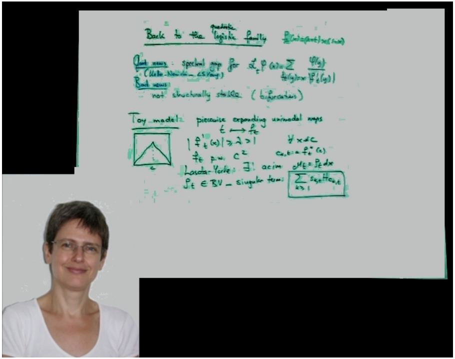
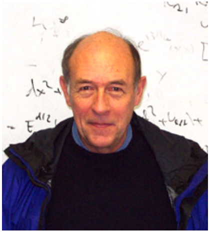
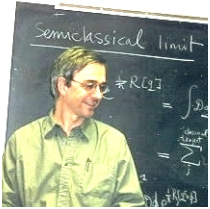


Figure: Initially, I learnt about the problem from the ICM(2014) talk by prof. Viviane Baladi, “Linear Response or Else”. It was written on transparences and presented in a dark room, so the speaker appeared like a ghost.

Physical Settings (Ruelle)





Linear response theory deals with the way a physical system reacts to a small change in the applied forces or the control parameters. The system starts in an equilibrium or a steady state ρ, and is subjected to a small perturbation x, which may depend on time. In first approximation, the change $\Delta\rho$ of ρ is assumed to be linear in the perturbation x.

Figure: two great interpreters: IHES prof. David Ruelle and Georgia Tech prof. Predrag Cvitanović.

Mathematical Settings (Baladi)

- a one-paramter family T_λ of diffeomorphisms of a *compact manifold* M, continuously depending on the parameter λ;
- a large set Λ of paramter values, with accumulation point at 0, such that for any $\lambda \in \Lambda$ the transformation T_λ admits a unique SRB measure μ_λ;

Question

How smooth is the map $\lambda \to \mu_\lambda$?

Definition (Linear Response)

The dynamical system (T_0, M, μ_0) has *linear response*, if the map $\lambda \to \int g d\mu_\lambda$ is differentiable for any $g \in C^1(M)$.

The invariant measure

Definition (Sinai-Ruelle-Bowen Measures)

Let f be a C^2 diffeomorphism of a compact manifold M with an Axiom A attractor S. The *SRB* measure is a unique f-invariant Borel probability measure μ on S such that

1. μ gives absolutely continuous conditional measures on unstable manifolds;
2. the metric entropy $h_\mu(f)$ satisfies $\int |\det(Df|_{E_u})| d\mu = h_\mu(f)$;
3. there is a set $V \subset M$ having full Lebesgue measure such that for every continuous observable $\varphi \colon M \to \mathbb{R}$ and any $x \in V$ we have $\frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k x) \to \int \varphi d\mu$.

First result (flows)

Theorem (Ruelle, generalised by Dolgopyat later)

Let a C^3 vector field $v_0 + \lambda v$ define an axiom A flow f_λ^t on M with an attractor S_λ, depending continuously on $\lambda \in (-\varepsilon, \varepsilon)$. Then $\exists!$ SRB measure μ_λ with $\mathrm{supp}\mu_\lambda = S_\lambda$. Furthermore

1. *for any C^2 function $g\colon M \to \mathbb{R}$ the map $\lambda \mapsto \int g d\mu_\lambda$ is C^1 on $(-\varepsilon, \varepsilon)$;*
2. $\frac{\partial}{\partial \lambda} \int g d\mu_\lambda = \lim_{\omega \to +0} \kappa_\lambda(\omega)$, *where*

$$\kappa_\lambda(\omega) = \int_0^\infty e^{i\omega t} \int v(x) \cdot \nabla_x (g \circ f_\lambda^t) \mu_\lambda(dx)$$

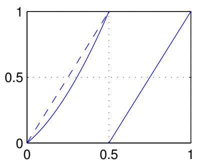
3. *The function $\kappa_\lambda(\omega)$ is holomorphic for $\Im\omega > 0$, extends meromorphically to $\Im\omega > -a$ and has no pole at 0.*

Recent result

Theorem (Baladi-Todd,Korepanov)

Consider a family $f_\lambda\colon (0,1) \to (0,1)$ of Pomeau-Manneville type maps with slow decay of correlations given by

$$f_\lambda(x) = \begin{cases} x(1+2^\lambda x^\lambda) & \text{if } x \in (0,1/2) \\ 2x-1 & \text{if } x \in (1/2,1) \end{cases}$$



where $\lambda \in [0,1)$. Then each f_λ admits a unique a.c. invariant probability measure μ_λ and for any $\varphi \in C^1[0,1]$ the map $\lambda \mapsto \int \varphi d\mu_\lambda$ is continuously differentiable on $(0,1)$.

However, explicit quontitative estimates can be useful for constructing (contr)examples.

Aim

Our goal

Given a family f_λ of transformations of a compact manifold M, admitting a unique SRB measure μ_λ, provide an efficient algorithm for numerical computation of the power series coefficients A and B in expansion

$$\int g d\mu_\lambda = \int g d\mu_0 + A\lambda + B\lambda^2 + o(\lambda^2)$$

for any test function $g \in C^\omega(M)$, whenever the linear response holds.

We will consider two cases

1. Expanding maps of the unit circle $\mathbb{T}^1$;
2. Anosov diffeomorphisms of the torus $\mathbb{T}^2$.

Main Result

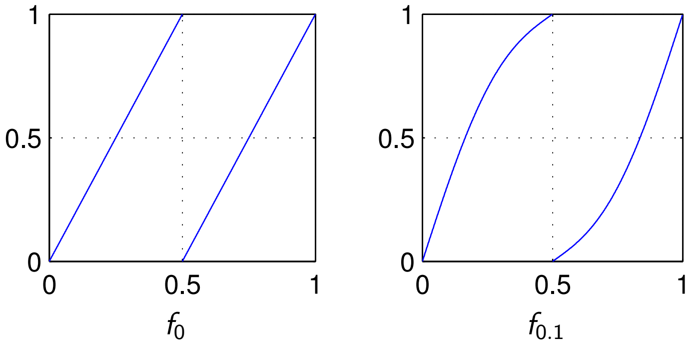
Theorem

Let f_λ be a C^2 family of expanding maps of the circle, (or Anosov diffeomorphisms of a torus) let μ_λ be the a.c. invariant probability measure and let g be a C^ω observable. Then

1. *The partial derivatives $A = \sum_{k=0}^{\infty} a_k$ and $B = \sum_{k=1}^{\infty} b_k$ may be computed as sums absolutely convergent series;*
2. *The k'th terms of both series are defined in terms of periodic points of period $\leq k$;*
3. *The partial sums $A_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} a_k$ and $B_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} b_k$ of the first n terms converge superexponentialy to A and B, respectively.*

Expanding maps of the circle (I)

$f_\lambda(x) = 2x + \lambda \sin(2\pi x) \mod 1$, where $\lambda \in (-1/2\pi, 1/2\pi)$
An observable $g = \cos(2\pi x)$.



Expanding maps of the circle (II)

The method suggested gives both (rather scary) analytic formulae for A and B and numerical approximations.

n	A_n	B_n
4	20.2762085	−1256.3094
5	−1.5659504	113.473941
6	0.0757309	1.12546977
7	−0.0018976	7.84724909
8	$2.503 \cdot 10^{-5}$	7.65567051
9	$-1.73 \cdot 10^{-7}$	7.65805840
10	$6.24 \cdot 10^{-10}$	7.65805063
11	$-1.15 \cdot 10^{-12}$	7.65805056
12	$1.42 \cdot 10^{-13}$	7.65805056

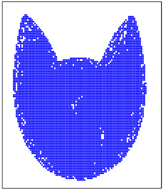
$$\frac{\partial}{\partial \lambda} \int g d\mu_\lambda = \lim_{n\to\infty} A_n = 0;$$

$$\frac{\partial^2}{\partial \lambda^2} \int g d\mu_\lambda = \lim_{n\to\infty} B_n = 7.66\ldots$$

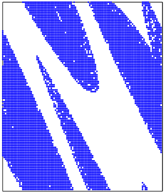
Anosov diffeomorphisms of the torus

$$f_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \lambda \begin{pmatrix} \cos(2\pi x) \\ 0 \end{pmatrix} \text{ mod } 1, \lambda \in \left(-\frac{1}{2\pi}, \frac{1}{2\pi}\right)$$

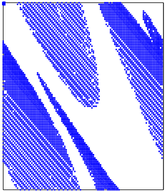
An observable $g(x, y) = \sin(19 \sin(2\pi x) + 41 \cos(2\pi y))$.



$D \subset \mathbb{T}^2$



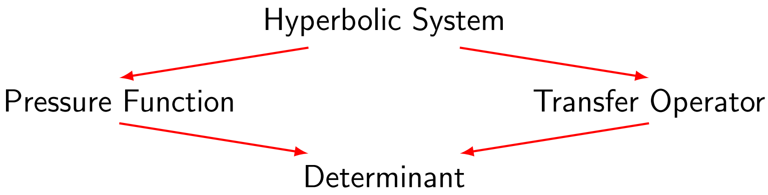
$f_0(D)$



$f_{0.1}(D)$

$$A = \frac{\partial}{\partial \lambda} \int g d\mu_\lambda = \lim_{n\to\infty} A_n = 0.002790864776\ldots$$

Thermodynamic formalism



Definition

Let $F_\lambda(x) \overset{\text{def}}{=} -\log|f_\lambda'(x)|$ be a C^ω function. The *pressure function* is $P(F_\lambda) \overset{\text{def}}{=} \sup_{m\in\mathcal{M}}\{h(m) + \int F_\lambda dm\}$ where $\mathcal{M}$ is the set of f_λ-invariant probability measures $h(m)$ is the entropy. Supremum is achieved at SRB measure μ_λ.

For any $g \in C^\omega$, the pressure $P(F_\lambda + tg)$ is analytic and

$$\left.\frac{\partial P(F_\lambda + tg)}{\partial t}\right|_{t=0} = \int g d\mu_\lambda$$

Transfer operator

Definition

We let B be the Banach space of complex-valued bounded analytic functions on $U \supset \mathbb{T}^1$ with supremum norm $\|\cdot\|_\infty$. To a family of maps $F_\lambda \in B$ and a test function $g \in B$ we associate a family of transfer operators $\mathcal{L}_{u,\lambda,g} : B \to B$:

$$(\mathcal{L}_{u,\lambda,g} f)(x) = \sum_k e^{(F_\lambda - ug)(T_k x)} f(T_k x), \quad u \in \mathbb{R},\ \lambda \in (-\varepsilon, \varepsilon);$$

where $T_k : U \to U$ are C^ω contractions $\overline{T_k(U)} \subset U$, such that $F_\lambda \circ T_k$ is the identity map.

Determinant

Theorem (Grothendieck–Ruelle)

The transfer operator is nuclear. Its determinant is an entire function in z. $d\colon \mathbb{C}\times\mathbb{R}\times(-\varepsilon,\varepsilon)\times C^\omega(U)\to\mathbb{C}$ is given by

$$d(z,u,\lambda,g) \stackrel{\text{def}}{=} \det(I - z\mathcal{L}_{u,\lambda,g}) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n} trace(\mathcal{L}^n_{u,\lambda})\right)$$

Lemma (Ruelle)

$$d(z,u,\lambda,g) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{T^n_\lambda x_\lambda = x_\lambda}\frac{\exp(-ug^n(x_\lambda))}{|(T^n_\lambda)'(x_\lambda)|-1},\right)$$

where $g^n(x_\lambda) = \sum_{k=0}^{n-1} g(T^k_\lambda x_\lambda)$.

Magic of theromodynamics

Lemma (Ruelle)

For any $z \in \mathbb{C}$, $\lambda \in (-\varepsilon, \varepsilon)$, $u \in \mathbb{R}$, and $g \in C^\omega(U)$ we have that:

1. $d(z, u, \lambda, g)$ *converges to an analytic function for* $|z| < e^{-P(F_\lambda - ug)}$;
2. $d(z, u, \lambda, g)$ *has an analytic extension in* $z \in \mathbb{C}$ *to the entire complex plane* $\mathbb{C}$;
3. $z \mapsto d(z, u, \lambda, g)$ *has a simple zero at* $z(u, \lambda, g) = e^{-P(F_\lambda - ug)}$.

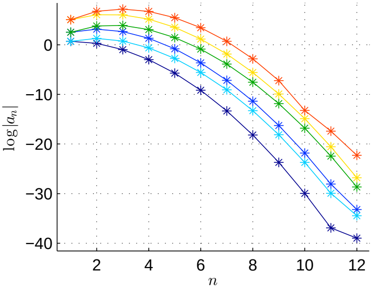
Lemma (Grothendieck–Ruelle)

The powerseries coefficients of the determinant decrease superexponentially and uniformly in $u \in \mathbb{R}$ *and* $\lambda \in (-\varepsilon, \varepsilon)$.

Coefficients of the power series

$$d(z, u, \lambda, g) = 1 + \sum_{n=1}^{\infty} a_n(u, \lambda, g) z^n$$

Using the method presented, an 8 years old (dob March 2007) coffee-fed laptop can compute (in about 2 minutes)...



The plot in logarithmic scale of sums of coefficients $|a_n|$ (dark blue) and partial derivatives $\left|\frac{\partial a_n}{\partial u}\right|$ (blue), $\left|\frac{\partial a_n}{\partial \lambda}\right|$ (light blue), $\left|\frac{\partial^2 a_n}{\partial u \partial \lambda}\right|$ (green), $\left|\frac{\partial^2 a_n}{\partial \lambda^2}\right|$ (yellow), and $\left|\frac{\partial^3 a_n}{\partial u \partial^2 \lambda}\right|$ (red) evaluated at $\lambda = 0$, $u = 0$.

Sometimes, linear response brakes down, but...

Pamela May as the Princess Aurora in Sleeping Beauty at the Royal Opera House in Covent Garden, 1960s.

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