There are computable measures which are comparable but are not computably comparable

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The question

- Inifinite binary sequences
- Probabilistic measures
- Comparison: couplings (measures on pairs) $\begin{matrix}0&1&1\\0&0&1\end{matrix}\quad\xcancel{\begin{matrix}0\\1\end{matrix}}$
- Computable measures
- Can we compare computable measures using only computable couplings?
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Probabilistic measures

Minimal σ-algebra that allows statements $\omega_i = c$

It is enough to specify measures for all prefixes

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Measure comparison

- Symbols: $1 > 0$
- Words: pointwise
- Measures on finite words:
  coupling is a measure on pairs $\begin{smallmatrix} x \\ y \end{smallmatrix}$
  Coupling, consistent with the ordering $\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}$ $\quad \xcancel{\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}}$
  $\mu \geqslant \nu$ when there is a coupling of μ and ν, consistent with the ordering
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Computable measures

We need to specify measures of all possible finite prefixes

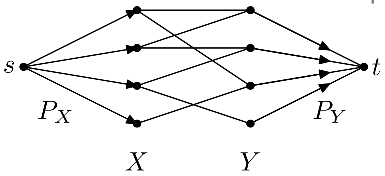
Strong definition: the measure values are rational and can be calculated precisely

Weak definition: given a value $\varepsilon > 0$, the approximation algorithm gives an upper and a lower bound with difference less than ε.

Computable couplings

The question: is there a computable coupling consistent with $\geqslant$ for every two comparable computable measures?

For finite words the answer is positive



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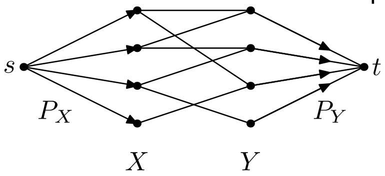
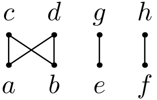


Construction

Construct in a larger ordered alphabet first



Locally, there should be many couplings

c—a d—b   c—b d—a

Long-range correlations based on long-running computations

Let g—e and h—f make us to choose c—a or c—b

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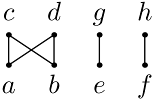

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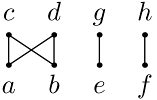

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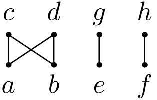

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Construction details

Enumerable inseparable sets

Symbols on the positions $2n$ and $2m + 1$ are dependent if the number n is printed at the step number m

Measures are computable in the strong sense, all comparing couplings are undecidable and there exist $0'$-decidable one.

Local choice of coupling describes the long-range correlation that happened

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Thanks for your attention

Questions?