Desingularizations of Schubert varieties in double Grassmannians

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Abstract

Let $X = \mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$ be the direct product of two Grassmann varieties of k- and l-planes in a finite-dimensional vector space V, and let $B \subset \mathrm{GL}(V)$ be the isotropy group of a complete flag in V. One can consider B-orbits in X in analogy with Schubert cells in Grassmannians. We describe this set of orbits combinatorially and construct desingularizations for the closures of these orbits, analogous to the Bott–Samelson desingularizations for Schubert varieties.

1 Introduction

Let V be a finite-dimensional vector space. We are interested in describing pairs of subspaces in V of fixed dimensions k and l up to a change of coordinates given by the group $B \subset \mathrm{GL}(V)$ of non-degenerate upper-triangular matrices. So, what we describe is the decomposition into B-orbits of the direct product of two Grassmann varieties $X = \mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$. This decomposition is analogous to the Schubert decomposition for Grassmannians, or to the Ehresmann–Bruhat decomposition for complete flags.

The combinatorial description of orbits in X was given (as a particular case of some very general problem) in the paper by Magyar, Weyman and Zelevinsky [MWZ]. The description given below does not refer to these results — in this case everything can be done using only some elementary linear algebra. This is a generalization of the description of orbits in the symmetric space $\mathrm{GL}_{k+l}/(\mathrm{GL}_k \times \mathrm{GL}_l)$, that was obtained by Stéphane Pin in his thesis [P].

We also regard the closures of these B-orbits. They can be considered as analogues of Schubert varieties in Grassmannians. We are interested in their singularities. The singularities of Schubert varieties are well-known objects. They admit nice desingularizations, constructed by Bott and Samelson. They are normal, rational, their singular loci can be described explicitly. Good references on this topic are, for instance, [B2] and [M]. So, it is natural to ask the same questions (resolutions of singularities, normality, rationality) for the case of B-orbit closures in X. In this paper we construct desingularizations of these varieties.

Our interest in this problem is motivated by the recent paper [BZ] by G. Bobinski and G. Zwara, when they prove that the singularities of orbit closures in representations of quivers of type D are equivalent to the singularities of Schubert varieties in double Grassmannians.

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2 Description of orbits

2.1 Notation

Let V be an n-dimensional vector space over a field $\mathbb{K}$. The results of Section 2 are valid over an arbitrary ground field; however, in Sections 3 and 4 we assume $\mathbb{K}$ be algebraically closed. Let $k, l < n$ positive integers. The direct product $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$ is denoted by X. Usually we do not make any difference between points of X and the corresponding configurations of subspaces (U, W), where $U, W \subset V$, $\dim U = k$, $\dim W = l$.

We fix a Borel subgroup B in $\mathrm{GL}(V)$. Let $V_\bullet = (V_1, \ldots, V_n = V)$ be the complete flag in V stabilized by B.

2.2 Combinatorial description

In this section we will introduce some combinatorial objects that parametrize pairs of subspaces up to B-action. Namely, orbits will be parametrized by triples consisting of two Young diagrams contained in the rectangles of size $k \times (n-k)$ and $l \times (n-l)$, respectively, and an involutive permutation of S_n.

Together with constructing these data we will also construct some "canonical" bases in subspaces U, W, and V, respectively.

Proposition 1. *(i). There exist ordered bases $(u_1, \dots, u_k)$, $(w_1, \dots, w_l)$, and $(v_1, \dots, v_n)$ of U, W, and V, respectively, such that:*

- *$V_i = \langle v_1, \dots, v_i \rangle$ for each $i \in \{1, \dots, n\}$ (angle brackets stand for the linear span of vectors);*
- *$u_i = v_{\alpha_i}$, where $i \in \{1, \dots, k\}$, and $\{\alpha_1, \dots, \alpha_k\} \subset \{1, \dots, n\}$;*
- *The w_i are either basic vectors of V or vectors with two-elementary "support": $w_i = v_{\beta_i}$ or $w_i = v_{\gamma_i} + v_{\delta_i}$, where $\gamma_i > \delta_i$; moreover, in the latter case $v_{\gamma_i} \in U$ (that is, $\{\gamma_1, \dots, \gamma_r\} \subset \{\alpha_1, \dots, \alpha_k\}$).*
- *All the β_i, γ_i and δ_i are distinct; moreover, all the δ_i are distinct from the α_i.*

(ii). With the notation of (i), define a permutation $\sigma \in S_n$ as the product of all the transpositions (γ_i, δ_i). Their supports do not intersect, so this product does not depend of their order.

Then for the given pair (U, W) the subsets $\bar{\alpha} = \{\alpha_1, \dots, \alpha_k\}$, $\bar{\beta} = \{\beta_1, \dots, \beta_{l-r}\}$, $\bar{\gamma} = \{\gamma_1, \dots, \gamma_r\}$ of $\{1, \dots, n\}$, and the permutation σ are independent of the choice of bases in U, W, and V.

Proof. (i) We will prove this by induction over n.

If $n = 1$, there is nothing to prove.

For arbitrary n, take a nonzero vector $v_1 \in V_1$, and consider the following cases:

- $v_1 \notin U$, $v_1 \notin W$. Take the quotient $\bar{V} = V/\langle v_1 \rangle$ with the flag $\bar{V}_\bullet = \bar{V}_2 \subset \dots \subset \bar{V}_n$, consider the image of our configuration, that consists of the subspaces $\bar{U} \cong U$ and $\bar{W} \cong W$, and apply the induction hypothesis to this configuration. Let us choose ordered bases $\{\bar{u}_1, \dots, \bar{u}_k\}$, $\{\bar{w}_1, \dots, \bar{w}_l\}$, and $\{\bar{v}_1, \dots, \bar{v}_{n-1}\}$ in $\bar{U}$, $\bar{W}$, and $\bar{V}$. Then we choose a lift $\imath \colon \bar{V} \hookrightarrow V$. Now take the pre-images of these basis vectors in V in the following way: $u_i = \imath(\bar{u}_i)$, $w_i = \imath(\bar{w}_i)$, $v_i = \imath(\bar{v}_{i-1})$. We get the required triple of bases.
- $v_1 \in U$, $v_1 \notin W$. Set $u_1 = v_1$ and again apply the induction hypothesis to the quotient $\bar{V} = V/\langle v_1 \rangle$ with the flag $\bar{V}_\bullet$ and the configuration $(\bar{U}, \bar{W})$. The only difference is that in this case $\dim \bar{U} = \dim U - 1$. After that we take the pre-images of the bases of $\bar{U}$, $\bar{W}$, and $\bar{V}$ in V in a similar way.

- The case when $v_1 \notin U$, $v_1 \in W$, is analogous to the previous one (we set $w_1 = v_1$).
- If $v_1 \in U \cap W$, let us set $u_1 = w_1 = v_1$ and again apply the induction.
- The most interesting case is the last one: $v_1 \in U+W$, but it does not belong to any of these two subspaces. Consider then the set of vectors $S = \{v \mid v \in U, v_1 + v \in W\}$. Since v_1 belongs to the sum $U + W$, this set is nonempty. Now let j be the minimal number such that V_j contains vectors from S, and $v_j \in V_j \cap S$. Let us set $u_1 = v_j$, $w_1 = v_1 + v_j$. Now apply the induction hypothesis to the $(n-2)$-dimensional space $\bar{V} = V/\langle v_1, v_j\rangle$ and to the configuration of two subspaces $\bar{U}$, $\bar{W}$, and the flag

  $$\begin{aligned}\bar{V}_\bullet = V_2/V_1 \subset \cdots \subset V_{j-1}/V_1 = \\ = V_j/\langle v_1, v_j\rangle \subset V_{j+1}/\langle v_1, v_j\rangle \subset \cdots \subset V_n/\langle v_1, v_j\rangle.\end{aligned}$$

  We take the pre-images of vectors from $\bar{V}$ to V as follows:

  $$v_i = \imath(\bar{v}_{i-1}),\ \text{if } i \in [2, j-1]; \qquad v_i = \imath(\bar{v}_{i-2}) \text{ if } i \in [j+1, n],$$

  where, as above, $\imath$ is an embedding of $\bar{V}$ into V. We have already defined the vectors v_1 and v_j.

(ii) Take a configuration (U, W) and assume that there exist two triples of ordered bases $((u_1, \ldots, u_k), (w_1, \ldots, w_l), (v_1, \ldots, v_n))$ and $((u'_1, \ldots, u'_k), (w'_1, \ldots, w'_l), (v'_1, \ldots, v'_n))$, satisfying the conditions of (i), such that either the triples of sets $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ and $(\bar{\alpha}', \bar{\beta}', \bar{\gamma}')$, or the permutations σ and σ', corresponding to the first and the second triple of bases, respectively, are not equal.

The set $\bar{\alpha}$ can be described as follows. $i \in \bar{\alpha}$ iff $\dim U \cap V_i > \dim U \cap V_{i-1}$. This means that $\bar{\alpha} = \bar{\alpha}'$.

By the same argument we can prove that $\bar{\beta} \cup \bar{\gamma} = \bar{\beta}' \cup \bar{\gamma}'$.

Now let us prove that $\sigma = \sigma'$. This will complete the proof, since $\bar{\beta} = \{j \in \bar{\beta} \cup \bar{\gamma} \mid \sigma(j) = j\}$.

Let j be the minimal number from $\beta \cup \gamma$, such that $\sigma(j) \neq \sigma'(j)$. Suppose that $\sigma(j) < \sigma'(j)$. Two cases may occur:

a) $i := \sigma'(j) \neq j$. First observe that $i \notin \bar{\alpha}$. Consider the subspace

$$\begin{aligned}\tilde{V} = (U \cap V_j) + V_{i-1} = \langle v_s, v_{\alpha_i} \mid s \leq i-1, \alpha_i \in \bar{\alpha} \cup [i, j]\rangle = \\ = \langle v'_s, v'_{\alpha_i} \mid s \leq i-1, \alpha_i \in \bar{\alpha}' \cup [i, j]\rangle.\end{aligned}$$

Let R and R' denote respectively the sets $\{r \in \bar{\beta} \cup \bar{\gamma} \mid r, \sigma(r) \in [1, i-1] \cup (\bar{\alpha} \cap [i, j])\}$ and $\{r \in \bar{\beta} \cup \bar{\gamma} \mid r, \sigma'(r) \in [1, i-1] \cup (\bar{\alpha} \cap [i, j])\}$. One can easily see that

$$\dim \tilde{V} \cap W = \#R = \#R'.$$

But $\sigma(r) = \sigma'(r)$ for all $r \in [1, j-1]$, and j belongs to R and does not belong to R'. That means that the cardinalities of these two sets are different, that gives us the desired contradiction.

b) If $\sigma'(j) = j$, set $i = \sigma(j)$, and proceed as in a). □

Let us now introduce a combinatorial construction that parametrizes configuration types. Namely, having a configuration, we will construct a pair of Young diagrams with some boxes distinguished.

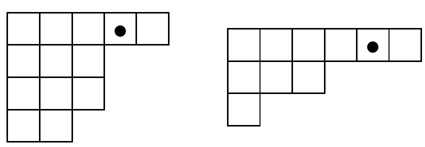
Suppose we have a configuration (U, W) with bases $(u_1, \ldots, u_k)$, $(w_1, \ldots, w_l)$, and $(v_1, \ldots, v_n)$, chosen as in Prop. 1, the sets $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, and the involution σ corresponding to this configuration. Consider a rectangle of size $k \times (n-k)$ and construct a path from its bottom-left to upper-right corner, such that its j-th step is vertical if j belongs to $\bar{\alpha}$ (that is, v_j is equal to some u_i), and horizontal otherwise. This path bounds (from below) the first Young diagram.

The second diagram will be contained in the rectangle of size $l \times (n-l)$. Again, we will construct a path bounding it. Let the j-th step of this path be vertical if $j \in \bar{\beta} \cup \bar{\gamma}$, and horizontal otherwise.

If $j \in \bar{\gamma}$, then the $\sigma(j)$-th step of this path is horizontal. This also means that the j-th and $\sigma(j)$-th steps of the path bounding the first diagram are also vertical and horizontal, respectively. In each diagram, take the box located above the $\sigma(j)$-th step and to the left of the j-th step, and put a dot into this box.

Let us call this pair of diagrams with dots a *marked pair*.

Example. Let $n = 9$, $k = 4$, $l = 3$. Suppose that $\bar{\alpha} = \{3, 5, 6, 9\}$, $\bar{\beta} = \{2, 5\}$, $\bar{\gamma} = \{9\}$, $\sigma = (7, 9)$. Then the corresponding marked pair of diagrams is the following:



Remark. Note that the constructed diagrams (without dots) are the same as the diagrams that correspond to the Schubert cells containing the points $U \in \mathrm{Gr}(k, V)$ and $W \in \mathrm{Gr}(l, V)$. (The correspondence between Schubert cells and Young diagrams is described, for example, in [F], [M], or any other textbook on this subject).

2.3 Stabilizers and dimensions of orbits

Now let us find the stabilizer $\mathrm{GL}(V)_{(U,W)}$ for a given configuration.

Proposition 2. *With the notation of Prop. 1, the stabilizer of a configuration (U, W) written w.r.t. basis $(v_1, \ldots, v_n)$, consists of the upper-triangular matrices $A = (a_{ij}) \in \mathrm{GL}(n)$ satisfying the following conditions:*

1. $a_{\gamma\gamma} = a_{\sigma(\gamma)\sigma(\gamma)}$ *for each* $\gamma \in \bar{\gamma}$;
2. $a_{i\alpha} = 0$ *for each* $\alpha \in \bar{\alpha}$, $i \notin \bar{\alpha}$;
3. $a_{j\beta} = 0$ *for each* $\beta \in \bar{\beta}$ *and* $j \notin \bar{\beta} \cup \bar{\gamma} \cup \sigma(\bar{\gamma})$;
4. $a_{\gamma\beta} = a_{\sigma(\gamma)\beta}$ *for each* $\beta \in \bar{\beta}$ *and* $\gamma \in \bar{\gamma}$, $\gamma < \beta$;
5. $a_{j\gamma} = -a_{j\sigma(\gamma)}$ *for each* $j \notin \bar{\beta} \cup \bar{\gamma} \cup \sigma(\bar{\gamma})$ *and* $\gamma \in \bar{\gamma}$;
6. *for each* $\gamma_1, \gamma_2 \in \bar{\gamma}$, $\gamma_1 < \gamma_2$, *one of the following cases occurs:*
   - $\sigma(\gamma_2) < \sigma(\gamma_1) < \gamma_1 < \gamma_2$: *then* $a_{\gamma_1\gamma_2} = a_{\sigma(\gamma_1)\gamma_2} = a_{\sigma(\gamma_2)\gamma_1} = a_{\sigma(\gamma_1)\sigma(\gamma_2)} = 0$;
   - $\sigma(\gamma_1) < \sigma(\gamma_2) < \gamma_1 < \gamma_2$: *then* $a_{\sigma(\gamma_2)\gamma_1} = a_{\sigma(\gamma_1)\gamma_2} = 0$, $a_{\gamma_1\gamma_2} = a_{\sigma(\gamma_1)\sigma(\gamma_2)}$;
   - $\sigma(\gamma_1) < \gamma_1 < \sigma(\gamma_2) < \gamma_2$: *then* $a_{\sigma(\gamma_1)\gamma_2} = 0$, $a_{\gamma_1\gamma_2} + a_{\gamma_1\sigma(\gamma_2)} = a_{\sigma(\gamma_1)\sigma(\gamma_2)}$.

Corollary 3. *The stabilizer of a configuration (U, W) is a semidirect product of a toric and a unipotent part:*

$$\mathrm{GL}(V)_{(U,W)} = T_{(U,W)} \ltimes U_{(U,W)},$$

where $T_{(U,W)}$ is the subgroup in the group of diagonal matrices defined by the equations 1., *so that* $\dim T_{(U,W)} = n - \#\bar{\gamma}$, *and $U_{(U,W)}$ is the subgroup in the group of unitriangular matrices, defined by the equations* 2.–6.

Definition. The codimension of the toric part of the stabilizer is said to be the *rank* of a configuration (or its corresponding orbit):

$$\mathrm{rk}\ (U, W) := n - \dim T_{(U,W)} = \#\bar{\gamma}.$$

Proof of the proposition. First of all, the stabilizer $B_{(U,W)}$ is formed by upper-triangular matrices, as a subgroup of B.

Next, it preserves the subspace $U = \langle v_{\alpha_1}, \ldots, v_{\alpha_k} \rangle$. This means that a transformation $A \in B_{(U,W)}$ maps each v_{α_i} into a linear combination of v_{α_j}, so all the elements $a_{i\alpha}$, where $\alpha \in \bar{\alpha}$, $i \notin \bar{\alpha}$, vanish. (Note that the zeros in A obtained in this way also form a Young diagram corresponding to the subspace U, rotated 90° clockwise — this proves, in particular, that the dimension of a Schubert cell in a Grassmannian is equal to the number of boxes in the corresponding diagram).

So, the boxes of the first Young diagram are in a one-to-one correspondence with the linear equations defining B_U as a subgroup of the group of upper-triangular matrices: the box located above the i-th (horizontal) step and to the left of the j-th (vertical) step of the corresponding path (denote this box by (i, j)) corresponds to the equation $a_{ij} = 0$.

Similarly, the stabilizer of our configuration preserves the subspace W. This gives us a set of linear equations on the elements a_{ij}, and the number of them is equal to the number of boxes in the second diagram of the corresponding marked pair. Again, we establish a one-to-one correspondence between the boxes of this diagram and these equations, denoting boxes as in the previous paragraph. Here they are:

- $a_{j\beta} = 0$ for each $\beta \in \bar{\beta}$ and $j \notin \bar{\beta} \cap \bar{\gamma} \cap \sigma(\bar{\gamma})$, $j < \beta$. The corresponding box is (j, β);
- $a_{j\gamma} = -a_{j\sigma(\gamma)}$ for each $j \notin \bar{\beta} \cup \bar{\gamma} \cup \sigma(\bar{\gamma})$ and $\gamma \in \bar{\gamma}$, $j < \gamma$. The corresponding box is (j, γ);
- $a_{\sigma(\gamma)\gamma} + a_{\gamma\gamma} - a_{\sigma(\gamma)\sigma(\gamma)} = 0$ for each $\gamma \in \bar{\gamma}$. The corresponding box is $(\sigma(\gamma), \gamma)$;
- $a_{\gamma\beta} = a_{\sigma(\gamma)\beta}$ for each $\beta \in \bar{\beta}$ and $\gamma \in \bar{\gamma}$, $\gamma < \beta$. The corresponding box is $(\sigma(\gamma), \beta)$;
- $a_{\sigma(\gamma_1)\sigma(\gamma_2)} + a_{\sigma(\gamma_1)\gamma_2} = a_{\gamma_1\sigma(\gamma_2)} + a_{\gamma_1\gamma_2}$ for each $\gamma_1 < \gamma_2$. This equation corresponds to the box $(\sigma(\gamma_1), \gamma_2)$.

Bringing all these equations together completes the proof of the proposition. $\square$

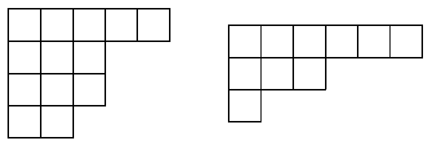
Once we know the stabilizer of a configuration, we can calculate its dimension (and hence the dimension of the orbit $B(U, W) \subset X$).

Analyzing the equations above, one can deduce a combinatorial interpretation of dimension in terms of Young diagrams with dots.

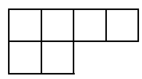
To do this, we have to introduce one more combinatorial notion. Suppose we have two rectangles of size $k \times (n-k)$ and $l \times (n-l)$, respectively, and a path in each of these rectangles bounding a Young diagram (so both paths are of the length n). Consider the set of all numbers i, such that the i-th steps in the paths bounding both diagrams are horizontal, and take the columns in the diagrams lying above these steps. After that let us do the same for those pairs of steps that are "simultaneously vertical", and take the rows to the left of these steps.

The intersection of columns and rows we have taken also forms a Young diagram. Let us call it a *common diagram* corresponding to the given pair of diagrams.

Example. The pair of Young diagrams



has the following common diagram:



By our construction of marked pairs, dots can only be situated in the boxes of the common diagram of a marked pair.

Corollary 4. *Let (U, W) be a configuration of subspaces, and let (Y_1, Y_2) be the corresponding marked pair of Young diagrams, with dots in some boxes of its common diagram Y_{com}.*

Now take the diagram Y_{com}. Take all its boxes with dots and consider all the hooks with spikes in these boxes. Let H be the set of boxes that belong to at least one of these hooks. Then the dimension of the B-orbit of (U, W) equals

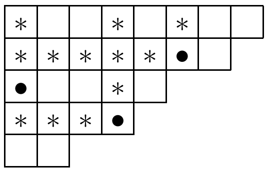
$$\dim B(U, W) = \#Y_1 + \#Y_2 - \#Y_{com} + \#H,$$

where $\#Y$ denotes the number of boxes in Y.

Remark. $\#H$ equals the total number of boxes contained in all the hooks, not the sum of all the hooks' lengths. That means that a box included into two hooks must be counted once, not twice!

Proof. In the proof of Prop. 2 we deal with two systems of linear equations on the matrix entries (a_{ij}), that correspond to stabilizing the subspaces U and W and consist of $\#Y_1$ and $\#Y_2$ equations, respectively. One can easily see that the equations corresponding to the box (i,j) coincide in both systems iff the box (i,j) of the common diagram does not belong to any hook, and also that the system obtained by eliminating these "double" equations is linearly independent. So, the codimension of $B_{(U,W)}$ in B (that is, the dimension of $B(U,W)$) equals $\#Y_1 + \#Y_2 - \#Y_{com} + \#H$. □

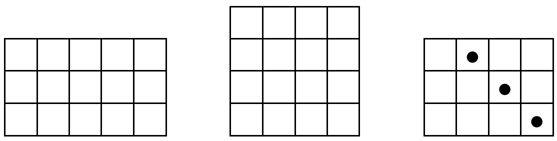
Example. Let the common diagram for a marked pair be as follows:



Then $\#Y_{com} = 26$, $\#H = 15$ (boxes belonging to H are the non-empty ones).

In particular, the dimension formula allows us to describe the minimal, or the most special, and the maximal (open) orbit. The most special orbit is zero-dimensional and corresponds to $Y_1 = Y_2 = \varnothing$. It is the point $(\langle v_1, \dots, v_k\rangle, \langle v_1, \dots, v_l\rangle) \in X$. Both Young diagrams corresponding to the most generic orbit are rectangular, of size $k \times (n-k)$ and $l \times (n-l)$, respectively. So, their common diagram is also a rectangle of size $\min\{k,l\} \times (n - \max\{k,l\})$, with dots situated on a diagonal starting from the bottom-right corner.

Example. For $n = 8$, $k = 3$, and $l = 4$, the combinatorial data corresponding to the maximal orbit are as follows:



Y_1 $\qquad$ Y_2 $\qquad$ Y_{com}

2.4 Decomposition of X into the union of $\mathrm{GL}(V)$-orbits

$\mathrm{GL}(V)$-orbits in X have a much simpler description: the $\mathrm{GL}(V)$-orbit is given only by one natural number, namely, the dimension of the intersection of a k-plane and an l-plane. For this number (denote it by i) we have the inequality

$$\max\{0, k+l-n\} \leq i \leq \min\{k, l\}.$$

Denote the corresponding $\mathrm{GL}(V)$-orbit by X_i:

$$X = \bigsqcup_{i \in \{\max(0,k+l-n),\dots,\min(k,l)\}} X_i.$$

For each B-orbit the dimension of the intersection of the corresponding subspaces is equal to $\#(\bar{\alpha} \cap \bar{\beta})$. This follows from our construction of the combinatorial data corresponding to an orbit.

3 The weak order on the set of orbits

Starting from this point, we work over an algebraically closed ground field $\mathbb{K}$.

In the previous section we described the set of B-orbits in $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$. There exist several partial order structures on this set. The first, and the most natural one, is defined as follows:

Definition. Let $\mathcal{O}$ and $\mathcal{O}'$ be two B-orbits in $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$. We say that $\mathcal{O}$ is less or equal than $\mathcal{O}'$ w.r.t. the *strong* (or *topological*) *order*, iff $\mathcal{O} \subset \bar{\mathcal{O}}'$. (Saying "topological", we speak about the Zariski topology). Notation: $\mathcal{O} \leq \mathcal{O}'$.

There exists another order on this set, usually called the weak order. Here notation and terminology is taken from [B1].

Let W be the Weyl group for $\mathrm{GL}(n)$, and let Δ be the corresponding root system. Denote the simple reflections by $s_1, \dots, s_{n-1}$, and the corresponding simple roots by $\alpha_1, \dots, \alpha_{n-1}$. Let $P_i = B \cup Bs_iB$ be the minimal parabolic subgroup in $\mathrm{GL}(V)$ corresponding to the simple root α_i.

We say that α_i *raises* an orbit $\mathcal{O}$ to $\mathcal{O}'$, if $\bar{\mathcal{O}}' = P_i\bar{\mathcal{O}} \neq \bar{\mathcal{O}}$. In this case, $\dim \mathcal{O}' = \dim \mathcal{O} + 1$. This notion allows us to define the weak order.

Definition. An orbit $\mathcal{O}$ is said to be less or equal than $\mathcal{O}'$ w.r.t. the *weak order* (notation: $\mathcal{O} \preceq \mathcal{O}'$), if $\bar{\mathcal{O}}'$ can be obtained as the result of several consecutive raisings of $\bar{\mathcal{O}}$ by minimal parabolic subgroups:

$$\mathcal{O} \preceq \mathcal{O}' \quad \Leftrightarrow \quad \exists (i_1, \dots, i_r) \colon \bar{\mathcal{O}}' = P_{i_r} \dots P_{i_1} \bar{\mathcal{O}}.$$

Let us represent this relation of order by an oriented graph. Consider a graph $\Gamma(X)$ with vertices indexed by B-orbits in X. Join $\mathcal{O}$ and $\mathcal{O}'$ with an edge of label i, leading to $\mathcal{O}'$, if P_i raises $\mathcal{O}$ to $\mathcal{O}'$.

It is clear that the connected components of $\Gamma(X)$ consist of the B-orbits contained in the same $\mathrm{GL}(V)$-orbit X_i, and that every connected component has a unique maximal element (the B-orbit that is open in X_d).

Our next aim will be to describe minimal elements w.r.t. the weak order in each connected component.

3.1 Combinatorial description of minimal parabolic subgroup action

Consider an orbit $\mathcal{O}$ and the corresponding combinatorial data: the sets $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, and the involution $\sigma \in S_n$. Let the minimal parabolic subgroup $P_i = B \cup Bs_iB$ raise the orbit $\mathcal{O}$ to the orbit $\mathcal{O}' \neq \mathcal{O}$. Now we will describe the combinatorial data $(\bar{\alpha}', \bar{\beta}', \bar{\gamma}', \sigma')$ of $\mathcal{O}'$.

Denote the transposition $(i, i+1) \in S_n$ by τ_i.

The following cases may occur:

1. Suppose that

$$i \in \bar{\alpha}, \quad i \notin \bar{\beta}, \quad i+1 \notin \bar{\alpha}, \quad i+1 \in \bar{\beta},$$

or, vice versa,

$$i \notin \bar{\alpha}, \quad i \in \bar{\beta}, \quad i+1 \in \bar{\alpha}, \quad i+1 \notin \bar{\beta}.$$

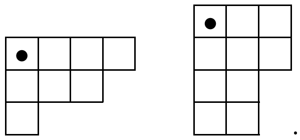
These two variants correspond to two orbits that could be risen to $\mathcal{O}'$. In this case, the new combinatorial data is given as follows:

$$\begin{aligned} \bar{\alpha}' &= \bar{\alpha} \cup \{i+1\} \setminus \{i\}; \\ \bar{\beta}' &= \bar{\beta} \setminus \{i, i+1\}; \\ \bar{\gamma}' &= \bar{\gamma} \cup \{i+1\} \\ \sigma' &= \sigma \cdot \tau_i. \end{aligned}$$

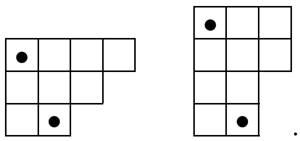
Note that $\mathrm{rk}\ \tilde{\mathcal{O}} = \mathrm{rk}\ \mathcal{O} + 1$, $\dim \tilde{\mathcal{O}} = \dim \mathcal{O} + 1$.

In the language of marked pairs of diagrams, this is represented as follows. If the i-th and the $i+1$-th steps of the path bounding the first diagram form a ravine, and the corresponding intervals of the second diagram form a spike (or, vice versa, we have a spike in the first diagram and a ravine in the second), both these pairs of steps can be replaced by spikes bounding a marked box.

Example. Apply the minimal parabolic subgroup P_2 to the orbit $\mathcal{O} \subset \mathrm{Gr}(3,7) \times \mathrm{Gr}(4,7)$ defined by the following marked pair:



The orbit $\mathcal{O}'$ obtained as the result of this raising is defined by the marked pair



2. In all the other cases $\bar{\alpha}' = \tau_i(\bar{\alpha})$, $\bar{\beta}' = \tau_i(\bar{\beta})$, $\bar{\gamma}' = \tau_i(\bar{\gamma})$, and the permutation $\tilde{\sigma}$ is the result of the conjugation of σ by τ_i:

$$\tilde{\sigma} = \tau_i \sigma \tau_i.$$

The ranks of these orbits are equal: $\mathrm{rk}\ \mathcal{O}' = \mathrm{rk}\ \mathcal{O}$.

3.2 Minimal orbits

Lemma 5. *All minimal B-orbits w.r.t. the weak order in a given $\mathrm{GL}(V)$-orbit have rank 0.*

Proof. Assume the converse. Let $\mathcal{O}$ be a minimal orbit with a nonzero rank, and let $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \sigma)$ the corresponding combinatorial data, such that $\sigma \neq Id$. Let $p \in \bar{\gamma}$, $p' = \sigma(p)$. Without loss of generality we can suppose that there is no other $q \in \bar{\gamma}$, such that $p < q < \sigma(q') < p'$.

Let C_1 denote the set of ravines in the first diagram, situated between p and p' — that is, the set of indices i, such that the i-th step in the first diagram is horizontal, and the $i+1$-th is vertical:

$$C_1 = H_1 \cap (V_1 - 1) \cap \{p, \ldots, p'\}.$$

Similarly, let D_1 denote the set of spikes, — that is, the set of i, such that the i-th step is vertical, and the $i+1$-st is horizontal:

$$D_1 = V_1 \cap (H_1 - 1) \cap \{p, \ldots, p'\}.$$

Denote the same sets for the second diagram by C_2 and D_2. Note that $\#C_1 = \#D_1 + 1$, and $\#C_2 = \#D_2 + 1$ — since $p \in H_{1,2}$, $p' \in V_{1,2}$.

Now take a j, such that $j \in (C_1 \setminus D_2) \cup (C_2 \setminus D_1)$. Let us show that there exists an orbit $\mathcal{O}'$, such that $\bar{\mathcal{O}} = P_j \bar{\mathcal{O}}'$. We describe the combinatorial data for this orbit.

If the permutation σ contains the transposition $(j, j+1)$, then the combinatorial data for $\mathcal{O}'$ is as follows:

$$\begin{aligned} \bar{\alpha}' &= \bar{\alpha} \cup \{j\} \setminus \{j+1\}; \\ \bar{\beta}' &= \bar{\beta} \cup \{j\}; \\ \bar{\gamma}' &= \bar{\gamma} \setminus \{j+1\} \\ \sigma' &= \sigma \cdot \tau_j. \end{aligned}$$

Otherwise $\bar{\alpha}' = \tau_j(\bar{\alpha})$, $\bar{\beta}' = \tau_j(\bar{\beta})$, $\bar{\gamma}' = \tau_j(\bar{\gamma})$, $\sigma' = \tau_j \sigma \tau_j$.

The calculation of the dimensions shows that $\dim \mathcal{O}' = \dim \mathcal{O} - 1$.

To complete the proof, we have to show that the set $(C_1 \setminus D_2) \cup (C_2 \setminus D_1)$ is nonempty:

$$\begin{aligned} \#((C_1 \setminus D_2) \cup (C_2 \setminus D_1)) &\geq \max(\#(C_1 \setminus D_2), \#(C_2 \setminus D_1)) \geq \\ &\geq \max(\#C_1 - \#C_2 + 1, \#C_2 - \#C_1 + 1) \geq 1. \end{aligned}$$

□

After that we can find all the minimal orbits in X_d. One can easily see that each minimal orbit has the following combinatorial data:

$$\begin{aligned} \bar{\alpha} \cup \bar{\beta} &= \{1, \dots, k+l-d\}; \\ \bar{\alpha} \cap \bar{\beta} &= \{1, \dots, d\}; \\ \bar{\gamma} &= \varnothing; \\ \sigma &= Id. \end{aligned}$$

The dimension of all minimal orbits in X_d equals $(k-d)(l-d)$. In particular, that means that they all are closed in X_d. They correspond to decompositions of the set $\{d+1, \dots, k+l-d\}$ into two parts, $\bar{\alpha} \setminus \bar{\beta}$ and $\bar{\beta} \setminus \bar{\alpha}$, so their number is equal to $\binom{k+l-2d}{k-d}$.

Also note that the pair of Young diagrams that corresponds to a minimal orbit is complementary: one can put these two diagrams together so that they will fill a rectangle of size $(k-d) \times (l-d)$.

It is also clear that no other B-orbit corresponds to such pair of Young diagrams. That means that all the minimal orbits are stable under the $(B \times B)$-action, that is, they are direct products of two Schubert cells in two Grassmannians.

These results can be summarized as the following theorem.

Theorem 6. *Each X_d, where $d \in \{\max(k+l-n, 0), \dots, \min(k,l)\}$, contains $\binom{k+l-2d}{k-d}$ minimal orbits. All these orbits are closed in X_d and have dimension $(k-d)(l-d)$. They are direct products of Schubert cells.*

4 Desingularizations of the orbit closures

In this section we construct desingularizations for the B-orbit closures in X. Given a minimal parabolic subgroup P_i and an orbit closure $\bar{\mathcal{O}}$, consider the morphism

$$F_i : P_i \times^B \bar{\mathcal{O}} \to P_i\bar{\mathcal{O}},$$

$$(p, x) \mapsto px.$$

Knop [K] and Richardson–Springer [RS] showed that the following three cases may occur:

- Type U: $P_i\mathcal{O} = \mathcal{O}' \sqcup \mathcal{O}$, and F_i is birational;
- Type N: $P_i\mathcal{O} = \mathcal{O}' \sqcup \mathcal{O}$, and F_i is of degree 2;
- Type T: $P_i\mathcal{O} = \mathcal{O}' \sqcup \mathcal{O} \sqcup \mathcal{O}''$, and F_i is birational. In this case $\dim \mathcal{O}'' = \dim \mathcal{O}$.

It turns out that in our situation the case N never occurs.

Proposition 7. *Let $\mathcal{O}$ be a B-orbit in X and let P_i be a minimal parabolic subgroup raising this orbit. Then the map $F_i : P_i \times^B \mathcal{O} \to P_i\mathcal{O}$ is birational.*

Proof. Choose the canonical representative $x \in \mathcal{O}$ as in Prop. 1. A direct calculation shows that the isotropy group of x in P_i equals the isotropy group of x in B, described in Prop. 2. This implies the birationality of F_i. □

Remark. The two remaining cases correspond to the two possible "raisings" described in the subsection 3.1: (T) corresponds to (1), and (U) corresponds to (2). In the first case, the rank of the orbit is increased by one, and in the second case, it does not change. So, the weak order is compatible with the rank function: if $\mathcal{O} \preceq \mathcal{O}'$, then $\operatorname{rk} \mathcal{O} \leq \operatorname{rk} \mathcal{O}'$. This is true in general for spherical varieties (cf., for instance, [B1]). Note that the strong order is *not* compatible with the rank function.

Proposition 7 together with Theorem 6 allows us to construct desingularizations for $\bar{\mathcal{O}}$'s similar to Bott–Samelson desingularizations of Schubert varieties in Grassmannians.

Given an orbit $\mathcal{O}$, consider a minimal orbit $\mathcal{O}_{min}$ that is less that $\mathcal{O}$ w.r.t. the weak order. That means that there exists a sequence of minimal parabolic subgroups $(P_{i_1}, \dots, P_{i_r})$, such that

$$\bar{\mathcal{O}} = P_{i_r} \dots P_{i_1} \bar{\mathcal{O}}_{min}.$$

So, we can consider the map

$$F\colon P_{i_r} \times^B \cdots \times^B P_{i_1} \times^B \bar{\mathcal{O}}_{min} \to \bar{\mathcal{O}},$$

$$F\colon (p_{i_r}, \dots, p_{i_1}, x) \mapsto p_{i_r} \dots p_{i_1} x.$$

According to Proposition 7, it is birational. But this is not yet a desingularization, because $\bar{\mathcal{O}}_{min}$ can be singular.

The second step of the desingularization consists in constructing a B-equivariant desingularization for $\bar{\mathcal{O}}_{min}$. We have already proved in Theorem 6 that $\bar{\mathcal{O}}_{min}$ can be presented as the direct product

$$\bar{\mathcal{O}}_{min} = X_w \times X_v$$

for some Schubert varieties $X_w \subset \mathrm{Gr}(k, V)$ and $X_v \subset \mathrm{Gr}(l, V)$.

For X_w and X_v one can take Bott–Samelson desingularizations

$$F_w\colon Z_w \to X_w \quad \text{and} \quad F_v\colon Z_v \to X_v.$$

(Details can be found, for instance, in [B2]). So, we get a desingularization

$$F_w \times F_v\colon \quad Z_w \times Z_w \to X_w \times X_v = \bar{\mathcal{O}}_{min}.$$

Having this, we can combine this map with the map F and get the main result of this paper:

Theorem 8. *The map*

$$\tilde{F} = F \circ (F_w \times F_v)\colon \quad P_{i_r} \times^B \cdots \times^B P_{i_1} \times^B (Z_w \times Z_v) \to \bar{\mathcal{O}}$$

is a desingularization of $\bar{\mathcal{O}}$.

Proof. We have already seen that both maps F and $F_w \times F_v$ are proper birational morphisms. The space $P_{i_r} \times^B \cdots \times^B P_{i_1} \times^B (Z_w \times Z_v)$ is a homogeneous B-bundle over a nonsingular variety, hence it is nonsingular itself. □

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