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SINGULARITIES OF DIVISORS ON FLAG VARIETIES VIA HWANG'S PRODUCT THEOREM

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Abstract. We give an alternative proof of a recent result by B. Pasquier stating that for a generalized flag variety $X = G/P$ and an effective $\mathbb{Q}$-divisor D stable with respect to a Borel subgroup the pair (X, D) is Kawamata log terminal if and only if $\lfloor D \rfloor = 0$.

1. Introduction

Let G be a connected reductive algebraic group over $\mathbb{C}$. Recall that a horospherical G-variety X is a normal G-variety with an open G-orbit isomorphic to a torus fibration G/H over a flag variety G/P, where P is a parabolic subgroup in G and P/H is a torus. In [7] Boris Pasquier shows that for a horospherical variety X and an effective $\mathbb{Q}$-divisor D stable with respect to a Borel subgroup the pair (X, D) is Kawamata log terminal if and only if $\lfloor D \rfloor = 0$.

An essential part of the proof is the case when X itself is a flag variety G/P. In this case Pasquier uses a Bott–Samelson resolution to provide an explicit log resolution of the pair $(G/P, D)$, and to check that this pair is Kawamata log terminal using rather heavy combinatorics related to root systems. Further on, he uses the latter resolution to provide a log resolution of a general horospherical pair, and he uses Kawamata log terminality of $(G/P, D)$ to establish the same result in general.

The main purpose of this note is to prove a similar result for a variety G/P avoiding explicit log resolutions, and instead using the Product Theorem for log canonical thresholds due to Jun-Muk Hwang, see [5]. In particular, we will not assume that the $\mathbb{Q}$-divisor D is stable under the action of the Borel subgroup. Note that while this approach does not allow one to get rid of Bott–Samelson resolutions because they are needed for the case of general horospherical varieties, it does allow to avoid the computations from [7, §5] and to replace them by easier computations of Proposition 4.4 below.

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The plan of the paper is as follows. In §2 we recall definitions and properties of log canonical thresholds. In §3 we recall the basic facts about geometry of flag varieties and give a precise statement of our main result, which is Theorem 3.2. In §4 we introduce more notation and prove the main theorem.

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2. Log canonical thresholds

In this section we recall definitions and some properties of log canonical thresholds. We refer a reader to [6, §8] for (much) more details.

Let X be a smooth complex algebraic variety, D be an effective $\mathbb{Q}$-divisor. Choose a point $x \in X$.

Definition 2.1. If D is a Cartier divisor locally defined by the equation $f = 0$, then the log canonical threshold $\operatorname{lct}_x(D)$ of D near x is defined by

$$\operatorname{lct}_x(D) = \sup\left\{ c > 0 \;\middle|\; \frac{1}{|f|^c} \in L^2_{loc} \right\}.$$

In particular, if x is not contained in the support of D, we put $\operatorname{lct}_x(D) = +\infty$.

If D is an arbitrary $\mathbb{Q}$-divisor, then we define $\operatorname{lct}_x(D) = \operatorname{lct}_x(rD)/r$ for sufficiently divisible integer r. The log canonical threshold $\operatorname{lct}(X, D)$ of D is defined as the infimum of $\operatorname{lct}_x(D)$ over all $x \in X$.

Definition 2.2. The pair (X, D) is said to be *Kawamata log terminal* if the inequality $\operatorname{lct}(X, D) > 1$ holds, and *log canonical* if the inequality $\operatorname{lct}(X, D) \geqslant 1$ holds.

Let $L \in \operatorname{Pic}(X)$ be a line bundle such that the linear system $|L|$ is non-empty. We define $\operatorname{lct}(X, L)$ as the infimum $\inf_{\Delta \in |L|} \operatorname{lct}(X, \Delta)$. The following theorem is taken from [5].

Theorem 2.3 (see [5, §2])**.** *Let $f\colon X \to Y$ be a smooth projective morphism between two smooth projective varieties, $y \in Y$, and $X_y = f^{-1}(y)$ be the fiber over y. Let D be an effective divisor on X, and let L be the restriction of $\mathcal{O}_X(D)$ to X_y. Then one of the following holds*:

(i) *either* $\operatorname{lct}_x(D) \geqslant \operatorname{lct}(X_y, L)$ *for each* $x \in X_y$,

(ii) *or* $\operatorname{lct}_{x_1}(D) = \operatorname{lct}_{x_2}(D)$ *for any two points* $x_1, x_2 \in X_y$.

One can define the *global log canonical threshold*

$$\operatorname{lct}(X) = \inf\left\{\operatorname{lct}(X, \Delta) \mid \Delta \sim_{\mathbb{Q}} -K_X \text{ is an effective } \mathbb{Q}\text{-divisor}\right\},$$

where $-K_X$ is the anticanonical class of X. This definition makes sense if some positive multiple of $-K_X$ is effective; for example, this holds for Fano varieties and for spherical varieties (see [1, §4]). For more properties of $\operatorname{lct}(X)$ see [3]; for its relation to the α-invariant of Tian see [3, Appendix A].

3. Flag varieties

Let G be a connected reductive algebraic group. We fix a Borel subgroup B in G and a maximal torus $T \subset B$. Denote by R the root system of G, and by $S \subset R$ the set of simple roots in R, where the positive roots are the roots of (B, T). The Weyl group of R will be denoted by W; let $\ell\colon W \to \mathbb{Z}_{\geqslant 0}$ be the length function on W.

Let $P \supset B$ be a parabolic subgroup in G. Then G/P is a (generalized) partial flag variety. Denote by I the set of simple roots of the Levi subgroup of P; in particular, for $P = B$ we have $I = \varnothing$ and for P maximal the set I is obtained from S by removing exactly one simple root. For example, if $G = \mathrm{SL}_n(\mathbb{C})$, then for the maximal parabolic subgroup P corresponding to $S \setminus \{\alpha_k\}$, $1 \leqslant k \leqslant n-1$, the homogeneous space G/P is the Grassmannian $\mathrm{Gr}(k, n)$.

For a subset $I \subset S$ of the set of simple roots, let $W_P \subset W$ be the subgroup of W generated by the simple reflections s_α, where $\alpha \in I$. In each left coset from W/W_P there exists a unique element of minimal length. Denote the set of such elements by W^P; we will identify it with W/W_P. It is well-known (see, for instance, [2, §1.2]) that the partial flag variety G/P admits a Schubert decomposition into orbits of B, and the orbits are indexed by the elements of W^P:

$$G/P = \bigsqcup_{w \in W^P} BwP/P.$$

Moreover, the dimension of the cell BwP/P equals the length of w. The closures of these cells are called *Schubert varieties*; we denote them by $Y_w = \overline{BwP/P}$.

Denote by w_0 and w_0^P the longest elements in W and W_P, respectively. Then the length $\ell(w_0 w_0^P)$ is the dimension of G/P. We shall also need the *opposite Schubert varieties*

$$Y^w = \overline{w_0 B w_0 w P/P} = w_0 Y_{w_0 w}.$$

If $w \in W^P$, then $w_0 w w_0^P$ also belongs to W^P (i.e., is the shortest representative in its left coset $w_0 w W_P = w_0 w w_0^P W_P$); in this case $\dim Y_w = \ell(w)$ and

$$\dim Y^w = \ell(w_0 w_0^P) - \ell(w).$$

From the definition of Y_w and Y^w we readily see that the cohomology classes $[Y_w]$ and $[Y^{w_0 w w_0^P}]$ in $H^\bullet(G/P, \mathbb{Z})$ are equal.

Irreducible B-stable divisors of G/P are the Schubert varieties of codimension 1. Denote them by

$$D_\alpha = \overline{B w_0 s_\alpha w_0^P P/P} = w_0 Y^{s_\alpha}.$$

The following proposition is a standard fact on Schubert varieties (cf. [2, §1.4]).

Proposition 3.1. (i) *The divisors D_α for $\alpha \in S \setminus I$ freely generate* $\mathrm{Pic}(G/P)$, *so one has* $\mathrm{rk}\,\mathrm{Pic}(G/P) = |S \setminus I|$. *In particular, for P*

maximal one has $\operatorname{Pic}(G/P) \cong \mathbb{Z}$*, and there is a unique* B*-stable prime divisor.*

(ii) *The classes of Schubert varieties* $[Y_w] \in H^\bullet(G/P,\mathbb{Z})$ *freely generate* $H^\bullet(G/P,\mathbb{Z})$ *as an abelian group. The elements of this basis are Poincaré dual to the classes of the corresponding opposite Schubert varieties: if* $w, v \in W^P$ *and* $\ell(w) = \ell(v)$*, then*

$$[Y^w] \smile [Y_v] = \delta_{wv} \qquad \textit{for each } w, v \in W.$$

In particular, the classes of one-dimensional Schubert varieties, that is, of B*-stable curves* $\overline{Bs_\alpha P/P} \subset G/P$*, are dual to the classes of divisors:*

$$D_\alpha \smile [\overline{Bs_\beta P/P}] = \delta_{\alpha\beta} \qquad \textit{for each } \alpha, \beta \in S \setminus I.$$

The purpose of our paper is to give a new proof of the following result.

Theorem 3.2 (see [7, Theorem 3.1])**.** *Let* $D \sim \sum a_\alpha D_\alpha$*, where* a_α *are non-negative rational numbers, be an effective non-zero* $\mathbb{Q}$*-divisor. Then*

$$\operatorname{lct}(G/P, D) \geqslant \frac{1}{\max a_\alpha}.$$

In particular, the pair $(G/P, D)$ *is Kawamata log terminal provided that all* a_α *are less than 1.*

Remark 3.3. One can show that every effective divisor on G/P is linearly equivalent to an *effective* B-stable divisor. In other words, the classes of divisors D_α in the $\mathbb{Q}$-vector space $\operatorname{Pic}(G/P) \otimes \mathbb{Q}$ span the cone of effective divisors. Thus the assumption of Theorem 3.2 requiring that a_α are non-negative is implied by effectiveness of D; we keep it just to make the assertion more transparent.

Remark 3.4. If the $\mathbb{Q}$-divisor D of Theorem 3.2 is B-stable, then in addition to the inequality given by Theorem 3.2 we have an obvious opposite inequality, because D_α is an effective divisor. Therefore, in this case we recover the equality given by [7, Theorem 3.1].

A by-product of Theorem 3.2 is the following assertion on global log canonical thresholds of complete flag varieties that is well known to experts.

Corollary 3.5. *One has* $\operatorname{lct}(G/B) = 1/2$.

Proof. One has $-K_{G/B} \sim \sum_{\alpha \in S} 2D_\alpha$. Thus $\operatorname{lct}(G/B) \geqslant 1/2$ by Theorem 3.2. The opposite inequality is implied by the fact that the divisor D_α is effective. □

4. Proof of the main theorem

In this section we prove Theorem 3.2.

Fix a simple root $\alpha \in S \setminus I$. Let $J = I \cup \{\alpha\}$, and let P' be the parabolic subgroup corresponding to J; then $P \subset P'$. There is a G-equivariant fibration $\pi_\alpha \colon G/P \to G/P'$.

Let X_α be a fiber of this fibration. Consider the Dynkin diagram of G; its vertices correspond to simple roots from S. Let $\overline{J}$ be the connected component containing α of the subgraph spanned by the vertices of J. This component is the Dynkin diagram of a connected simple algebraic group $\overline{G}$. Let $\overline{P}_\alpha$ be a maximal parabolic subgroup of $\overline{G}$ with the set of roots $\overline{J} \setminus \{\alpha\}$. Then X_α is isomorphic to the $\overline{G}$-homogeneous space $\overline{G}/\overline{P}_\alpha \cong P'/P$.

Example 4.1. Let $G = \mathrm{SL}_n(\mathbb{C})$. Its Dynkin diagram is A_{n-1}; denote its simple roots by $\alpha_1, \ldots, \alpha_{n-1}$. Put $I = S \setminus \{\alpha_{d_1}, \ldots, \alpha_{d_r}\}$, where $1 \leqslant d_1 < \cdots < d_r \leqslant n-1$. We also formally set $d_0 = 0$ and $d_{r+1} = n$. Then G/P is a partial flag variety

$$\mathrm{Fl}(d_1, \ldots, d_r) \cong \{U_1 \subset \cdots \subset U_r \subset \mathbb{C}^n \mid \dim U_j = d_j\}.$$

Let $\alpha = \alpha_{d_s}$, where $1 \leqslant s \leqslant r$, and $J = I \cup \{\alpha\}$. Then G/P' is an $(r-1)$-step flag variety $\mathrm{Fl}(d_1, \ldots, \widehat{d_s}, \ldots, d_r)$, and the map $\pi_\alpha \colon G/P \to G/P'$ is given by forgetting the s-th component of each flag. The fibers of this projection are isomorphic to the Grassmannian $\mathrm{Gr}(d_s - d_{s-1}, d_{s+1} - d_{s-1})$.

According to Proposition 3.1(i), since $\overline{P}_\alpha \subset \overline{G}$ is a maximal parabolic subgroup, one has $\operatorname{Pic} X_\alpha \cong \mathbb{Z}$. Let H_α be the ample generator of $\operatorname{Pic} X_\alpha$.

Theorem 4.2 ([5, Theorem 2]; see also [4])**.** *Let k be a positive integer. Then one has* $\mathrm{lct}(X_\alpha, kH_\alpha) = 1/k$.

Remark 4.3. The assertion of [5, Theorem 2] is that the inequality $\mathrm{lct}(X_\alpha, kH_\alpha) \geqslant 1/k$ holds. This is equivalent to Theorem 4.2 since the linear system $|H_\alpha|$ is always non-empty by Proposition 3.1(i).

The following computation will be the central point of our proof of Theorem 3.2.

Proposition 4.4. *For each $\beta \in S \setminus I$ one has*

$$D_\alpha|_{X_\beta} \sim \begin{cases} H_\alpha & \text{if } \alpha = \beta; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As we discussed above, the fiber X_β can be identified with the variety $P'/P \subset G/P$. It is a flag variety with the Picard group of rank one; since the classes of B-stable curves are dual to the classes of (B-stable) divisors (see Proposition 3.1(ii)), X_β contains a unique B-stable curve. This curve has the form $\overline{Bs_\beta P/P} \subset P'/P \cong X_\beta$. Its class in $H^\bullet(X_\beta, \mathbb{Z})$ is Poincaré dual to the ample generator H_β of $\mathrm{Pic}(X_\beta, \mathbb{Z})$.

At the same time, as it was stated in Proposition 3.1(ii), the intersection of $\overline{Bs_\beta P/P}$ with D_α equals the class of a point if $\alpha = \beta$ and zero otherwise. □

Proof of Theorem 3.2. Replacing D by its appropriate multiple, we may assume that it is a Cartier divisor, not just a $\mathbb{Q}$-divisor. Put $a = \max a_\alpha$. Suppose that $\mathrm{lct}(X, D) < 1/a$.

Pick a point $x \in X$ such that $\mathrm{lct}_x(D) < 1/a$. Choose an index α from $S \setminus I$, and let $[D|_{X_\alpha}]$ be the class of the restriction of $\mathcal{O}_X(D)$ to X_α in $\mathrm{Pic}(X_\alpha)$. According to Proposition 4.4, one has $[D|_{X_\alpha}] \sim a_\alpha H_\alpha$. Theorem 4.2 implies that

$$\mathrm{lct}(X_\alpha, [D|_{X_\alpha}]) \geqslant \frac{1}{a_\alpha} \geqslant \frac{1}{a};$$

this trivially includes the case when $a_\alpha = 0$. Without loss of generality we can suppose that the fiber X_α passes through the point x. The above means that the alternative (i) in Theorem 2.3 never holds.

Let $\alpha_1, \dots, \alpha_r$ be the simple roots from $S \setminus I$. Let $\widetilde{X}_1$ be the fiber of π_{α_1} passing through the point x. For each $i = 2, \dots, r$ let $\widetilde{X}_i$ be the union of all fibers of π_{α_i} passing through the points of $\widetilde{X}_{i-1}$. In particular, one has $\widetilde{X}_r = X$.

First apply Theorem 2.3 to the point x and the fibration π_{α_1}. It implies that for each $x_1 \in \widetilde{X}_1$ we have $\mathrm{lct}_{x_1}(D) < 1/a$. Now apply it to each point $x_1 \in \widetilde{X}_1$ and the fibration π_{α_2}. We see that for each $x_2 \in \widetilde{X}_2$ the inequality $\mathrm{lct}_{x_2}(D) < 1/a$ holds. Proceeding by induction, we obtain the same inequality for every point in $\widetilde{X}_r = X$. In particular, each point of X is contained in the support of D, which is a contradiction. $\square$

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