Schubert polynomials and pipe dreams

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Topology of Torus Actions and Applications to Geometry and Combinatorics
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Outline

1. General definitions
   - Flag varieties
   - Schubert varieties and Schubert polynomials
   - Pipe dreams and Fomin–Kirillov theorem
2. Numerology of Schubert polynomials
   - Permutations with many pipe dreams
   - Catalan numbers and Catalan–Hankel determinants
3. Combinatorics of Schubert polynomials
   - Pipe dream complexes
   - Generalizations for other Weyl groups
4. Open questions

Flag varieties

- $G = \mathrm{GL}_n(\mathbb{C})$
- $B \subset G$ upper-triangular matrices
- $Fl(n) = \{V_0 \subset V_1 \subset \dots \subset V_n \mid \dim V_i = i\} \cong G/B$

Theorem (Borel, 1953)

$$\mathbb{Z}[x_1, \dots, x_n]/(x_1 + \dots + x_n, \dots, x_1 \dots x_n) \cong H^*(G/B, \mathbb{Z}).$$

This isomorphism is constructed as follows:

- $\mathcal{V}_1, \dots, \mathcal{V}_n$ tautological vector bundles over G/B;
- $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$ $(1 \le i \le n)$;
- $x_i \mapsto -c_1(\mathcal{L}_i)$;
- The kernel is generated by the symmetric polynomials without constant term.

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Schubert varieties

- $G/B = \bigsqcup_{w \in S_n} B^- wB/B$ — *Schubert decomposition*;
- $X^w = \overline{B^- wB/B}$, where B^- is the opposite Borel subgroup;
- $H^*(G/B, \mathbb{Z}) \cong \bigoplus_{w \in S_n} \mathbb{Z} \cdot [X^w]$ as abelian groups.

Question

Are there any "nice" representatives of $[X^w]$ in $\mathbb{Z}[x_1, \dots, x_n]$?

Answer: Schubert polynomials

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- $\mathfrak{S}_w \mapsto [X^w] \in H^*(G/B, \mathbb{Z})$ under the Borel isomorphism;
- Defined by A. Lascoux and M.-P. Schützenberger, 1982;
- Combinatorial description: S. Billey and N. Bergeron, S. Fomin and An. Kirillov, 1993–1994.

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Pipe dreams

Let $w \in S_n$. Consider a triangular table filled by $+$ and ⌟⌜, such that:

- the strands intertwine as prescribed by w;
- no two strands cross more than once (*reduced* pipe dream).

Pipe dreams for $w = (1432)$

Pipe dream $P \rightsquigarrow$ monomial $x^{d(P)} = x_1^{d_1} x_2^{d_2} \dots x_{n-1}^{d_{n-1}}$,
$d_i = \#\{+\text{'s in the } i\text{-th row}\}$

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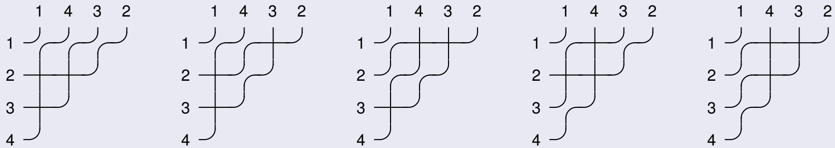

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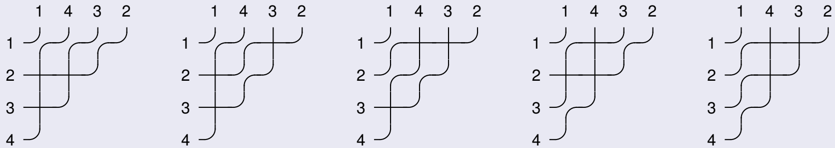

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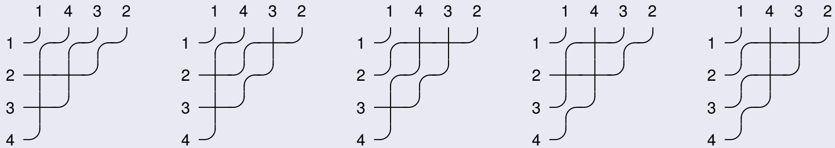

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Pipe dreams and Schubert polynomials

Theorem (S. Fomin, An. Kirillov, 1994)

Let $w \in S_n$. Then

$$\mathfrak{S}_w(x_1, \dots, x_{n-1}) = \sum_{w(P)=w} x^{d(P)},$$

where the sum is taken over all reduced pipe dreams P corresponding to w.

Example

$$\mathfrak{S}_{1432}(x_1, x_2, x_3) = x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1^2 x_2.$$

Corollary

$$\mathfrak{S}_w(1, \dots, 1) = \#\{P \mid \text{ pipe dream } P \text{ corresponds to } w\}.$$

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Pipe dreams and torus actions

Toric degeneration of a flag variety (N. Gonciulea, V. Lakshmibai)

$$Fl(n) \to \widetilde{Fl}(n)$$

- $\widetilde{Fl}(n)$ is a *singular* (but still irreducible!) toric variety.
- It corresponds to Gelfand-Zetlin polytope $GZ(n)$.

Degenerate Schubert varieties (A. Knutson, M. Kogan, E. Miller)

$$X^w \to \widetilde{X}^w \subset \widetilde{Fl}(n)$$

- $\widetilde{X}^w$ may be reducible!
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Permutations with the maximal number of pipe dreams

How many pipe dreams can a permutation have?

Find $w \in S_n$, such that $\mathfrak{S}_w(1, \ldots, 1)$ is *maximal*.

Answers for small n

- $n = 3$: $w = (132)$, $\mathfrak{S}_w(1) = 2$;
- $n = 4$: $w = (1432)$, $\mathfrak{S}_w(1) = 5$;
- $n = 5$: $w = (15432)$ and $w = (12543)$, $\mathfrak{S}_w(1) = 14$;
- $n = 6$: $w = (126543)$, $\mathfrak{S}_w(1) = 84$;
- $n = 7$: $w = (1327654)$, $\mathfrak{S}_w(1) = 660$.

Definition

$w \in S_n$ is a *Richardson permutation*, if for $(k_1, \ldots, k_r)$, $\sum k_i = n$,

$$w = \begin{pmatrix} 1 & 2 & \ldots k_1 & k_1+1 & \ldots & k_1+k_2 & k_1+k_2+1 & \ldots \\ k_1 & k_1-1 & \ldots 1 & k_1+k_2 & \ldots & k_1+1 & k_1+k_2+k_3 & \ldots \end{pmatrix}.$$

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Counting pipe dreams of Richardson permutations

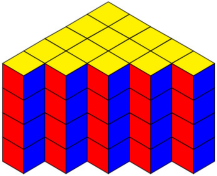
Let $w^0_{k,m} = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & k+m \\ 1 & 2 & \dots & k & k+m & \dots & k+1 \end{pmatrix}$.

Theorem (A. Woo)

Let $w = w^0_{1,m}$. Then $\mathfrak{S}_w(1) = Cat(m)$.

Theorem (S. Fomin, An. Kirillov)

Let $w = w^0_{k,m}$. Then $\mathfrak{S}_w(1)$ is equal to the number of "Dyck plane partitions of height k", i.e., subdiagrams of the prism of height k and side length m.



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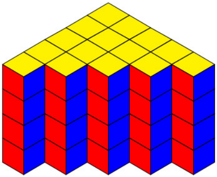

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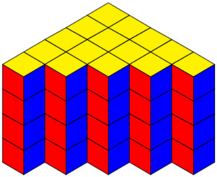
Let $w = w^0_{k,m}$. Then $\mathfrak{S}_w(1)$ is equal to the number of "Dyck plane partitions of height k", i.e., subdiagrams of the prism of height k and side length m.



Determinantal formulas for Schubert polynomials

Theorem (G. Merzon, E. S.)

Let $w = w^0_{k,m}$. Then the following "Jacobi–Trudi type" formula holds:

$$\frac{\mathfrak{S}_w(x_1,\dots,x_{m+k-1})}{x_1^m\dots x_k^m x_{k+1}^{m-1}\dots x_{m+k-1}} = \det\left(\frac{\mathfrak{S}_{w^0_{1,m+i+j}}(x_{i+1},\dots,x_{m+i+j-1})}{x_{i+1}^{m+j-1}x_2^{m+j-2}\dots x_{m+i+j-1}}\right)_{i,j=0}^{k-1}$$

Corollary

$\mathfrak{S}_w(1)$ is equal to a $(k\times k)$ Catalan–Hankel determinant:

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Pipe dream complex (A. Knutson, E. Miller)

- To each permutation $w \in S_n$ one can associate a shellable CW-complex $PD(w)$;
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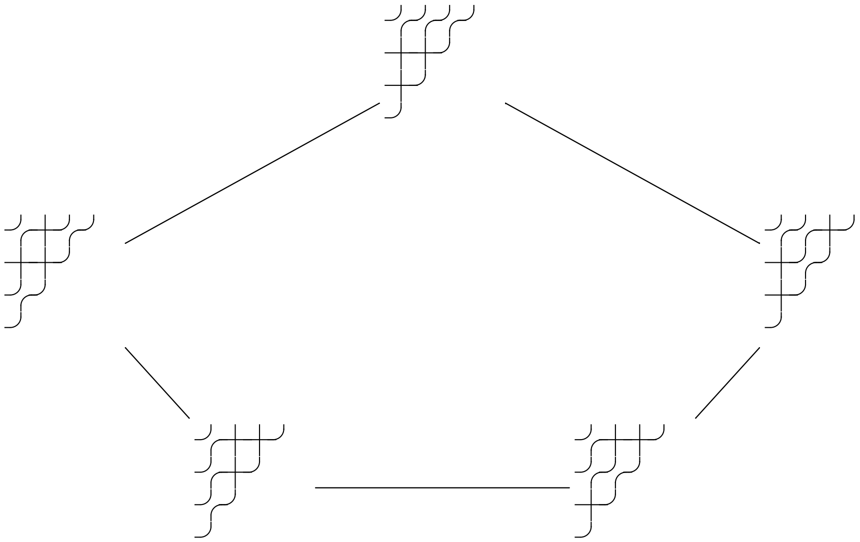

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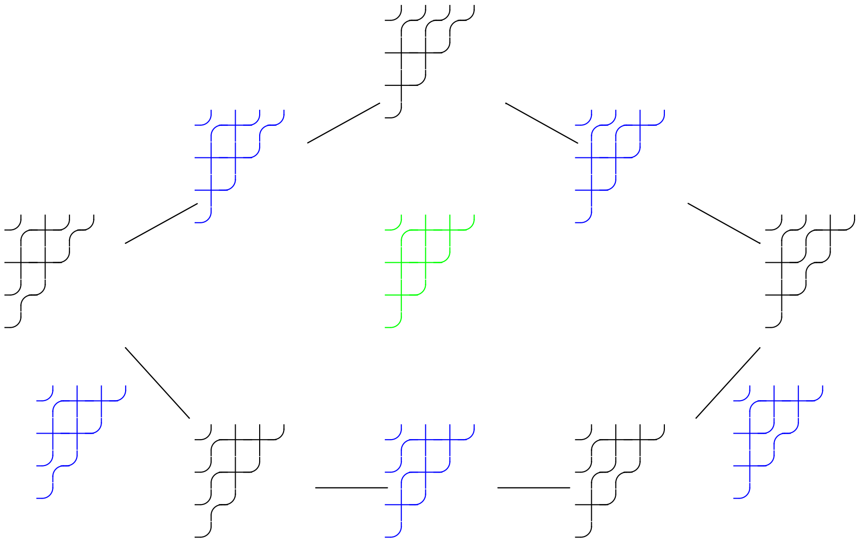
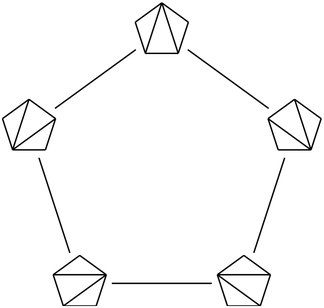
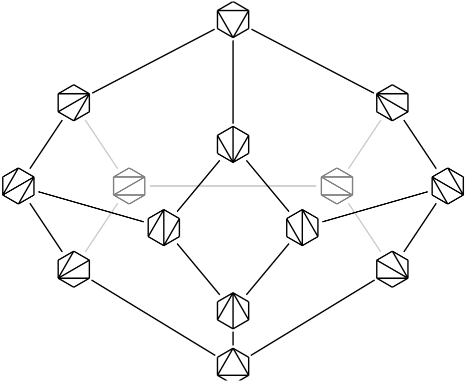


Associahedra are PD-complexes

Theorem (probably folklore? also cf. V. Pilaud)

Let $w = w^0_{1,n} = (1, n+1, n, \ldots, 3, 2) \in S_{n+1}$ be as in Woo's theorem. Then $PD(w)$ is the Stasheff associahedron.





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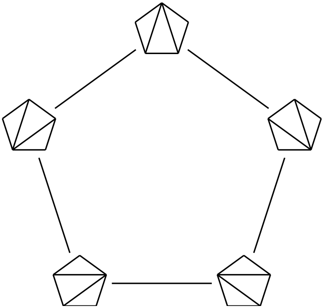

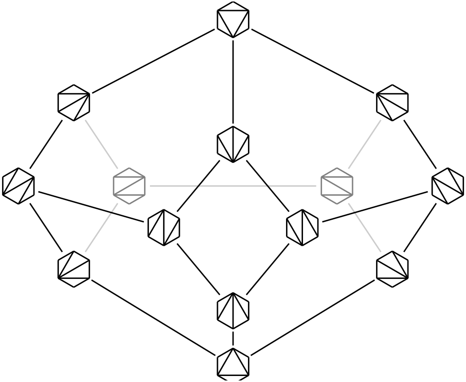

Zoo of pipe dream complexes

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$$C(n, d) = Conv((t_i, t_i^2, \dots, t_i^d))_{i=1}^n \subset \mathbb{R}^d.$$

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Generalization: other Weyl groups

- G semisimple group, W its Weyl group;
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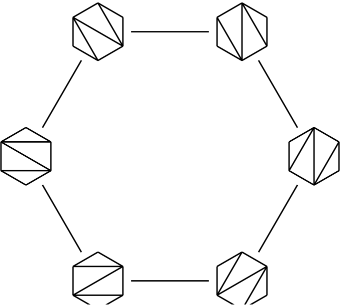
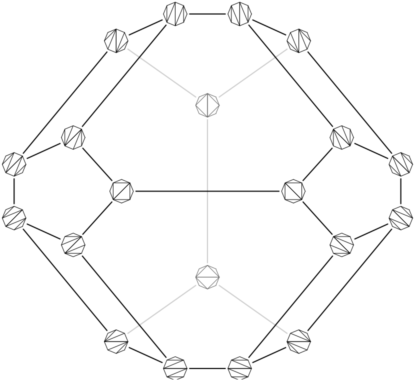
Cyclohedra are subword complexes

Theorem

Let W be of type C_n, generated by $s_1, \ldots, s_n$, where s_1 corresponds to the longest root α_1. Consider a Richardson element $w = (s_1 s_2 \ldots s_{n-1})^{n-1}$. Then $PD(w)$ is a cyclohedron.





Questions about $PD(w)$

- Is it true that $PD(w)$ is always a polytope?
- At least, is it true when w is a Richardson element?
- If yes, what is the combinatorial meaning of this polytope?
- Possible answer: associahedra, cyclohedra etc. are examples of *2-truncated cubes* (cf. V. Buchstaber's works). Is it true that $PD(w)$ are 2-truncated cubes?

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