Maps that take all lines to circles

V. Timorin*

*State University of New York
at Stony Brook

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Some classical theorems

Theorem (Möbius, 1827)

Suppose that $f : \mathbb{RP}^n \to \mathbb{RP}^n$ is a one-to-one map taking all straight lines to straight lines. Then f is a projective transformation.

Theorem (Möbius, 1820s)

Suppose that $f : S^n \to S^n$ is a one-to-one map taking all circles to circles. Then f is a Möbius transformation (i.e. an element of the group generated by inversions).

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August Möbius 1790–1868

Automorphisms of geometric structures

Möbius' theorems are important because they describe all automorphisms of the most fundamental geometric structures:

- the projective structure and
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A problem

Problem

Describe all (sufficiently smooth) (one-to-one) maps from an open subset of $\mathbb{RP}^n$ to an open subset of S^n that take all lines to circles.

Definition

We say that a map $f : U \subset \mathbb{RP}^n \to V \subset S^n$ takes all lines to circles if the image of each straight segment contained in U is an arc of Euclidean circle contained in V.

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Theorem (Beltrami, 1880s)

Let g be a Riemannian metric on an open subset of $\mathbb{RP}^n$ such that all geodesics are straight segments. Then g is a classical metric, i.e. has constant sectional curvature.

Theorem (Segre, 1950s)

Let g be a Riemannian metric on an open subset of S^2 such that all geodesics are arcs of circles. Then g is a classical metric, i.e. has constant Gaussian curvature.

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Eugenio Beltrami 1835–1900

Beniamino Segre 1903–1977

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Motivation: Nomography

Nomograms:

A nomogram is a planar picture representing a function of many variables. Usually, it consists of several curves equipped with scalings. One uses a straightedge or a compass to read the output.

Compass vs Straightedge:

Compass is more accurate than a straightedge, because it draws round circles even when deformed. Thus circular nomograms are more practical, while nomograms with aligned points (those using a straightedge) are easier theoretically.

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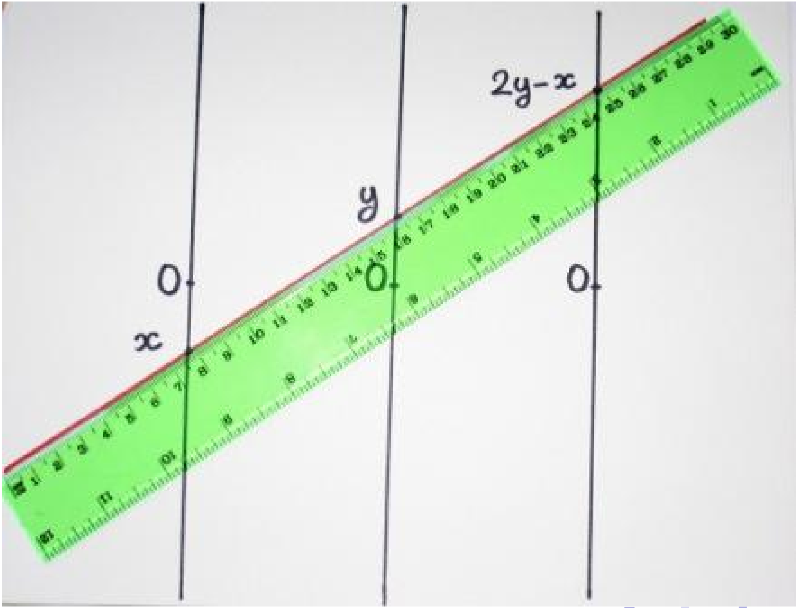
A simple nomogram computing $(x, y) \mapsto 2y - x$



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Results in dimensions 2 and 3

Theorem (Khovanskii, 70s)

Suppose that a diffeomorphism $f : U \subset \mathbb{RP}^2 \to V \subset S^2$ *takes all lines to circles. Up to projective transformations in the source and Möbius transformations in the target, there are only 3 such maps* f*, and they correspond to classical models of classical geometries (i.e. Euclidean, spherical or hyperbolic geometry).*

Theorem (Izadi,2003)

The same result is true for diffeomorphisms $f : U \subset \mathbb{RP}^3 \to V \subset S^3$.

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Example

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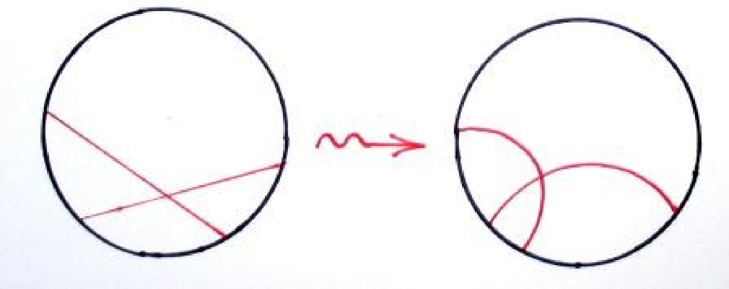
A map establishing an isomorphism between the Klein model and the Poincaré model of the classical hyperbolic geometry.



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Example

Complex projective transformations $U \subset \mathbb{C}^2 \to V \subset \mathbb{C}^2$ take all lines to circles.

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The (left and right) quaternionic Hopf fibrations $\mathbb{RP}^7 \to S^4$ take all lines to circles, so do their restrictions to $\mathbb{RP}^4 \subset \mathbb{RP}^7$.

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Quaternionic Hopf fibrations

Definition

Consider the map

$$\mathbb{H}^2 \to \mathbb{HP}^1, \quad (q_1, q_2) \mapsto [q_1 : q_2],$$

where $\mathbb{HP}^1$ is the (left or right) quaternionic projective line. Note that $\mathbb{HP}^1 = S^4$. This map factors through the real projectivization

$$\mathbb{H}^2 \to \mathbb{RP}^7$$

to give a map

$$\mathbb{RP}^7 \to S^4$$

called a quaternionic Hopf fibration.

Results in dimension 4

Theorem (VT)

Let $f : U \subset \mathbb{RP}^4 \to V \subset S^4$ be a diffeomorphism taking all lines to circles. Then

- *either f corresponds to one of the three classical geometries*
- *or f is of the form $\mathbb{RP}^4 \hookrightarrow \mathbb{RP}^7 \to S^4$, where the first arrow is a projective embedding, and the second is a quaternionic Hopf fibration.*

There are much more maps of the second kind.

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Let $f : U \subset \mathbb{R}\mathrm{P}^4 \to V \subset S^4$ be a diffeomorphism taking all lines to circles. Then

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Results in dimension 4

Theorem (VT)

Let g be a Kähler metric on an open subset of $\mathbb{C}^2$ such that all geodesics are arcs of circles (or straight segments). Then g has constant holomorphic sectional curvature, i.e. g is a "complexification" of one of the classical geometries.

Higher dimensions

In higher dimensions, the problem is still OPEN, but there are remarkable relations with classical problems in algebra, including:

- Hurwitz problem on sums of squares,
- quadratic maps between spheres,
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Results in higher dimensions

Theorem (VT)

Suppose that $f : U \subset \mathbb{RP}^n \to V \subset S^n$ *takes all lines passing through a particular point* $p \in U$ *to circles. Also, let* f *be differentiable sufficiently many times and satisfy* $\mathrm{rank}(d_p f) > 1$. *Then there is a fractional quadratic map* $Q : \mathbb{RP}^n \dashrightarrow S^m$ *such that* $f(l) = Q(l)$ *for all lines* $l \ni p$.

Open problems in algebra

Problem

Describe all fractional quadratic maps $\mathbb{RP}^n \dashrightarrow S^m$.

Remark:

This problem is very difficult. A special case of it is the following

Problem (Hurwitz, 1898)

Describe all triples of integers (r, s, n) *such that*

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2,$$

where z_i *are some bilinear combinations of* x_j *and* y_k.

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(2,2,2) = multiplication of complex numbers.

(4,4,4) = multiplication of quaternions.

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(r, n, n) = representations of Clifford algebras.

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Fractional quadratic transformations in terms of Hurwitz formulas

Set

$$X = (x_1, \dots, x_r), \quad Y = (y_1, \dots, y_s), \quad Z = (z_1, \dots, z_n).$$

Then

$$Q[X, Y] = \left(\frac{2Z}{|X|^2 + |Y|^2}, \frac{|X|^2 - |Y|^2}{|X|^2 + |Y|^2} \right)$$

is a fractional quadratic map from $\mathbb{RP}^{r+s-1}$ to S^n.

Some similar problems

Problem

Describe all maps that take one nice class of curves to another nice class of curves.

E.g.

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Describe all maps that take all lines to conics.

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A sample result

Theorem (VT)

Suppose that a local analytic diffeomorphism $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *takes all lines through 0 to conics and satisfies a minor non-degeneracy assumption. Then, for almost all lines* $l \ni 0$*, the conic* $f(l)$ *has 3 points of tangency with a curve of class 3 (i.e. dual curve to a cubic).*

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